The Non-Linear Oscillation of the Centre of Mass of the System in Elliptic Orbit under the Influence of the Shadow of the Earth due to Solar Radiation Pressure, Magnetic Force and Oblateness of the Earth.

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Abstract
We have studied the non-linear oscillation of the system of the satellites connected by light, flexible and extensible cable under the influence of Earth's magnetic force, the shadow of the earth due to solar radiation pressure and earth oblateness in the case of elliptic orbit of the Centre of mass of the system. The non-linear terms present in the equations of motion of the system are taken into consideration. First of all we have derived equations of motion for non-linear oscillations and a system of equation representing almost periodic oscillations due to Malkin. An attempt has been made to analyse the motion and stability of the system analytically. As there is no periodic terms in the equation of motion, so only non-resonant solution have been obtained and shown to be stable.

Keywords: Stability, Non-linear oscillation ,Solar radiation pressure, Earth Magnetic force, Satellites, Elliptic orbit.

1. Introduction
This paper deals with the study the effect of shadow of the earth due to solar radiation pressure, magnetic force and earth's oblateness on non-linear oscillation and stability of two satellites connected by light, flexible and extensible cable in the central gravitational field of earth in anelliptic orbit of the centre of mass of the system in case of two dimensional motion**.** Beletsky, V.V. is the pioneer worker in this field. This paper is an attempt towards the generalization of works done by him.

2. Equations of Motion for Non-Linear oscillation of the Centre of Mass of the System.
The equations of motion in elliptic orbit of the centre of mass of a system of two satellites connected by a light, flexible and extensible cable under the influence of the shadow of the earth due to solar radiation pressure, magnetic force and oblateness of the earth in two dimension case in Nechvile's coordinates system are given by

$$x'' - 2y' - 3x\rho - \frac{4Bx}{\rho} + A\rho^3\psi\cos\in\cos(\nu-\alpha) = -\bar{\lambda}_\alpha\rho^4\left[1-\frac{\ell_0}{\rho r}\right]x - \frac{C\cos i}{\rho}$$

and $$y'' + 2x' + \frac{By}{\rho} - A\rho^3\psi\cos\in\sin(\nu-\alpha) = -\bar{\lambda}_\alpha\rho^4\left[1-\frac{\ell_o}{\rho r}\right]y - \frac{\rho^1}{\rho^2}\cos i$$

…………(2.1)

Where, $r=\sqrt{x^2+y^2}$, $\rho=\frac{1}{1+e\cos\nu}$, $A=\frac{p3}{\mu}\left[\frac{B_1}{m_1}-\frac{B_2}{m_2}\right]$ = Solar pressure parameter

ψ = The shadow function parameter

$B=\frac{3K_2}{p^2}$ = Oblateness parameter

$C=\left[\frac{Q_1}{m_1}-\frac{Q_2}{m_2}\right]\frac{\mu_E}{\sqrt{\mu\rho}}$ = Magnetic force parameter

$$\overline{\lambda_\alpha} = \frac{p^3}{\mu}\lambda_\alpha = \frac{p_3}{\mu \ell_o}\frac{(m_1+m_2)}{m_1 m_2}\lambda$$

…………….(2.2)

Where λ denotes modulus of elasticity and μ denotes the product of gravitational constant and mass of the earth. Here B_1 and B_2 are the absolute values of forces due to the direct solar pressure exerted on the masses m_1 and m_2 and α be the angular separation of solar position vector projected on the orbital plane from the orbit perigee. Here $\in$ is the inclination of the osculating plane of the orbit of the centre of mass of the system with the plane of ecliptic and p and e are focal parameter and eccentricity of the earth.

Here, dashes denote the differentiation with respect to the true anomaly v of the elliptic orbit of the centre of mass.

We have

$$\rho = \frac{1}{(1+e\cos v)} = (1+e\cos v)^{-1} = = 1 - e\cos v + e^2\cos^2 v - \ldots\ldots\ldots$$

$$\rho^2 = (1+e\cos v)^{-2} = 1 - 2e\cos v + \ldots\ldots\ldots$$

$$\rho^3 = (1+e\cos v)^{-3} = 1 - 3e\cos v + \ldots\ldots\ldots$$

$$\rho^4 = (1+e\cos v)^{-4} = 1 - 4e\cos v + \ldots\ldots\ldots$$

$$\rho' = \frac{e\sin v}{(1+e\cos v)^2} = e\sin v[1+e\cos v]^{-2} = e\sin v - 2e^2\sin v\cdot\cos v + \ldots\ldots\ldots$$

…………..(2.3)

Putting the value of ρ, ρ^2 , ρ^3 and ρ^4 from [2.3] in [2.1] and neglecting the 2nd higher order terms containing e in their expansions, we get

$$x'' - 2y' - 3(1-e\cos v)x - 4B(1+e\cos v)x + A(1-3e\cos v)\psi\cos\in\cos(v-\alpha)$$

$$= -\overline{\lambda_\alpha}\left[(1-4e\cos v) - \frac{\ell_0}{r}(1-3e\cos v)\right]x - c(1+e\cos v)\cos i$$

and $$y'' + 2x' + B(1+e\cos v)y - A(1-3e\cos v)\psi\cos\in\sin(v-\alpha)$$

$$= -\overline{\lambda_\alpha}\left[(1-4e\cos v) - \frac{\ell_0}{r}(1-3e\cos v)\right]y - ce\cos v\cos i$$

…………… (2.4)

Now, we want to examine the effect of the shadow of the earth due to the solar radiation pressure, magnetic force and oblateness of the earth on the equilibrium position (a, o) for the non-linear oscillation of the system.

For this, Let η_1 and η_2 be the small variations in x and y coordinates at the given equilibrium point (a, o) of the system. Then we have

$$\left.\begin{aligned} x &= a + \eta_1 \text{ and } y = \eta_2 \\ x' &= \eta_1' \text{ and } y' = \eta_2' \\ x'' &= \eta_1'' \text{ and } y'' = \eta_2'' \end{aligned}\right\}$$

…….......... (2.5)

and $$r^2 = x^2 + y^2 = (a+\eta_1)^2 + \eta_2^2 = a^2 + 2a\eta_1 + \eta_1^2 + \eta_2^2 = a^2\left[1 + \frac{(2a\eta_1 + \eta_1^2 + \eta_2^2)}{a^2}\right]$$

$$\therefore r = a\left[1 + \frac{(2a\eta_1 + \eta_1^2 + \eta_2^2)}{a^2}\right]^{1/2}$$

……......... (2.6)

But at the equilibrium point, we have -

$r_o = a$ .... (2.7)

From (2.6) and (2.7), we have

$$\frac{1}{r}=\frac{1}{r_O}\left[1+\frac{(2r_O\eta_1+\eta_1^2+\eta_2^2)}{r_O^2}\right]^{-1/2} \qquad \text{............ (2.8)}$$

Now expanding the right hand side of [2.8] and retaining terms only up to third order in infinitesimals η_1 and η_2 , we get after some simplifications.

$$\frac{1}{r}=\frac{1}{r_0}-\frac{\eta_1}{r_0^2}+\frac{\eta_1^2}{r_0^3}-\frac{\eta_2^2}{2r_0^3}+\frac{3\eta_1^3}{2r_0^4}+\frac{3}{2}\frac{\eta_1\eta_2^2}{r_0^4} \qquad \text{.................. (2.9)}$$

Substituting the values of x and y and their derivatives from [2.5] in [2.4], we get the variational equations of motion in the form:

$$\eta_1''-2\eta_2'-\left[(3+4B)+(4B-3)e\cos v\right](r_O+\eta_1)=-\overline{\lambda_\alpha}\left[(1-4e\cos v)-\frac{\ell_0}{r}(1-3e\cos v)\right](r_O+\eta_1)$$

$$-A(1-3e\cos v)\psi_1\cos\in\cos(v-\alpha)-c(1+e\cos v)\cos i$$

$$\eta_2''-2\eta_1'+B(1+e\cos v)\eta_2=-\overline{\lambda_\alpha}\left[(1-4e\cos v)-\frac{\ell_0}{r}(1-3e\cos v)\right]\eta_2$$

$$+A\psi_1\cos\in(1-3e\cos v)\sin(v-\alpha)-Ce\sin v\cos i$$

$$\text{............(2.10)}$$

Putting the value of $\frac{1}{r}$ from [2.9] in [2.10], we get after neglecting the higher order terms than the third in infinitesimals η_1 and η_2 and after some simplifications.

$$\eta_1''-2\eta_2'-m_1^2\eta_1=\frac{\overline{\lambda_\alpha}}{r_O^3}\left[\frac{3r_O^4}{\overline{\lambda_\alpha}}+\frac{4Br_O^4}{\overline{\lambda_\alpha}}-r_O^4-\frac{\ell_O}{2}(1-3e\cos v)(2r_O^3-r_O\eta_2^2+5\eta_1^2+2\eta_2^2)\right.$$

$$+e\left\{\frac{r_O^3}{\overline{\lambda_\alpha}}(3A\psi_1\cos\in\cos(v-\alpha)+(4\overline{\lambda_\alpha}+4B-3)\eta_1\right.$$

$$\left.\left.-\cos i+(4\overline{\lambda_\alpha}+B-3)\,r_O)\right\}\frac{r_O^{\ 3}}{\overline{\lambda_\alpha}}\cos v\right]-Q\cos i(v-\alpha)$$

and
$$\eta_2''+2\eta_1'-m_2^2\eta_2=\frac{\overline{\lambda_\alpha}}{r_O^3}\left[\frac{\ell_O}{2}(3e\cos v-1)(r_O\eta_1\eta_2-\eta_1^2\eta_2+\eta_2^3)\right.$$

$$\left.+e\left\{(4r_O^2-B-3\ell_Or_O^2)\eta_2\cos v-\frac{3r_O^3}{\overline{\lambda_\alpha}}A\psi_1\cos\in\cos v\sin(v-\alpha)-\frac{cr_O^3}{\overline{\lambda_\alpha}}\sin v\cos i\right\}\right]+Q\sin(v-\alpha)$$

$$\text{............ (2.11)}$$

Where $Q = A\psi_1\cos\in$

[2.11] can be re-written as

$$\eta_1''-2\eta_2'-m_1^2\eta_1=\mu f_1-Q\cos(v-\alpha)$$

and

$$\eta_2''+2\eta_1'-m_2^2\eta_2=\mu H_1+Q\sin(v-\alpha) \qquad \text{................(2.12)}$$

Where

$$m_1^2=3+4B-\overline{\lambda_\alpha}\,,\quad m_2^2=\frac{\overline{\lambda_\alpha}\ell_O}{r_O^3}-B-\overline{\lambda_\alpha}\,,\quad Q=A\psi_1\cos\in,\quad \mu=\frac{\overline{\lambda_\alpha}}{r_O^3}\langle\langle 1$$

$$f_1=\left[\frac{3r_O^4}{\overline{\lambda_\alpha}}+\frac{4Br_O^4}{\overline{\lambda_\alpha}}-r_O^4-\frac{r_O^3}{\overline{\lambda_\alpha}}c.\cos i-\frac{\ell_O}{2}(1-3e\cos v)(2r_O^3-r_O\eta_2^2+5\eta_1^2+2\eta_2^2)\right.$$

$$+e\left\{3A\psi_1\cos\in\cos(\nu-\alpha)+\left(4\overline{\lambda_\alpha}+4B-3\right)\eta_1-\cos i+\left(4\overline{\lambda_\alpha}+B-3\right)r_O\right\}\frac{r_O^3}{\lambda_\alpha}\cos\nu\ \ ]$$

and $$H_1=\frac{\ell_O}{2}(3e\cos\nu-1)\left(r_O\eta_1\eta_2-\eta_1^2\eta_2+\eta_2^3\right)$$

$$+e\left\{\left(4r_O^2-B-3\ell_O r_O^2\right)\eta_2\cos\nu-\frac{3r_O^3}{\lambda_\alpha}A\psi_1\cos\in\cos\nu\sin(\nu-\alpha)-\frac{cr_O^3}{\lambda_\alpha}\sin\nu\cos i\right\}$$

…..……..... (2.13)

Thus, the system of equations given by [2.12] represents the non-linear oscillation of the system at the equilibrium point (a_o, o). We see that it represents the almost periodic oscillation due to Malkin.

3. Non-resonant solution of the equations and its stability

The general solution of linear part of [2.12] which can be obtained by putting μ=0 , can be written in the form :

$$\left.\begin{aligned}\eta_1 &= a_1\sin\varphi_1+a_2\sin\varphi_2+A_1\cos(\nu-\alpha)\\ \eta_2 &= a_1w_1\cos\phi_1+a_2w_2\cos\phi_2-A_1\sin(\nu-\alpha)\\ \eta^{'} &= a_1K_1\cos\phi_1+a_2K_2\cos\phi_2+A_2\sin(\nu-\alpha)\\ \eta_2^{''} &= -a_1K_1w_1\sin\phi_1-a_2K_2w_2\sin\phi_2+A_2\cos(\nu-\alpha)\end{aligned}\right\} \quad \text{…..…...... (3.1)}$$

Where

$$\phi_1=\omega_1\nu+\alpha_1,\quad \phi_2=\omega_2\nu+\alpha_2$$

Here α_1 , α_2 ; a_1 and a_2 are constants to be determined from the initial conditions and ω_1 , ω_2 are the roots of the characteristic equation.

$$\omega^4+\left(m_1^2+m_2^2-\theta\right)\omega^2+m_1^2+m_2^2=0 \quad \text{…….…... (3.2)}$$

From (4.3.1), we have -

$$\left.\begin{aligned}\eta_1'' &= -a_1w_1^2\sin\phi_1-a_2w_2^2\sin\phi_2-A_1\cos(\nu-\alpha)\\ \eta_2'' &= -a_1w_1^2K_1^2\cos\phi_1-a_2K_2w_2^2\cos\phi_2-A_2\sin(\nu-\alpha)\end{aligned}\right\} \quad \text{…….......(3.3)}$$

Thus, on putting the values of $\eta_1, \eta_2, \eta'_1, \eta'_2, \eta''_1$ and η''_2 from [3.1] and [3.3] respectively in [2.12] when μ = 0, we get

$$-\left[a_1\left(w_1^2-2K_1w_1+m_1^2\right)\right]\sin\phi_1-\left[a_2\left(w_2^2-2K_2w_2+m_1^2\right)\right]\sin\phi_2$$
$$-\left[A_1+2A_2+m_1^2A_1\right]\cos(\nu-\alpha)=-Q\cos(\nu-\alpha)$$

and $$-\left[a_1\left(K_1w_1^2-2w_1+m_2^2K_1\right)\right]\cos\phi_1-\left[a_2\left(K_2w_2^2-2w_2+m_2^2K_2\right)\right]\cos\phi_2$$
$$-\left[A_2+2A_1+m_2^2A_2\right]\sin(\nu-\alpha)=Q\sin(\nu-\alpha) \quad \text{…..............(3.4)}$$

Equations of (3.4) will be identically satisfied if the coefficients of $\sin\phi_1$, $\sin\phi_2$, cos (v - α), $\cos\phi_1$, $\cos\phi_2$ and sin (v - α) vanish separately, so we get

$$\left.\begin{aligned}w_1^2-2K_1w_1+m_1^2 &= 0\\ w_2^2-2K_2w_2+m_1^2 &= 0\\ A_1+2A_2+m_1^2A_1 &= Q\end{aligned}\right\} \quad \text{…............(3.5)}$$

and $$\left.\begin{aligned}K_1w_1^2-2w_1+m_2^2K_1 &= 0\\ K_2w_2^2-2w_2+m_2^2K_2 &= 0\\ A_2+2A_1+m_2^2A_2 &= -Q\end{aligned}\right\} \quad \text{…........... (3.6)}$$

From (4.3.5) and (4.3.6), we get

$$\left.\begin{aligned} K_1 &= \frac{w_1 + m_1^2}{2w_1} = \frac{2w_1}{w_1^2 + m_2^2},\ K_2 = \frac{w_2^2 + m_1^2}{2w_2} = \frac{2w_2}{w_2^2 + m_2^2} \\ A_1 &= \frac{\left(m_2^2 + 3\right)Q}{m_1^2 + m_2^2 + m_1^2\ m_2^2 - 3},\ A_2 = \frac{-\left(m_1^2 + 3\right)}{m_1^2 + m_2^2 + m_1^2\ m_2^2 - 3} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (3.7)$$

Now, we shall study the general solution of the entire non-linear equations (2.12) with $\mu \neq 0$ (i.e. $f_1 \neq 0$, $H_1 \neq 0$). For this, we exploit the method of variation of arbitrary constants in our further studies.
Here the amplitude and the phase will now be taken as functions of v but not constants as in linear case.

Here, $a_1 = a_1(v),\ a_2 = a_2(v),\qquad \alpha_1 = \alpha_1(v),\ \alpha_2 = \alpha_2(v)$

$$\therefore \qquad \eta_1 = a_1 \sin\phi_1 + a_2 \sin\phi_2 + A_1 \cos(v - \alpha)$$

$$\eta_2 = a_1 k_1 \cos\phi_1 + a_2 k_2 \cos\phi_2 + A_2 \sin(v - \alpha)$$

Thus, we get

$$\eta_1' = a_1' \sin\phi_1 + a_1 \phi_1' \cos\phi_1 + a_2' \sin\phi_1 + a_2 \phi_2' \cos\phi_2 - A_1 \sin(v - \alpha)$$

$$\eta_2' = a_1' K_1 \cos\phi_1 - a_1 K_1 \phi'_1 \sin\phi_1 + a_2' K_2 \cos\phi_2 - a_2 K_2 \phi_2 '\sin\phi_2 + A_2 \cos(v - \alpha)$$

$$\eta_1'' = a_1' w_1 \cos\phi_1 - a_1 w_1 \phi'_1 \sin\phi_1 + a_2' w_2 \cos\phi_2 - a_2 w_2 \phi_2 '\sin\phi_2 - A_1 \cos(v - \alpha)$$

$$\eta_2'' = -a_1' K_1 w_1 \sin\phi_1 - a_1 K_1 w_1 \phi'_1 \cos\phi_1 - a_2' K_2 w_2 \sin\phi_2 - a_2 K_2 w_2 \phi_2 '\cos\phi_2 - A_2 \sin(v - \alpha) \qquad \ldots\ldots\ldots\ldots(3.8)$$

Comparing the values of η_1 and η_2 in the system of equations [4.3.8] and [4.3.1], we get by subtraction:

$$a_1' \sin\phi_1 + a_1 \phi_1 '\cos\phi_1 - a_1 w_1 \cos\phi_1 + a_2' \sin\phi_2 + a_2 \phi_2 '\cos\phi_2 - a_2 w_2 \cos\phi_2 = 0$$

and $$a_1' K_1 \cos\varphi_1 - a_1 K_1 \varphi'_1 \sin\varphi_1 + a_1 K_1 w_1 \sin\varphi_1 + a_2' K_2 \sin\varphi_2 - a_2 K_2 \varphi'_2 \sin\varphi_2 + a_2 K_2 w_2 \sin\varphi_2 = 0 \qquad \ldots\ldots\ldots\ldots (3.9)$$

In two cases when $f_1 \neq 0$, $H_1 \neq 0$ and $f_1 = 0$, $H_1 = 0$, substituting the values of η_1 and η_2 and their derivatives from [3.1], [3.3] and [3.8], we get on using [3.7].

$$a_1' w_1 \cos\phi_1 - a_1 w_1 \phi_1' \sin\phi_1 + a_1' w_1^2 \sin\phi_1 + a_2' w_2 \cos\phi_2 - a_2 w_2 \phi_2' \sin\phi_2 + a_2 w_2^2 \sin\phi_2 = \mu\ f_1$$

and $$- a_1' K_1 w_1 \sin\phi_1 - a_1 K_1 w_1 \phi'_1 \cos\phi_1 + a_1 K_1 w_1^2 \cos\phi_1 - a_2' K_2 w_2 \sin\phi_2 - a_2 K_2 w_2 \phi'_2 \cos\phi_2 + a_2 K_2 w_2^2 \cos\phi_2 = \mu H_1 \qquad \ldots\ldots\ldots\ldots (3.10)$$

Multiplying the first equation of [3.9] by $k_1 w_1$ and then adding it to the 2nd equation of (3.10), we get

$$a_2' \sin\phi_2 \left[w_1 K_1 - w_2 K_2\right] + a_2 \left(\phi'_2 - w_2\right)\left(w_1 K_1 - w_2 K_2\right)\cos\phi_2 = \mu H_1 \qquad \ldots\ldots\ldots\ldots (3.11)$$

Again, multiply the 2nd equation of [3.9] by $k_2 w_2$ and adding it the 2nd equation of [3.10], we get

$$a_1' \left(w_2 K_2 - w_1 K_1\right)\sin\phi_1 + a_1 \left(\phi'_1 - w_1\right)\left(w_2 K_2 - w_1 K_1\right)\cos\phi_1 = \mu H_1 \qquad \ldots\ldots\ldots\ldots (3.12)$$

Again, multiplying the 2nd equation of [3.9] by w_1 and subtracting it from k_1 times the first equation of [3.10], we get -

$$a_2' \left(w_2 K_1 - w_1 K_2\right)\cos\phi_2 + a_2 \left(\phi'_2 - w_2\right)\left(w_2 K_1 - w_1 K_2\right)\sin\phi_2 = \mu K_1 f_1 \qquad \ldots\ldots\ldots\ldots(3.13)$$

Lastly, multiply the 2nd equation of (3.9) by w_2 and then subtracting it from w_2 times the first equation of (3.10), we get

$$a_1' \left(w_1 K_2 - w_2 K_1\right)\cos\phi_1 - a_1 \left(\phi'_1 - w_1\right)\left(w_1 K_2 - w_2 K_1\right)\sin\phi_1 = \mu K_2 f_1 \qquad \ldots\ldots\ldots\ldots (3.14)$$

Now, putting the value of w_1 and w_2 from (3.7) in (3.11), (3.12), (3.13) and (3.14), we get,

$$a_1' \left(\frac{w_2^2 - w_1^2}{2}\right)\sin\phi_1 + a_1 \left(\phi'_1 - w_1\right)\left(\frac{w_2^2 - w_1^2}{2}\right)\cos\phi_1 = \mu H_1$$

$$a_1' \cdot \left[m_1^2 \frac{\left(w_1^2 - w_2^2\right)}{2w_1 w_2} \right] \cos\phi_1 - a_1 (\phi'_1 - w_1) m_1^2 \frac{\left(w_1^2 - w_2^2\right)}{2w_2} \sin\phi_1 = \mu K_2 f_1$$

$$a_2' \left(\frac{w_1^2 - w_2^2}{2} \right) \sin\phi_2 + a_2 (\phi'_2 - w_2) \left(\frac{w_1^2 - w_2^2}{2} \right) \cos\phi_2 = \mu H_1$$

and $$a_2' \left[m_1^2 \frac{\left(w_2^2 - w_1^2\right)}{2w_1 w_2} \right] \cos\phi_2 - a_2 (\phi'_2 - w_2) \left(\frac{w_1^2 - w_2^2}{2w_1} \right) \sin\phi_2 = \mu K_1 f_1$$

………….. (3.15)

After solving these four equations of [3.15] for and, we get

$$a_1' = -\mu \left[H_1^* \sin\phi_1 + K_2 f_1^* \cos\phi_1 \right]$$

$$a_2' = \mu \left[H_1^* \sin\phi_2 + K_1 f_1^* \cos\phi_2 \right]$$

$$\phi'_1 = w_1 + \frac{\mu}{a_1} \left[-H_1^* \cos\phi_1 + K_2 f_1^* \sin\phi_1 \right]$$

$$\phi'_2 = w_2 + \frac{\mu}{a_2} \left[H_1^* \cos\phi_2 - K_1 f_1^* \sin\phi_2 \right] \qquad \text{…............ (3.16)}$$

Where,

$$H_1^* = \frac{-2H_1}{w_2^2 - w_1^2}, \quad f_1^* = \frac{-2m_2 f_1}{m_1 \left(w_2^2 - w_1^2\right)} \quad \text{and} \quad \frac{w_1 w_2}{m_1^2} = \frac{m_2}{m_1} \qquad \text{…...........(3.17)}$$

Thus, on considering a_1, a_2, ϕ_1 and ϕ_2 as variables, we get a new system of four variations equations of motion given in [3.16].

It we put on the right hand Side of the system of equations [3.16], the values of f_1^* and H_1^* in terms of f_1 and H_1 respectively from [2.13] and then the values of η_1 and η_2 from [3.1], then the right hand side forms of the expression are expanded into trigonometrical sums and averaged values of the variables are taken after dripping all the terms in the system of equations [3.16] except the free terms, we get the system of equations for first approximation as :

$$\left. a_1' = 0, \quad a_2' = 0, \quad \varphi'_1 = w_1^*, \quad \varphi'_2 = w_2^* \right\} \qquad \text{…….......... (3.18)}$$

Where w_1^* and w_2^* are the new frequencies depending on w_1, w_2 and constant quantities a_1, a_2, A_1, A_2, r_o, m_1, m_2, k_1 and k_2 and hence on integration, we get from (3.18).

$$\left. \begin{array}{ll} a_1 = \text{constant} = a_1^*, & a_2 = \text{constant} = a_2^* \\ \varphi_1 = w_1^* v + \in_1, & \varphi_2 = w_2^* v + \in_2 \end{array} \right\} \qquad \text{............. (3.19)}$$

where $\in_1$ and $\in_2$ are constants, Thus we see that in the relation [3.19] ; a_1 and a_2 remain constant where as the values of ϕ_1 and ϕ_2 are slightly changed in the first approximation which indicates the change in the frequencies. But it has no effect on stability.

Thus, in the first approximation, the solutions of the equations of non-linear oscillation [2.12] can be written as -

$$\eta_1 = a_1^* \sin\left(w_1^* v + \in_1\right) + a_2^* \sin\left(w_2^* v + \in_2\right) + A_1 \cos(v - \alpha)$$

$$\eta_2 = a_1^* K_1 \cos\left(w_1^* v + \in_1\right) + a_2^* K_2 \cos\left(w_2^* v + \in_2\right) + A_2 \sin(v - \alpha)$$

…….…... (3.20)

Where a_1^*, a_2^*, $\in_1$ and $\in_2$ are arbitrary constant and w_1^* and w_2^* will be new frequencies, the values of A_1 and A_2 are given in [3.6].

Finally, we conclude that the solutions given in [3.20] will be stable.

4. <u>RESONANT SOLUTION OF THE EQUATION AND ITS STABILITY</u>

In this section, we shall examine the system of equations [2.12] with the supposition that the oscillation is of resonance type.

In case of resonance oscillation, we suppose that

$$w_2 = 1 \text{ and } Q = \mu Q^* \qquad \text{.............. (4.1)}$$

Which are customary in the resonance case of oscillation. In its absence, the generating system will have no almost periodic solution.

Thus, the system of equations [2.12] can be put in the form:

$$\left.\begin{aligned} \eta_1'' - 2\eta_2' - m_1^2\eta_1 &= \mu f_1 - \mu Q^* \cos(v-\alpha) \\ \eta_2'' - 2\eta_1' - m_2^2\eta_2 &= \mu H_1 + \mu Q^* \sin(v-\alpha) \end{aligned}\right\} \qquad \text{............ (4.2)}$$

Where , f_1 and H_1 have their usual meanings given in [2.13].

In this case, the particular solutions of [4.2] for $f_1 = 0$ and $H_1 = 0$ can be assumed to be in the form -

$$\left.\begin{aligned} \eta_1 &= a\sin\phi + M_1\cos(v-\alpha) + M_2\sin(v-\alpha) \\ \eta_2 &= aK_1\cos\phi + M_2K_2\cos(v-\alpha) - M_1K_2\sin(v-\alpha) \end{aligned}\right\} \qquad \text{............(4.3)}$$

where $\emptyset = w_1 v + \alpha_3$; a, α_3 , M_1, M_2 are constants.

From (4.4.3), we get

$$\left.\begin{aligned} \eta_1' &= aw_1\cos\phi - M_1\sin(v-\alpha) + M_2\cos(v-\alpha) \\ \eta_2' &= -aK_1w_1\sin\phi - M_2K_2\sin(v-\alpha) - M_1K_2\cos(v-\alpha) \end{aligned}\right\} \qquad \text{.............. (4.4)}$$

In a similar way just as in the preceding sections of this chapter and keeping in mind that $w_2 = 1$, we get from [4.3.7]

$$\left. K_1 = \frac{w_1 + M_1^2}{2w_1} = \frac{2w_1}{w_1^2 + M_2^2}, \; K_2 = \frac{1+M_1^2}{2} = \frac{2}{1+M_2^2} \right\} \qquad \text{.............. (4.5)}$$

From [4.4.4], we get

$$\left.\begin{aligned} \eta_1'' &= -aw_1^2\sin\phi - M_1\cos(v-\alpha) - M_2\sin(v-\alpha) \\ \eta_2'' &= -aK_1w_1^2\cos\phi + M_1K_2\sin(v-\alpha) - M_2K_2\cos(v-\alpha) \end{aligned}\right\} \qquad \text{............... (4.6)}$$

Now, similar to the non-resonance case, we shall investigate the general solution to the system of equations [4.2] representing the non-linear oscillation, when $f_1 \neq 0$ and $H_1 \neq 0$.

We shall assume that a, M_1, M_2 and ϕ are new variables like the previous section of this chapter, we get on solving the system of equations for M'_1, M'_2, a' and ϕ' obtained in the form :

$$M_1' = \mu\left[H_1^{**}\cos(v-\alpha) - K_1 f_1^{**}\sin(v-\alpha)\right], \quad M_2' = \mu\left[H_1^{**}\sin(v-\alpha) + K_2 f_1^{**}\cos(v-\alpha)\right]$$

$$a' = \mu\left[-H_1^{**}\sin\phi - K_2 f_1^{**}\cos\phi\right], \qquad \phi' = w_1 + \frac{\mu}{a}\left[H_1^{**}\cos\theta + K_2 f_1^{**}\sin\theta\right]$$

$$\text{...................... (4.7)}$$

Where, $$\left. H_1^{**} = \frac{2Q^*\sin(v-\alpha)}{\omega_1^2 - 1} + H_1^*, \;\; f_1^{**} = \frac{2M_2Q^*\cos(v-\alpha)}{\omega_1^2 - 1} + f_1^* \right\} \qquad \text{..................... (4.8)}$$

The values of H_1^* and f_1^* can be given from [3.17] in case of resonance oscillation where $\omega_2 = 1$ as

$$\left. H_1^* = \frac{-2H_1}{1-\omega_1^2}, \;\; f_1^* = \frac{-2M_2 f_1}{M_1\left(1-\omega_1^2\right)} \right\} \qquad \text{.............. (4.9)}$$

In order to get the first approximate solution of the system of equation [4.7], we shall put in the right hand sides of [4.7], the different values from [4.8], [4.9] and [2.13] and then the values of η_1 and η_2 from [3.1]. Now after dropping the other terms except the free terms, we take the averaged terms into trigonometrical sums as mentioned in the previous section of this chapter, the set of equations [4.7] can be written as :

$$M_1' = \frac{\mu}{4\left(w_1^2 - 1\right)}\left[\left[\left\{M_1^2\left(K_2 + 3K_2^3\right) - \frac{6K_1m_2}{m_1} + \frac{K_1K_2^2m_2}{m_1}\right] + M_2^2\left(-2K_2 + \frac{3}{2}K_2^3 - \frac{3K_1m_2}{m_1} + \frac{K_1K_2^2m_2}{m_1}\right)\right.\right.$$

$$+a^2\left(-2K_2+3K_1^2K_2-\frac{6K_1m_2}{m_1}+\frac{K_1^3m_2}{m_1}\right)\Bigg\}M_2+6er_0\left\{M_1^2\left(K_2-\frac{K_1m_2}{m_1}\right)\sin\alpha-M_2^2\left(K_2+\frac{K_1m_2}{m_1}\right)\sin\alpha\right.$$

$$\left.-\frac{2K_1m_2}{m_1}M_1M_2\cos\alpha\right\}\Bigg]$$

$$M_2'=\frac{\mu}{4(w_1^2-1)}\left[\left\{M_1^2\left(2K_2-\frac{3}{2}K_2^3+\frac{3K_1m_2}{m_1}-\frac{K_1K_2^2m_2}{m_1}\right)+M_2^2\left(-K_2\;3K_2^3+\frac{6K_1m_2}{m_1}+\frac{K_1K_2^2m_2}{m_1}\right)\right.\right.$$

$$+a^2\left(2K_2\;3K_1^2K_2+\frac{6K_1m_2}{m_1}+\frac{K_1^3m_2}{m_1}\right)\Bigg\}M_1+6er_0\left\{M_2^2\left(K_2-\frac{K_1m_2}{m_1}\right)\cos\alpha-M_1^2\left(K_2+\frac{K_1m_2}{m_1}\right)\cos\alpha\right.$$

$$\left.\left.-2K_1\frac{m_2}{m_1}M_1M_2\sin\alpha\right\}\right]+\frac{Q^*}{(w_1^2-1)}\left(1+\frac{K_1m_2}{m_1}\right)$$

$$a'=\frac{-\mu 3er_0K_2}{2(w_1^2-1)}\left[M_1\sin\alpha+M_2\cos\alpha\right]a$$

$$\varphi'=w_1+\frac{\mu}{2(w_1^2-1)}\left[\left\{M_1^2\left(2K_1-3K_1K_2^2+\frac{6K_2m_2}{m_1}-\frac{K_2^3m_2}{m_1}\right)+M_2^2\left(2K_1-3K_1K_2^2+\frac{6K_2m_2}{m_1}+\frac{K_2^2m_2}{m_1}\right)\right.\right.$$

$$\left.+a^2\left(2K_1-\frac{3}{2}K_1^3+\frac{3K_2m_2}{m_1}-\frac{K_1^2K_2m_2}{m_1}\right)\text{-}12er_0\left\{M_1\left(\frac{K_2m_2}{m_1}+\frac{K_1}{2}\right)\cos\alpha-M_2\left(\frac{K_1m_2}{m_1}+\frac{K_1}{2}\right)\sin\alpha\right\}\right]$$

.... (4.10)

The set of equations [4.10] can be written in the form:

$$M_1'=\mu\left[b\left\{M_1^2E_1^2-M_2^2E_2^2-a^2E_3\right\}+b_1\left\{M_1^2\left(K_2-\frac{K_1m_2}{m_1}\right)\sin\alpha-M_2^2\left(K_2+\frac{K_1m_2}{m_1}\right)\sin\alpha\right.\right.$$

$$\left.\left.+\frac{2K_1m_2}{m_1}M_1M_2\cos\alpha\right\}\right]=\mu R_1[M_1,\,M_2,\,a]$$

$$M_2'=\mu\left[b\left\{M_1^2E_2-M_2^2E_1+a^2E_3\right\}M_1+b_1\left\{M_2^2\left(K_2-\frac{K_1m_2}{m_1}\right)\cos\alpha-M_1^2\left(K_2-\frac{K_1m_2}{m_1}\right)\cos\alpha\right.\right.$$

$$\left.\left.+\frac{2K_1m_2}{m_1}M_1M_2\sin\alpha\right\}\right]=\mu R_2[M_1,\,M_2,\,a]$$

$$a'=-\mu b_1K_2\left[M_1\sin\alpha+M_2+\cos\alpha\right]a=\mu R_3[M_1,\,M_2,\,a]$$

$$\phi'=w_1+\mu\left[b\left\{E_4M_1^2M_2^2+a^2E_5\right\}-b_1\left\{M_1\left(\frac{2K_2m_2}{m_1}+K_1\right)\cos\alpha-M_2\left(\frac{2K_2m_2}{m_1}+K_1\right)\sin\alpha\right\}\right]=w^{**}$$

...............(4.11)

Where the values of b, b_1, E_1, E_2, E_3, E_4 and E_5 are given by

$$b=\frac{1}{4\left(w_1^2-1\right)},\ b_1=\frac{3er_0}{2\left(w_1^2-1\right)},\ E_1=K_2+3K_2^3-\frac{6K_1m_2}{m_1}-\frac{K_1K_2^2m_2}{m_1}$$

$$E_2=2K_2-\frac{3}{2}K_2^3+\frac{3K_1m_2}{m_1}-\frac{K_1K_2^2m_2}{m_1},E_3=2K_2\ 3K_1^2K_2+\frac{6K_1m_2}{m_1}-\frac{K_1^2m_2}{m_1}$$

$$E_4=2K_1-\frac{3}{2}K_1K_2^2+\frac{6K_2m_2}{m_1}-\frac{K_2^3m_2}{m_1},\text{ and }E_5=2K_1-\frac{3}{2}K_1^3+\frac{3K_2m_2}{m_1}-\frac{K_1^2K_2m_2}{m_1} \qquad \text{.............(4.12)}$$

Here R_1, R_2 and R_3 and stand for functional notations.

Now let us examine the system of equations [4.11] we see that the solutions given in [4.3] can be stable only when the given conditions are satisfied.

$$R_1[M_1,\ M_2,\ a]=0,\ R_2[M_1,\ M_2,\ a]=0,\ R_3[M_1,\ M_2,\ a]=0 \qquad \text{................(4.13)}$$

If the conditions mentioned in (4.13) are satisfied then the values of M_1, M_2 and a will remain constants and in that case the value of $\phi' = \omega^{**}$ will also be a constant quantity.

Therefore, we shall have a new frequency ω^{**} in place of ω_1. But it will not affect the stability. Thus, in the first approximation, the stationary solutions of the system of equations [4.2] for non-linear oscillation can be written in the form:

$$\eta_1=a^{**}\sin\phi+M_1^{**}\cos(v-\alpha)+M_2^{**}\sin(v-\alpha)$$

$$\eta_2=a^{**}K_1\cos\theta-M_1^{**}\sin(v-\alpha)+M_2^{**}\cos(v-\alpha) \qquad \text{.............. (4.14)}$$

Where $\theta = \omega^{**} v + \alpha_3$ being arbitrary constant and a^{**}, M_2^{**} are the roots of the system of equations.

$$b\left\{M_1^2E_1-M_2^2E_1-a^2E_3\right\}M_2+b_1\left\{M_1^2\left(K_2+\frac{K_1m_2}{m_1}\right)\sin\alpha\right.$$

$$\left.-M_2^2\left(K_2+\frac{K_1m_2}{m_1}\right)\sin\alpha+\frac{2K_1m_2}{m_1}M_1M_2\cos\alpha\right\}=0$$

$$b\left\{M_1^2E_2-M_2^2E_1+a^2E_3\right\}M_1+\frac{Q^*}{w_1^2-1}\left(1+\frac{K_1m_2}{m_1}\right)$$

$$+b_1\left\{M_2^2\left(K_2+\frac{K_1m_2}{m_1}\right)\cos\alpha-M_1^2\left(K_2+\frac{K_1m_2}{m_1}\right)\cos\alpha+\frac{2K_1m_2}{m_1}M_1M_2\sin\alpha\right\}=0$$

and $$[M_1\sin\alpha+M_2\cos\alpha]a=0 \qquad \text{............. (4.15)}$$

Hence, we finally come to the conclusion that the stationary solution [4.14] can be stable for -

$M_1 = M_1^{**}$, $M_2 = M_2^{**}$ and $a = a^{**}$

Only when the roots of the following characteristic equation

$$\begin{vmatrix}\frac{\partial R_1}{\partial m_1}-\eta & \frac{\partial R_1}{\partial m_2} & \frac{\partial R_1}{\partial a}\\ \frac{\partial R_2}{\partial m_1} & \frac{\partial R_2}{\partial m_2}-\eta & \frac{\partial R_2}{\partial a}\\ \frac{\partial R_3}{\partial m_1} & \frac{\partial R_3}{\partial m_2} & \frac{\partial R_3}{\partial a}-\eta\end{vmatrix}=0 \qquad \text{............. (4.16)}$$

have negative real parts.

From what we have discussed above in this paper, it follows that the stationary solution in the non-resonance case is stable in the first approximation in elliptic motion of the system where as the stationary solution in the resonance case exists only when the roots of the characteristic equation [4.16] have negative real parts.

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