Common Fixed Point Theorem in G-Metric Space Using Weakly Compatible Mappings

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Abstract:- In this paper we present a common fixed point theorem for self mappings using the concept of weakly compatible mappings.

Key words:- Complete G-metric Space, Weakley compatible mappings.

I. Introduction

There have been various generalizations of metric space such as Gahler who gave the concept of 2-metric space, Dhage [1,2] who gave the concept of D-metric. Mustafa and Sims [14,15] introduced a new generalized version of metric space & called it G-metric space after they had shown that most of the results related to D-metric space are invalid, Here we prove a common fixed point theorem in G-metric space using pairs of weakly compatible mappings.

II. Definitions and Preliminaries

We here begin with some definitions and results for G- metric spaces that will be used in the following sections .

Definition 2.1 [15] Let X be a nonempty set and let G; X x X x X--- >R^+ be a function satisfying the following axioms

(G_1) G (x, y, z) = 0 if x = y = z

(G_2) G (x, x, y) > 0, for all x, y ε X with x ≠ y

(G_3) G (x, x, y) ≤ G (x, y, z) for all x, y, z ε X with z ≠ y .

(G_4) G (x, y, z) = G (x, z, y) = G (y, z, x) =….. (symmetry in all three variables)

(G_5) G (x, y, z) ≤ G (x, a, a) + G (a, y, z), for all x, y, z, a ε x (rectangle inequality)

Then the function G is called a generalized metric or more specifically a G- metric on X, and the pair (X, G) is called a G- metric space .

Definition 2.2 **[15]** Let (X,G) be a G- metric space , let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ converges to a point x in X

if $\lim_{n,m \to \infty} G(x, x_n, x_m) = 0$

In other words for e ε > 0 there exists n_o ε N such that G (x, x_n, x_m) < ε for all n, m ≥ n_o Then x is called the limit of sequence $\{x_n\}$.

Definition 2.3 [15] Let (X,G) be a G- metric space , a sequence $\{x_n,\}$ is called G - Cauchy sequence if for given $\varepsilon > o$, there is $n_o \varepsilon N$ such that $G (x_n, x_m, x_e) < \varepsilon$ for all n, m ,$l \geq n_o$ that is if. $G (x_n , x_m, x_e) \to 0$ as n, m , $l \to \infty$

Definition 2.4 [15] Let A, B be self mappings on a G-metric space X. Then the pair (A, B) is said to be weakly compatible if they commute at their coincidence point , that is Ax= Bx implies that ABx = BAx for all x ε X.

Preposition 2.5 [15] Let (X, G) be a G-metric space, Then, the following are equivalent

(i) $\{x_n\}$is G- convergent to x
(ii) $G (x_n, x_n, x) \to 0$, as $n \to \infty$
(iii) $G (x_n, x, x,) \to 0$, as $n \to \infty$
(iv) $G (x_m, x_n, x) \to 0$ as n, $m \to \infty$

Preposition 2.6 [15] In a G-metric space (X, G) the following are equivalent

(i) The sequence $\{x_n\}$ is G- Cauchy

(ii) For every $\varepsilon > 0$, there exists $n_o \varepsilon N$ such that $G (x_n, x_m, x_m) < \varepsilon$ for all n , $m \geq n_0$.

Definition 2.7 [16]. Let ϕ denote the set of alternating distance functions

$\phi : [0, \phi [\to [0, \infty [$ which satifies following conditions

(i) ϕ is strictly increasing

(ii) ϕ is upper semi continuous from the right.

(iii) $\sum_{n=0}^{\infty} \phi (t) < \infty$ for all $t > 0$

(iv) $\phi (t) = 0 \Leftrightarrow t = 0$

Main Result

Let f, g, h, s, r and t be self mappings of a complete G-metric space (X, G) and

(i) $f (x) \subseteq t (X), g(X) \subseteq s (X), h (X) \subseteq r (X)$ and f (X) or g (X) or h (X) is a closed subset of X.

(ii) G (fx, gy, hz)

$\leq \phi$ {max [α (G (gy, fx, rx)

+ G (hz, gy, ty) + G (fx, sz, hz),

β (G (fx, rx, gy) + G (sz, fx, rx)),

γ (G(gy, ty, hz) + G (fx, gy, ty))] }

Where α, β, γ, >, 0 and $3 \propto +4\beta + 2\gamma < 1$

(iii) $\phi : R^+$ is increasing function such that $\phi (a) < a$ for all $a > 0$ and $\Sigma \phi (a) < \infty$ as $a \to \infty$

(iv) The pairs (f, r) , (g, t) and (h, s) are weakly compatible pairs.

Then the mappings f, g, h, r, s and t have a common unique fixed point in X.

Proof: Let $X_0 \in X$ be an arbitrary point. Then by (i) there exist $x_{1,} x_2, x_3 \in X$ such that

$fx_0 = tx_1 = y_0$, $gx_1 = sx_2 = y_1$ and $hx_2 = rx_3 = y_2$

applying the concept of mathematical induction we can define a sequence $\{y_n\}$ in X such that

$fx_{3n} = tx_{3n+1} = y_{3n}$,

$gx_{3n+1} = sx_{3n+2} = y_{3n+1}$ and

$hx_{3n+2} = rx_{3n+3} = y_{3n+2}$ for n = 0, 1, 2....

Now we prove that the sequence is a Cauchy sequence and for this we define

$d_m = G(y_m, y_{m+1}, y_{m+2})$

So we have

$d_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2})$

$= G(fx_{3n}, gx_{3n+1}, hx_{3n+2})$

$< \phi\{\max[\alpha(G(gx_{3n+1}, fx_{3n}, rx_{3n}) + G(hx_{3n+2}, gx_{3n+1}, tx_{3n+1}) + G(fx_{3n}, sx_{3n+2}, hx_{3n+2}),$

$\beta[G(fx_{3n}, rx_{3n}, gx_{3n+1}) \quad +G(sx_{3n+2}, fx_{3n}, rx_{3n})], \gamma[G(gx_{3n+1}, tx_{3n+1}) + G(gx_{3n+1}, tx_{3n+1})]\}$

$< \phi\{\max[\alpha(G(y_{3n+1}, y_{3n}, y_{3n-1}) + G(y_{3n+2}, y_{3n+1}, y_{3n}) + G(y_{3n}, y_{3n+1}, y_{3n+2})], \beta[G(y_{3n}, y_{3n-1}, y_{3n+1})$

$+ G(y_{3n+2}, y_{3n}, y_{3n-1})], \gamma[G(y_{3n+1}, y_{3n}, y_{3n+2}) + G(y_{3n+0}, y_{3n}, y_{3n})]\}$

$< \phi\{\max[\alpha G(d_{3n-1}, + d_{3n}, + d_{3n})\ 2\ \beta d_{3n-1}, 2\gamma d_{3n}]\}$

$< \phi\{\max[\alpha G(d_{3n-1}, + 2d_{3n},), 2\beta d_{3n-1}, 2\gamma d_{3n}]\}$

from the above inequality we will have following

Case I- If max = $\alpha(d_{3n-1} + 2d_{3n})$,then

$d_{3n} \le \phi(\alpha(d_{3n-1} + 2d_{3n}))$ as $\phi(t) < t$, hence we get

$d_{3n} \le \alpha(d_{3n-1} + 2d_{3n})$

$(1 - 2\alpha)\, d_{3n} \le \alpha\, d_{3n-1}$

$d_{3n} \le d_{3n-1}$

Case - II If max = $2\beta d_{3n-1}$ then $d_{3n} \le \phi(2\beta d_{3n-1})$ as $\phi(t) < t$ hence we get $d_{3n} \le 2\beta d_{3n-1}$

i.e. $d_{3n} \le d_{3n-1}$

Case - III If max = $2\gamma d_{3n}$ then $d_{3n} \le \phi(2\gamma d_{3n})$ again as $\phi(t) < t$ we get $d_{3n} \le d_{3n}$ which is a contradiction. Hence $d_{3n} \le d_{3n-1}$

If m = 3n+1, then

$d_{3n+1} = G(y_{3n+1}, y_{3n+2}, y_{3n+3})$

$= G(fx_{3n+1}, gx_{3n+2}, hx_{3n+3})$

$\le \phi\{\max[\alpha[G(gx_{3n+2}, fx_{3n+2}, rx_{3n+1}) + G(hx_{3n+3}, gx_{3n+2}, tx_{3n+2}) + G(fx_{3n+1}, sx_{3n+3}, hx_{3n+3})],$

$\beta[G(fx_{3n+1}, rx_{3n+1}, gx_{3n+2}) + G(Sx_{3n+3}, fx_{3n+1}, rx_{3n+1})], \gamma[G(gx_{3n+2}, tx_{3n+2}, hx_{3n+3})$

$+ G(fx_{3n+1}, gx_{3n+2}, tx_{3n+2})]]\}$.

$\le \phi\{\max[\alpha[G(y_{3n+2}, y_{3n+1}, y_{3n}) + G(y_{3n+3}, y_{3n+2}, y_{3n+1}) + G(y_{3n+1}, y_{3n+2}, y_{3n+3})], \beta[G(y_{3n+1}, y_{3n}, y_{3n+2}) +$

$G(y_{3n+2}, y_{3n+1}, y_{3n})], \gamma\ [G(y_{3n+2}, y_{3n+1}, y_{3n+3}) + G(y_{3n+1}, y_{3n+2}, y_{3n+1})]]\}$

$\leq \phi\{\max[\alpha(d_{3n}+2d_{3n+1}), 2\beta d_{3n}, 2\gamma d_{3n+1})]\}$

From the above inequality we have following cases.

Case- I If max = $\alpha(d_{3n}+2d_{3n+1})$ then $d_{3n+1} \leq \phi(\alpha(d_{3n}+2d_{3n+1})$ as $\phi(t) < t$, we get

$d_{3n+1} \leq \alpha(d_{3n}+2d_{3n+1})$

$(1-2\alpha)\, d_{3n+1} \leq \alpha\, d_{3n}$

$d_{3n+1} \leq d_{3n}$

Case- II If max = $2\beta\, d_{3n}$, then we get $d_{3n+1} \leq \phi(2\beta\, d_{3n})$ as $\phi(t) < t$, we get $d_{3n+1} \leq 2\beta\, d_{3n}$ or $d_{3n+1} \leq d_{3n}$

Case III If max = $2\gamma\, d_{3n+1}$, then we get $d_{3n+1} \leq \phi(2\gamma\, d_{3n+1})$ as $\phi(t), < t$ we get $d_{3n+1} \leq 2\gamma\, d_{3n+1}$ or $d_{3n+1} \leq d_{3n+1}$

which is a contradiction. Hence $d_{3n+1} \leq d_{3n}$

If m = 3n+2, then

$d_{3n+2} = G(y_{3n+2}, y_{3n+3}, y_{3n+4})$

$=G(fx_{3n+2}, gx_{3n+3}, hx_{3n+4}),$

$\leq \phi\{\max[\alpha[G(gx_{3n+3}, fx_{3n+2}, rx_{3n+2}) + G(hx_{3n+4}, gx_{3n+3}, tx_{3n+3}) + G(fx_{3n+2}, sx_{3n+4}, hx_{3n+4})]$

$\beta[G(fx_{3n+2}, rx_{3n+2}, gx_{3n+3}) + G(Sx_{3n+4}, fx_{3n+2}, rx_{3n+2})], \gamma[G(gx_{3n+3}, tx_{3n+3}, hx_{3n+4})$

$+ G(fx_{3n+2}, gx_{3n+3}, tx_{3n+3})]]\}$

$\leq \phi\{\max[\alpha(G(y_{3n+3}, y_{3n+2}, y_{3n+1}) + G(y_{3n+4}, y_{3n+3}, y_{3n+2})\, G(y_{3n+2}, y_{3n+3}, y_{3n+4})), \beta(G(y_{3n+2}, y_{3n+1}, y_{3n+3})$

$+ G(y_{3n+3}, y_{3n+2}, y_{3n+1})), \gamma(G(y_{3n+3}, y_{3n+2}, y_{3n+4}) + G(y_{3n+2}, y_{3n+3}, y_{3n+2})]\}$

$\leq \phi\{\max[\alpha(d_{3n+1}, +d_{3n+2}, +d_{3n+2}), + \beta(d_{3n+1} + d_{3n+1}), \gamma(d_{3n+2} + d_{3n+2})]\}$

From the above inequality we have following cases.

Case. I If max = $\alpha(d_{3n+1}, + 2d_{3n+2})$ then $d_{3n+2} \leq \phi(\alpha(d_{3n+1}, +2d_{3n+2}))$ as $\phi(t) < t$ then we get

$d_{3n+2} \leq \alpha(d_{3n+1} + 2d_{3n+2})$ or $d_{3n+2} \leq \dfrac{\alpha}{1-2\alpha}\, d_{3n+1}$

$d_{3n+2} \leq d_{3n+1}$

Case II. If max = $2\beta\, d_{3n+1}$ then we get $d_{3n+2} \leq \phi(2\beta d_{3n+1})$ as $\phi(t) < t$, hence we get $d_{3n+2} \leq 2\beta d_{3n+1}$ or

$d_{3n+2} \leq d_{3n+1}$ which is the required result.

Case III. If max = $2\gamma\, d_{3n+2}$ then $d_{3n+2} \leq \phi(2\gamma d_{3n+2})$ as $\phi(t) < t$, then we get $d_{3n+2} \leq 2\gamma d_{3n+2}$ or $d_{3n+2} \leq d_{3n+2}$

which is a contradiction. From the above three cases we can say that $d_n \leq d_{n-1}$ for every $n \in N$. So by above inequality we get $d_n \leq q\, d_{n-1}$, where $q = 3\alpha + 4\beta + 2\gamma < 1$ i.e.

$d_n = G(y_n, y_{n+1}, y_{n+2})$

$\leq q\, G(y_{n-1}, y_n, y_{n+1})$

$\leq q^n\, G(y_0, y_1, y_2)$

also we have $G(x, x, y) \le G(x, y, z)$, hence we get $G(y_n, y_n\, y_{n+1}) \le q^n G(y_0, y_1, y_2)$ and

$G(y_n, y_n, y_m) \le (y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \text{-----} + G(y_{m-1}, y_{m-1}, y_m)$

i.e. we have $G(y_n, y_n, y_m) \le g^n G(y_0, y_1, y_2) + \text{------------------} + \text{------} + g^{m-1} G(y_0, y_1, y_2)$ hence we have

$$G(y_n.y_n,y_m) \le \frac{q^n - q^m}{1-q} G(y_0, y_1, y_2)$$

$$\le \frac{q^n}{1-q} G(y_0, y_1, y_2) \to 0$$

So the sequence $\{y_n\}$ is a Cauchy sequence in X and as X is complete $\{y_n\}$ will converge to y in X.

i.e. $\lim\limits_{n \to \infty} y_n = y$ $\lim\limits_{n,m \to \infty} fx_{3n} = \lim\limits_{n,m \to \infty} gx_{3n+1} = \lim\limits_{m,n \to \infty} hx_{3n+2} = \lim\limits_{m,n \to \infty} tx_{3n+1}$

$= \lim\limits_{m,n \to \infty} sx_{3n+2} = \lim\limits_{m,n \to \infty} r_{3n+3} = y$

Let h (X) is a closed subset of r(X). Then there exist $u \in X$ Such that $r u = y$.Now consider on

$G(fu,y,y) = G(fu, gx_{3n+1}, hx_{3n+2})$

$\le \phi\{\max[\alpha(G(gx_{3n+1}, fu, ru) + G(hx_{3n+2}, gx_{3n+1}, tx_{3n+1}) + G(fu, Sx_{3n+2}, hx_{3n+2})), \beta(G(fu, ru, gx_{3n+1})$

$+ G(Sx_{3n+2}, fu, ru)), \gamma(G(gx_{3n+1}, tx_{3n+1}, hx_{3n+2}) + G(fu, gx_{3n+1}, tx_{3n+1}))]\}$

$\le \phi\{\max[\alpha(G(y, fu, y) + G(y, y, y) + G(fu, y, y)), \beta(G(fu, y, y) + G(y, fu, ru)), \gamma(G(y, y, y) + G(fu, y, y)]\}$

$\le \phi\{\max[2\alpha G(fu, y, y), 2\beta G(fu, y, y), \gamma G(fy, y, y)]$

from the above inequality we can have two cases.

Case I If $\max = 2\alpha G(fu, y, y)$

then $G(fu, y, y) \le \phi(2\alpha G(fu, y, y))$ as $\phi(t) < t$, $G(fu, y, y,) \le 2\alpha G(fu, y, y)$ or $(1 - 2\alpha) G(fu, y, y) \le 0$.Hence $G(fu, y, y) = 0$ or $fu = y$

Case II If $\max = 2\beta G(fu, y, y)$

then $G(fu, y, y) \le \phi[2\beta G(fu, y, y)]$as $\phi(t)$, hence we get.

$G(fu, y, y) \le 2\beta G(fu, y, y)]$ or $(1-2\beta) G(fu, y, y) \le 0$, hence $G(fu, y, y) = 0$ or $fu = y$.

Case III If $\max = \gamma G(fu, y, y)$, then

$G(fu, y, y) \le \phi[\gamma G(fu, y, y)]$ as $\phi(t), < t$, hence we get.$G(fu, y, y) \le \gamma G(fu, y, y)$ or

$(1-\gamma) G(fu, y, y) \le 0$ or $G(fu, y, y) = 0$ this implies $fu = y$.Therefore $fu = ru = y$. Then by applying the definition of weak compatibility on the pair (r, f) we have $fru = rfu$.Hence $fy = ry$

Now we prove $fy = y$. On the contrary Let $fy \ne y$, then

$G(fy, gx_{3n+1}, hx_{3n+2}) \le \phi\{\max\{\alpha(G(gx_{3n+1}, fy, ry) + G(hx_{3n+2}, gx_{3n+1}, tx_{3n+1}) + G(fy, Sx_{3n+2}, hx_{3n+2})), \beta(G(fy, ry, gx_{3n+1})$

$+ G(Sx_{3n+2}, fy, gy)), + \gamma (G(gx_{3n+1}, tx_{3n+1}, hx_{3n+2}) + G(fy, gx_{3n+1}, tx_{3n+1})\}\}$

$\leq \phi\{ \max\{ \alpha (G(y, fy, fy) + G(y, y, y) + G(fy, y, y)), \beta (G(fy, fy, y) + G(y, fy, fy)), \gamma((G(y, y, y)$

$+ G(fy, y, y)\}\}$

$\leq \phi\{ \max\{ 3\alpha G(fy, y, y), 4\beta G(fy, y, y), \gamma G(fy, y, y)\}\}$

From the above inequality we have following cases

Case I If max = $3\alpha G(fy, y, y)$ then

$G(fy, y, y) \leq \phi\{3\alpha G(fy, y, y)\}$ as $\phi(t) < t$,$G(fy, y, y) \leq 3\alpha G(fy, y, y)$ then $G(fy, y, y) = 0$ hence we get $fy = y$.

Case II If max = $4\beta G(fy, y, y)$ then

$G(fy, y, y) \leq \phi[4\beta G(fy, y, y)]$as $\phi(t) < t$ then we get $G(fy, y, y) \leq 4\beta G(fy, y, y)$ which implies $fy=y$

Case III If max = $\gamma G(fy, y, y)$ then

$G(fy, y, y) \leq \phi[\gamma G(fy, y, y)]$ as $\phi(t) < t$ we get $G(fy, y, y) \leq \gamma G(fy, y, y)$ which implies $fy = y$

and as $fy = ry$ we have $fy = ry = y$. Hence y is common fixed point of f and r.As $y = fy \in f(X) \subseteq t(X)$ there exists w such that $tw = y$. We shall now prove that $gw = y$.Consider

$G(y, gw, hx_{3n+2}) = G(fy, gw, hx_{3n+2})$

$\leq \phi\{ \max[\alpha (G(gw, fy, ry) + G(hx_{3n+2}, gw, tw) + G(fy, sx_{3n+2}), hx_{3n+2}), \beta (G(fy, ry, gw)$

$+ G(Sx_{3n+2}, fy, ry)), \gamma (G(gw, tw, hx_{3n+2}) + G(fy, gw, tw))]$

$\leq \phi\{ \max[\alpha (G(gw, y, y) + G(y, gw, y) + G(y, y, y)), \beta(G(y, y, gw) + G(y, y, y)), \gamma (G(gw, y, y)$

$+ G(y, gw, y))]$

$\leq \phi\{ \max[2\alpha G(gw, y, y), \beta G(gw, y, y), 2\gamma G(gw, y, y)]$

From the above inequality we have following cases.

Case I If max = $2\alpha G(gw, y, y)$ then we get $G(gw, y, y) \leq \phi[2\alpha G(gw, y, y)]$as $\phi(t) < t$ we have

$G(gw, y, y) \leq 2\alpha G(gw, y, y)$ hence $G(gw, y, y) = 0 \Rightarrow gw = y$

Case II If max = $\beta G(gw, y, y)$ then we get $G(gw, y, y) \leq \phi[\beta G(gw, y, y)]$ as$\phi(t) < t$ we have

$G(gw, y, y) \leq \beta G(gw, y, y)$ hence $G(gw, y, y) = 0 \Rightarrow gw = y$

Case III If max = $2\gamma G(gw, y, y)$ we get $G(gw, y, y) \leq \phi[2\gamma G(gw, y, y)]$ as $\phi(t) < t$ we have

$G(gw, y, y) \leq 2\gamma G(gw, y, y)$ hence $G(gw, y, y) = 0 \Rightarrow gw = y$

Therefore we have $gw = tw = y$. As (g, t) are weakly compatible we get $tgw = gtw$ Hence $ty = gy$.

We shall now prove that $gy = y$. On the contrary Let $gy \neq y$, then

$G(fy, gy, hx_{3n+2}) \leq \phi\{ \max[\alpha(G(gy, fy, ry) + G(hx_{3n+2}, gy, ty) + G(fy, Sx_{3n+2}, hx_{3n+2})), \beta (G(fy, ry, gy)$

$+ G(Sx_{3n+2}, fy, ry)), \gamma (G(gy, ty, hx_{3n+2}) + G(fy, gy, ty)]\}$

As $n \to \infty$ we get

$G(y, gy, y) \leq \phi\{ \max[\alpha (G(gy, y, y) + G(y, y, gy) + G(y, y, y)), \beta (G(y, y, gy) + G(y, y, y)), \gamma (G(gy, gy, y)$

$+ G(y, gy, ty))]\}$

$\leq \phi\{\max[2\alpha G(gy, y, ry), \beta G(gy, y, y), 4\gamma G(gy, y, y)\}$

From the above inequality we have following three cases.

Case I If max = $2\alpha G(gy, y, y)$ then

$G(gy, y, y) \leq \phi[2\alpha G(gy, y, y)]$ as $\phi(t) < t$ we get $G(gy, y, y) \leq 2\alpha G(gy, y, y)$ hence $G(gy, y, y) = 0$ or $gy = y$.

Case II If max = $\beta G(gy, y, y)$ then

$G(gy, y, y) \leq \phi[\beta G(gy, y, y)]$ as $\phi(t) < t$ we get $G(gy, y, y) \leq \beta G(gy, y, y)$ hence $G(gy, y, y) = 0$ or $gy = y$.

Case III If max = $4\gamma G(gy, y, y)$ then

$G(gy, y, y) \leq \phi[4\gamma G(gy, y, y)]$ as $\phi(t) < t$ we get $G(gy, y, y) \leq 4\gamma G(gy, y, y)$hence $G(gy, y, y) = 0$ or $gy = y$.

Also $gy= ty = y$. Hence we get y is a common fixed point of g and t. Similarly since $y = gy \in g(X) \subset s(X)$,

there exist $v \in X$ such that $sv = y$. We now prove that $hv = y$ If $hv \neq y$, we have

$G(y, y, hv) = G(fy, gy, hv)$

$\leq \phi\{\max[\alpha(G(gy, fy, ry) + G(hv, gy, ty) + G(fy, sv, hv)), \beta(G(fy, ry, gy) + G(sv, fy, ry)), \gamma(G(gy, ty, hv)$

$+ G(fy, gy, ty))]\}$

$\leq \phi\{\max[\alpha(G(y, y, y) + G(hv, y, y) + G(y, y, hv)), \beta(G(y, y, y) + G(y, y, y)), \gamma(G(y, y, hv) + G(y, y, y))]$

$\leq \phi\{\max[2\alpha G(hv, y, y), \gamma(hv, y, y)]\}$

From the above inequality we have following case

Case I If max = $2\alpha G(hv, y, y)$ then

$G(hv, y, y) \leq \phi[2\alpha G(hv, y, y)]$ as $\phi(t) < t$ we get $G(hv, y, y) \leq 2\alpha G(hv, y, y)$ hence $G(hv, y, y) = 0$ i.e. $hv = y$.

Case II If max = $\gamma G(hv, y, y)$ then

$G(hv, y, y) \leq \phi[\gamma G(hv, y, y)]$ as $\phi(t) < t$ we get $G(hv, y, y) \leq \gamma G(hv, y, y)$ hence $G(hv, y, y) = 0$ i.e. $hv = y$.

Thus $hv = sv = y$. As the pair (h, s) are weakly compatible we have $shv = hsv$. Hence $sy = hy$.Now we shall prove that $hy = y$.

$G(y, y, hy) = G(fy, gy, hy)$

$\leq \phi\{\max[\alpha(G(gy, fy, ry) + G(hy, gy, ty) + G(fy, sy, hy)), \beta(G(fy, ry, gy) + G(sy, fy, ry)), \gamma(G(gy, ty, hy)$

$+G(fy, gy, ty)]\}$

$\leq \phi\{\max[\alpha(G(y, y, y) + (hy, y, y) + G(y, y, hy)), \beta(G(y, y, y) + G(y, y, y)), \gamma(G(y, y, hy) + G(y, y, y)]\}$

$\leq \phi\{\max[2\alpha G(hy, y, y), \gamma G(y, y, hy)]\}$

From the above inequality we have following cases.

Case I If max = $2\alpha G(hy, y, y)$ then

$G(y, y, hy) \leq \phi[2\alpha G(hy, y, y)]$ as $\phi(t) < t$ we get $G(y, y, hy) \leq 2\alpha G(hy, y, y)$ hence $G(y, y, hy) = 0$ i.e. $hy = y$.

Case II- If max = $\gamma G(y, y, hy)$ then

$G(y, y, hy) \leq \phi[\gamma G(y, y, hy)]$ as $\phi(t) < t$ we get $G(y, y, hy) \leq \gamma G(y, y, hy)$hence $G(y, y, hy) = 0$ i.e. $hy = y$.

Since sy = hy = y. We have y is a common fixed point of s and h. Thus f, g, h, s, t, r have a common fixed point y.

So fy = gy = hy = sy = ty = ry = y. We shall now prove that y is a unique fixed point of f, g, h, s, t, r Let y' is the another fixed point of f, g, h, r, s, t.

G (y, y, hy') = G (fy, gy, hy')

$\leq \phi$ {max [α (G (gy, fy, ry) + G (hy', gy, ty)+ G (fy, sy', hy')) , β (G (fy, ry, gy) + G (sy', fy, ry)) ,γ (G (gy, ty, hy') + G (fy, gy, ty))]}

G (y, y, y') $\leq \phi$ {max [α (G (y, y, y)+ G (y', y, y) + G (y', y', y')), β (G (y, y, y)+ G (y', y, y)) , γ (G (y, y, y') + G (y, y, y))]}

$\leq \phi$ {max [α G (y', y, y), β G (y', y, y),γ G (y', y, y)]}

From the above inequality we have three cases.

Case I . If max = α G (y', y, y) then

G (y, y, y') $\leq \phi$ [α G (y', y, y)] as ϕ (t) < t we get G (y, y, y') $\leq$ α G (y', y, y) hence G (y, y, y') = 0 or y = y'

Case II . If max = β G (y', y, y) then

G (y, y, y') $\leq \phi$ [β G (y', y, y)] as ϕ (t) < t we get G (y, y, y') $\leq \beta$ G (y', y, y) hence G (y, y, y') = 0 or y = y'

Case III . If max = γ G (y', y, y) then

G (y, y, y') $\leq \phi$ [γ G (y', y, y)] as ϕ (t) < t we get G (y, y, y') $\leq \gamma\, \alpha$ G (y', y, y) hence G (y, y, y') = 0 or y = y '. Hence y is unique common fixed point of f, g, h, s, t, r. This completes the proof of the theorem.

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