Application of Fractional Calculus to a Class of Multivalent β -Uniformly Starlike Functions

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Abstract:-**In this paper we introduce a new class of multivalent functions which is β- uniformly starlike in the unit disc. Characterization property exhibited and relation with other fractional calculus operators are given. Connections with β-uniformly starlike and parabolic starlike functions are defined. Results on modified Hadamard product, extreme points, growth and distortion theorems, class preserving integral operators, region of p-valency and radius of β-uniformly starlike functions are also derived.**

Keywords:-β-uniformly starlike functions, fractional calculus operator, region of p-valency, parabolic starlike functions, incomplete beta functions.

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I. INTRODUCTION AND PRELIMINARIES:

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(v) = z^p + \sum_{k=p+n}^{\infty} a_k v^k, \ \ n,p \in \mathbb{N} \tag{1.1}$$

which are analytic and multivalent in the open unit disc $U = \{v \in \mathbb{C}: |v| < 1\}$. Also consider the subclass $T(p)$ of $\mathcal{A}_p$ consisting of functions of the form

$$f(v) = v^p - \sum_{k=p+n}^{\infty} a_k v^k, (a_k \geq 0, n, p \in \mathbb{N}) \tag{1.2}$$

By $S^* or$ K we denote the subclass of $\mathcal{A}$ consisting of all functions which are starlike or convex respectively, while by $S^*(\delta)$ we denote the class of starlike functions of order δ, where $\delta \in [0,1)$.

In 1991, Goodman [9] introduced the class UCV of uniformly convex functions. A function $f(z) \in CV$ to be uniformly convex in U if $f(U)$ is convex in U and has the property that every circular arcγ, contained in U with center ξ in U, arc $f(\gamma)$ is convex with respect to (ξ) .

Meena More and S. M. Khairnar [1, 2] can further generalize the classes UST and UCV by introducing a $\alpha, (-p \leq \alpha < p)$.

A function $f(v) \in \mathcal{A}_p$is said to be β-uniformly starlike of order $\alpha, (-p \leq \alpha < p), \beta \geq 0 \ and \ z \in U,$ $denoted \ by \ UST(\alpha, \beta, p), if \ and \ only \ if$

$$Re\left\{v\frac{f'(v)}{f(v)} - \alpha\right\} \geq \beta \left|v\frac{f'(v)}{fv} - p\right| \tag{1.3}$$

A function $f(v) \in \mathcal{A}_p$is said to be β-uniformly convex of order $\alpha, (-p \leq \alpha < p), \beta \geq 0 \ and \ z \in U, denoted \ by \ UCV(\alpha, \beta, p), if \ and \ only \ if$

$$Re\left\{1 + v\frac{f''(v)}{f'(v)} - \alpha\right\} \geq \beta \left|1 + v\frac{f''(v)}{f'(v)} - p\right| \tag{1.4}$$

Notice that, $UST(\alpha,\beta,1) = UST(\alpha,\beta), UCV(\alpha,\beta,1) = UCV(\alpha,\beta), UST(\alpha,0) = S^*(\alpha) \ and \ UCV(\alpha,0) = K(\alpha),$ where $UST(\alpha,\beta) and \ UCV(\alpha,\beta)$ are the classes of β-uniformly starlike and β-uniformly convex functions of order α. $S(\alpha) \ and \ K(\alpha)$ are the popular classes of starlike and convex function s of order α, $(0 \leq \alpha < 1)$. We also note that $f \in UCV(\alpha,\beta,p)$, if and only if $vf' \in UST(\alpha,\beta,p)$.

Let the function $f(v)$ defined by (1.2) and $g(z)$ defined by

$$g(v) = v^p - \sum_{k=p+n}^{\infty} b_k v^k, \ \ (n,p \in \mathbb{N}) \tag{1.5}$$

belonging to $K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ and $K(\mu,\gamma,\eta,a,b,c,\sigma,\xi,\beta)$, respectively. Then modified Hadamard product of $f$and g is defined by

$$(f * g)(v) = v^p - \sum_{k=p+n}^{\infty} a_k b_k v^k, \ \ (n,p \in \mathbb{N}) \tag{1.6}$$

The incomplete beta function

$$\phi_p(a,c\,;v) = v^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} z^k, \tag{1.7}$$

$for \ a \in \mathbb{R}, \ c \in \mathbb{R}\backslash \bar{z}_0 \ where \ \bar{z}_0 = \{0,-1,-2,-3,\ldots\}, z \in U.$ $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & ; \quad k=1 \\ a(a+1)(a+2)\ldots(a+k-1); & k \in \mathbb{N} \end{cases}$$

Next, consider the Carlson-Shaffer operator [6] define by

$$L_p(a,c)f(v) = \phi_p(a,c\,;v) * f(v),$$

$$= v^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} a_k v^k, for\ f \in \mathcal{A}_p \quad (1.8)$$

The Gaussian hypergeometric function denoted by

$${}_2F_1(a,b;c;v) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{v^k}{k!}, v \in U \qquad (1.9)$$

And $a + b < c$.

Now, using the convolution theorem we can define the Hohlov operator $F_p(a,b;c): T(p) \to T(p)$ by the following relation:

$$F_p(a,b;c)\big(f(v)\big) = v^p\ {}_2F_1(a,b;c;z) * f(v)$$

$$= v^p - \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}(b)_{k-p}}{(c)_{k-p}} \frac{a_k v^k}{(k-p)!}, \qquad (1.10)$$

$a, b \in \mathbb{R}\ and\ c \in \mathbb{R} \setminus \bar{z}_0\ where\ \bar{z}_0 = \{0,-1,-2,-3,\ldots\}, z \in U$. Notice that, Hohlov operator reduces to Carlson-Shaffer operator if $b = 1$. Also for $a = m+1, b = c = 1$, we get Ruscheweyh derivative operator of order m.

We can write

$$F_p(a,b;c)f(v) = v^p - \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}(b)_{k-p}}{(c)_{k-p}} \frac{a_k v^k}{(k-p)!}, \qquad (1.11)$$

A. *Definition*:

Let $\mu > 0\ and\ \gamma, \eta \in \mathbb{R}$. Then the generalized fractional integral operator $I_{0,v}^{\mu,\gamma,\eta}$ of a function $f(z)$ is defined by

$$I_{0,v}^{\mu,\gamma,\eta} f(v) = \frac{v^{-\mu-\gamma}}{\Gamma(\mu)} \left\{ \int_0^z (v-t)^{\mu-1} f(t)\ {}_2F_1\left(\mu+\gamma, -\eta; \mu; 1-\frac{t}{v}\right) dt \right\}, \qquad (1.12)$$

where $f(v)$ is analytic in a simply-connected region of the z-plane containing the origin, with order

$$f(v) = 0(|v|^r), v \to 0,$$

$$\text{where } r > max\{0, \mu-\eta\} - 1 \quad (1.13)$$

and the multiplicity of $(v-t)^{\mu-1}$ is removed by requiring $\log(v-t)$ to be real, when $(v-t) > 0$ and is well defined in the unit disc.

B. *Definition:*

Let $0 \le \mu < 1\ and\ \gamma, \eta \in \mathbb{R}$. Then the generalized fractional derivative operator $J_{0,v}^{\mu,\gamma,\eta}$ of a function $f(v)$ is defined by

$$J_{0,v}^{\mu,\gamma,\eta} f(v) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dv} \left\{ v^{\mu-\gamma} \int_0^v (v-t)^{-\mu}\ {}_2F_1\left(\gamma-\mu, 1-\eta; 1-\mu; 1-\frac{t}{v}\right) dt \right\} \qquad (1.14)$$

where the function is analytic in the simply-connected region of z-plane containing the origin, with the order as given in (1.19) and multiplicity of $(v-t)^{-\mu}$is removed by requiring $\log(z-t)$ to be real, when $(v-t) > 0$. Notice that, we have the following relationships with the fractional integral and derivative operators of order μ.

$$I_{0,v}^{\mu,-\mu,\eta} f(v) = D_{0,v}^{-\mu} f(v)\ (\mu > 0),$$

$$J_{0,v}^{\mu,\mu,\eta} f(v) = D_{0,v}^{\mu} f(v)\ (0 \le \mu < 1)$$

Consider the fractional calculus operator $U_{0,v}^{\mu,\gamma,\eta}$ defined in terms of $J_{0,v}^{\mu,\gamma,\eta}$ as follows:

$$U_{0,v}^{\mu,\gamma,\eta} f(v) = \begin{cases} \frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)} z^{\gamma} J_{0,v}^{\mu,\gamma,\eta} f(v), 0 \le \mu < 1 \\ \frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)} v^{\gamma} I_{0,v}^{-\mu,\gamma,\eta} f(v), -\infty \le \mu < 0 \end{cases} \qquad (1.15)$$

Let $M_{0,z}^{\mu,\gamma,\eta,a,b,c} f(v) = F_p(a,b;c)f(v) * U_{0,v}^{\mu,\gamma,\eta} f(v)$

$$= v^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}(b)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p+\eta-\mu)_{k-p}(1+p-\gamma)_{k-p}} \frac{a_k v^k}{(k-p)!}, \qquad (1.16)$$

$a, b \in \mathbb{R}\ and\ c \in \mathbb{R} \setminus \bar{z}_0\ where\ \bar{z}_0 = \{0,-1,-2,-3,\ldots\}$, $-\infty < \mu, \gamma < 1, \eta \in \mathbb{R}^+,\ and\ f \in \mathcal{A}_p$.

For convenience, we will write$M_{0,z}^{\mu,\gamma,\eta,a,b,c} f(z)$ as follows:

$$M_{0,v}^{\mu,\gamma,\eta,a,b,c} f(v) = v^p + \sum_{k=p+n}^{\infty} g(k) a_k v^k, \qquad (1.17)$$

where

$$g(k) = \frac{(a)_{k-p}(b)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p+\eta-\mu)_{k-p}(1+p-\gamma)_{k-p}(k-p)!} \qquad (1.18)$$

And the subclass $T_p\ of\ \mathcal{A}_p$ we write

$$M_{0,v}^{\mu,\gamma,\eta,a,b,c} f(v) = v^p - \sum_{k=p+n}^{\infty} g(k) a_k v^k, \qquad (1.19)$$

where $g(k)$ is defined as (1.19)

Let $K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ denote the class of function $f \in \mathcal{A}_p$ satisfying

$$Re\left\{1+\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right\} \geq \beta\left|\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right|+\lambda \tag{1.20}$$

$a,b \in \mathbb{R}$ *and* $c \in \mathbb{R} \setminus \bar{z}_0$ *where* $\bar{z}_0 = \{0,-1,-2,-3,\ldots\}$, $-\infty < \mu,\gamma < 1, 0 \leq \lambda < 1, \eta \in \mathbb{R}^+, \beta \geq 0, \sigma \in C\setminus\{0\}$ *and* $z \in U$.

II. COEFFICIENT ESTIMATES:

A. *Theorem 2.1:*

A function $f(v)$ defined by $K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$if and only if

$$\sum_{k=p+n}^{\infty}[(k-p)(1+\beta)+(1-\lambda)|\sigma|]g(k)|a_k| \leq |\sigma|(1-\lambda) \tag{2.1}$$

with the limits
$a,b \in \mathbb{R}$ *and* $c \in \mathbb{R} \setminus \bar{z}_0$ *where* $\bar{z}_0 = \{0,-1,-2,-3,\ldots\}$, $-\infty < \mu,\gamma < 1, 0 \leq \lambda < 1, \eta \in \mathbb{R}^+, \beta \geq 0, \sigma \in C\setminus\{0\}$ *and* $z \in U$ *and* $g(k)$ *is define as* (1.23)

.Proof: Assume that $f \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ *and* z *is real*. Then we have from (1.25)

$$Re\left\{1+\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right\} \geq \beta\left|\frac{1}{\sigma}\left(\frac{v\left(M_{0,z}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,z}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right|+\lambda$$

$$Re\left\{1-\frac{1}{\sigma}\left(\frac{\sum_{k=p+n}^{\infty}(k-p)g(k)a_k v^{k-p}}{1-\sum_{k=p+n}^{\infty}g(k)a_k v^{k-p}}\right)\right\} \geq \beta\left|\frac{1}{\sigma}\left(-\frac{\sum_{k=p+n}^{\infty}(k-p)g(k)a_k v^{k-p}}{1-\sum_{k=p+n}^{\infty}g(k)a_k v^{k-p}}\right)\right|+\lambda$$

We know that $|Rev| < |v|$. Allowing $v \to 1$ along the real axis, we get

$$\sum_{k=p+n}^{\infty}[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)a_k \leq (1-\lambda)|\sigma|$$

We obtain the inequality (2.1).

Conversely, let us assume that (2.1) holds, then we show that

$$Re\left\{1+\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right\} \geq \beta\left|\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right|+\lambda$$

That is,

$$\beta\left|\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right| - Re\left\{1+\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right\}$$
$$\leq (1+\beta)\left|\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)}-p\right)\right|$$
$$\leq (1+\beta)\left[\frac{1}{|\sigma|}\left(\frac{\sum_{k=p+n}^{\infty}(k-p)g(k)|a_k|}{1-\sum_{k=p+n}^{\infty}g(k)|a_k|}\right)\right]$$

where $g(k)$ is given by (2.2). The last inequality is bounded above by $(1-\lambda)$ if

$$\sum_{k=p+n}^{\infty}g(k)a_k[(1+\beta)(k-p)+(1-\lambda)|\sigma|] \leq (1-\lambda)|\sigma|$$

B. *Corollary:*

Let the function $f(v)$ defined (1.2) be in the class $K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ then

$$a_k \leq \frac{(1-\lambda)|\sigma|}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}, k \geq p+n, n \in N \tag{2.2}$$

with equality for the function $f(z)$ given by

$$f(v) = v^p - \frac{(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(k)}v^{p+n}, n \in N \tag{2.3}$$

III. CONNECTIONS WITH OTHER FRACTIONAL CALCULUS OPERATORS

A. *Theorem 3.1:*

Let

$$\frac{a.b(1+p)(1+p+\eta-\gamma)}{(c)(1+p+\eta-\mu)(1+p-\gamma)} \leq 1 \tag{3.1}$$

for the limit on the parameters given by
$a,b \in \mathbb{R}$ *and* $c \in \mathbb{R} \setminus \bar{z}_0$ *where* $\bar{z}_0 = \{0,-1,-2,-3,\ldots\}$, $-\infty < \mu,\gamma < 1, 0 \leq \lambda < 1, \eta \in \mathbb{R}^+, \beta \geq 0, \sigma \in C\setminus\{0\}$ *and* $z \in U$

Also let the function $f(v)$defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty}\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]}{(1-\lambda)|\sigma|}g(k)a_k \leq \frac{a.b(1+p)(1+p+\eta-\gamma)}{c(1+p+\eta-\mu)(1+p-\gamma)} \quad (3.2)$$

Then $M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$where $g(k)$ is defined as (1.18)

Proof: We have

$$M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(z) = v^p - \sum_{k=p+n}^{\infty} g(k)a_k v^k, \quad (3.3)$$

$g(k)$ is defined as (1.18)

By hypothesis of the theorem, we observe that the function $g(k)$ is the nonincresing function, that is,

$g(p+n) \leq (p+1), n \in N.$

Thus

$$0 < g(p+n) \leq g(p+1) = \frac{a.b(1+p)(1+p+\eta-\gamma)}{c(1+p+\eta-\mu)(1+p-\gamma)} \quad (3.4)$$

Using (3.2) and (3.4), we get

$$\sum_{k=p+n}^{\infty}\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]}{(1-\lambda)|\sigma|}g(k)a_k \leq g(p+1)$$

Therefore $M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$

1. Remark:

The inequality in (3.2) is attained for the function

$$M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v) = v^p - \frac{c(1-\lambda)|\sigma|(1+p+\eta-\mu)(1+p-\gamma)}{a.b[(1+\beta)+(1-\lambda)|\sigma|](1+p)(1+p+\eta-\gamma)}v^{p+1}, \quad (3.5)$$

B. *Corollary:*

Let $\mu,\gamma,\eta \in \mathbb{R}$ such that $\mu \geq 0, \gamma < 1+p$,

$$max\{\mu,\gamma\} - (1+p) < \eta \leq \frac{\mu(\gamma-(2+p))}{\gamma} \quad (3.6)$$

Also let the function $f(v)$ by (1.2) satisfy

$$\sum_{k=p+n}^{\infty}\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]}{(1-\lambda)|\sigma|}g(k)a_k \leq \frac{(1+p+\eta-\mu)(1+p-\gamma)}{(1+p)(1+p+\eta-\gamma)} \quad (3.7)$$

For $0 \leq \lambda < 1, ,\beta \geq 0, \sigma \in c\backslash\{0\}$, then

$$M_{0,z}^{\mu,\gamma,\eta,a,b,c}f(v) = J_{0,z}^{\mu,\gamma,\eta}f(v) \in UST(\alpha,\beta,p)$$

Proof: The corollary follows from theorem (3.1) by setting $a = c$.

C. *Corollary:*

Let $\mu,\gamma,\eta \in \mathbb{R}$ such that $\mu \geq 0, \gamma < 1+p$,

$$max\{\mu,\gamma\} - (1+p) < \eta \leq \frac{\mu(\gamma-(2+p))}{\gamma}$$

Also let the function $f(v)$ by (1.2) satisfy

$$\sum_{k=p+n}^{\infty}\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]}{(1-\lambda)|\sigma|}g(k)a_k \leq \frac{c(1+p+\eta-\mu)(1+p-\gamma)}{ab(1+p)(1+p+\eta-\gamma)} \quad (3.8)$$

for $0 \leq \lambda < 1, ,\beta \geq 0, \sigma \in c\backslash\{0\}, a = c, b = 1$, then

$$M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v) = D_0^{\mu}f(v) \in UCV(\alpha,\beta,p)$$

Proof: The corollary follows from theorem (3.1) by setting $a = c, b = 1, \mu = \gamma$.

D. *Corollary:*

Let $-\infty < \mu,\gamma < 1$ *and* ηbe real. Also let function defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty}\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]}{(1-\lambda)|\sigma|}a_k \leq \frac{c(1+p-\mu)}{ab(1+p)} \quad (3.9)$$

for $0 \leq \lambda < 1, \beta \geq 0, \sigma \in c\backslash\{0\}$,Then

$$M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v) = D_{0,v}^{\mu}f(v) \in UST(\alpha,\beta,p)$$

Proof: The corollary follows from theorem (3.1) by setting $b = 1, \mu = \gamma$.

E. *Corollary:*

Let $-\infty < \mu,\gamma < 1$ *and* η be real. Also let function defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty}\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]}{(1-\lambda)|\sigma|}a_k \leq \frac{(1+p-\mu)}{(1+p)}, \quad (3.10)$$

for $0 \leq \lambda < 1, \beta \geq 0, \sigma \in c/\{0\}$,

Then $M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v) = D_{0,v}^{\mu}f(v) \in UCV(\alpha,\beta,p)$

Proof: The corollary follows from theorem (3.1) by setting $a = c, b = 1, \mu = \gamma$.

F. *Corollary:*

Let $-\infty < \mu,\gamma < 1$ *and* η be real such that $a,b \in \mathbb{R}$ *and* $c \in \mathbb{R}\backslash\bar{z}_0$ *where* $\bar{z}_0 = \{0,-1,-2,-3,\ldots\}$. Also let function defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty}\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]}{(1-\lambda)|\sigma|}a_k \leq \frac{c}{ab}, \quad (3.11)$$

for $0 \leq \lambda < 1, \beta \geq 0, \sigma \in c\backslash\{0\}$,

Then $M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v) = F_p(a,b;c)f(v) * U_{0,v}^{\mu,\gamma,\eta}f(v) \in UST(\alpha,\beta,p)$

Proof: The corollary follows from theorem (3.1) by setting $\mu = \gamma = 0$.

G. *Corollary:*

Let $-\infty < \mu, \gamma < 1$ *and* η be real such that $a, b \in \mathbb{R}$ *and* $c \in \mathbb{R} \setminus \bar{z}_0$ *where* $\bar{z}_0 = \{0, -1, -2, -3, \ldots\}$. Also let function defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty} \frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g_k}{(1-\lambda)|\sigma|} a_k \le \frac{c}{ab}, \quad (3.12)$$

for $0 \le \lambda < 1$, $\beta \ge 0, \sigma \in c\setminus\{0\}$,

Then

$M_{0,v}^{\mu,\gamma,\eta,a,b,c} f(v) = F_p(a,b;c) f(v) * U_{0,v}^{\mu,\gamma,\eta} f(v) \in UCV(\alpha,\beta,p)$

Proof: The corollary follows from theorem (3.1) by setting $\mu = \gamma = 0$.

IV. EXTREME POINTS OF THE CLASS $\boldsymbol{K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)}$

A. *Theorem 4.1:*

Let $f_p(v) = z^p$ *and*

$$f_k(v) = z^p - \frac{(1-\lambda)|\sigma|}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)} v^k, \quad (k \ge p+1) \quad (4.1)$$

Then $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ if and only if $f(v)$ can be express in the form

$$f(v) = \sum_{k=p}^{\infty} \lambda_k f_k(v), \text{ where } \lambda_k \ge 0 \text{ and } \sum_{k=p}^{\infty} \lambda_k = 1$$

Proof: Let $f(z)$ expressible in the form

$$f(v) = \sum_{k=p}^{\infty} \lambda_k f_k(v)$$

Then,

$$f_k(v) = v^p - \sum_{k=p+n}^{\infty} \frac{(1-\lambda)|\sigma|}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)} \lambda_k v^k$$

But

$$\sum_{k=p+n}^{\infty} \frac{(1-\lambda)|\sigma|\lambda_k}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)} \frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}{(1-\lambda)|\sigma|}$$

$$= \sum_{k=p+1}^{\infty} \lambda_k = 1 - \lambda_p \le 1.$$

Thus, $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$

Conversely, suppose that $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$

Thus,

$$a_k \le \frac{(1-\lambda)|\sigma|}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}$$

Setting

$$\lambda_k = \frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}{(1-\lambda)|\sigma|} a_k$$

and $\lambda_p = 1 - \sum_{k=p+1}^{\infty} \lambda_k$, we see that $f(v)$ can be expressed in the form (4.1)

B. *Corollary:*

The extreme points of the class $K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ are $f_p(v) = v^p$ and

$$f_k(v) = v^p - \sum_{k=p+n}^{\infty} \frac{(1-\lambda)|\sigma|}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)} v^k, \quad k \ge p+1. \quad (4.2)$$

V. GROWTH AND DISTORTION THEOREM:

A. *Theorem 5.1:*

Let the function $f(v)$ defined by (1.2) be in the class $K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. Then

$$\left| \left| M_{0,v}^{\mu,\gamma,\eta,a,b,c} f(v) \right| - |v|^p \right| \le \frac{c(1+p+\eta-\mu)(1+p-\gamma)(p-\alpha)}{a.b(1+p)(1+p+\eta-\gamma)(1+p+\beta-\alpha)} |v|^{p+1} \quad (5.1)$$

And

$$\left| \left| \left(M_{0,v}^{\mu,\gamma,\eta,a,b,c} f(v) \right)' \right| - p|v|^{p-1} \right| \le \frac{c(1+p+\eta-\mu)(1+p-\gamma)(p-\alpha)}{a.b(1+p+\eta-\gamma)(1+p+\beta-\alpha)} |v|^p \quad (5.2)$$

Remark: The result (5.1) and (5.2) are sharp for the extremal function $f(v)$ given by

$$f(v) = v^p - \frac{c(1+p+\eta-\mu)(1+p-\gamma)(p-\alpha)}{a.b(1+p)(1+p+\eta-\gamma)(1+p+\beta-\alpha)} v^{p+1} \quad (5.3)$$

B. *Corollary:*

Let $M_{0,v}^{\mu,\gamma,\eta,a,b,c} f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ then the disc $|z| < 1$ is mapped onto a domain that contains the disc

$$|w| < 1 - \frac{c(1+p+\eta-\mu)(1+p-\gamma)(p-\alpha)}{a.b(1+p)(1+p+\eta-\gamma)(1+p+\beta-\alpha)}$$

Also $\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c} f(v) \right)'$ maps the disc $|z| < 1$ is mapped onto a domain that contains the disc

$$|w| < p - \frac{pc(1+p+\eta-\mu)(1+p-\gamma)(p-\alpha)}{a.b(1+p+\eta-\gamma)(1+p+\beta-\alpha)}$$

1. Remark:

We can obtain Growth and Distortion theorems for $J_{0,v}^{\mu,\gamma,\eta} f(v), D_{0,v}^{\mu} f(v), F_p(a,b;c;f(v))$ *and* $L_p(a,b)f(v))$ by accordingly initializing the parameters.

VI. FAMILY OF CLASS PRESERVING INTEGRAL OPERATORS:

Consider $F(v)$ defined by

$$F(v) = (J_{c,p}f)(v) = \frac{c+p}{v^c}\int_0^v t^{c-1}f(t)dt \quad (6.1)$$

$$for\ f \in A_p; c > -p$$

Let the integral operator $G(v)$ be defined by

$$G(v) = z^{p-1}\int_0^z \frac{f(t)}{t^p}dt \quad (6.2)$$

is class preserving.

The Komatu operator [5] is defined by

$$H(v) = p_{c,p}^d f(v) = \frac{(c+p)^d}{\Gamma(d)v^c}\int_0^v t^{c-1}\left(log\frac{v}{t}\right)^{d-1} f(t)dt,$$

$$for\ d > 0; c > -p, v \in E, \quad (6.3)$$

Another integral operator, which is generalized Jung-Kim-Srivastava integral operator defined by

$I(v) = Q_{c,p}^d f(v) =$

$$\binom{d+c+p-1}{c+p-1}\frac{d}{v^c}\int_0^v t^{c-1}\left(1-\frac{t}{v}\right)^{d-1} f(t)dt \quad (6.4)$$

$for\ d > 0; c > -p, v \in E,$ is also class preserving.

A. *Theorem 6.1:*

Let $d > 0$, $c > -p$ and $f(v)$ belong to the class $K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. Then the function $H(v)$ defined by (6.3) is p-valent in the disc $|v| < R_1$, where

$$R_1 = \inf_k\left\{\frac{p[(1+\beta)(k-p)+(1-\lambda)|\sigma|](c+k)^d g(k)}{(1-\lambda)|\sigma|(c+p)^d}\right\}^{\frac{1}{k-p}} \quad (6.5)$$

The result is sharp for the function $f(v)$ given by

$$f(v) = v^p - \sum_{k=p+n}^{\infty}\frac{(1-\lambda)|\sigma|(c+p)^d}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)(c+k)^d}v^{p+n}, \quad n \in N.$$

Proof: Notice that $H(z) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ and has the form

$$H(v) = v^p - \sum_{k=p+n}^{\infty}\left(\frac{c+p}{c+k}\right)^d a_k v^k, n \in N. \quad (6.6)$$

In order to prove assertion it is enough to prove that

$$\left|\frac{H'(z)}{v^{p-1}} - p\right| \le p \text{ in } |v| < R_1. \quad (6.7)$$

Now,

$$\left|\frac{H'(v)}{v^{p-1}} - p\right| = \left|-\sum_{k=p+n}^{\infty} k\left(\frac{c+p}{c+k}\right)^d a_k v^{k-p}\right|$$

$$\le \sum_{k=p+n}^{\infty} k\left(\frac{c+p}{c+k}\right)^d a_k|v|^{k-p}.$$

The last inequality is bounded above by p if

$$\sum_{k=p+n}^{\infty} k\left(\frac{c+p}{c+k}\right)^d a_k|v|^{k-p} \le p.$$

$$\sum_{k=p+n}^{\infty}\frac{k\left(\frac{c+p}{c+k}\right)^d a_k|v|^{k-p}}{p} \le 1. \quad (6.8)$$

Given that $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ and so, by theorem (2.2)

$$\sum_{k=p+n}^{\infty}\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)a_k}{(1-\lambda)|\sigma|} \le 1. \quad (6.9)$$

Thus inequality (6.8) will hold if

$$k\left(\frac{c+p}{c+k}\right)^d a_k|v|^{k-p} \le \frac{p[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}{(1-\lambda)|\sigma|},$$

for $k \ge p+n$.

That is if

$$|v| \le \left\{\frac{p[(1+\beta)(k-p)+(1-\lambda)|\sigma|](c+k)^d g(k)}{k(1-\lambda)|\sigma|(c+p)^d}\right\}^{\frac{1}{k-p}}$$

for $k \ge p+n, n \in N$.

The result follows by setting $|v| = R_1$.

B. *Theorem 6.2:*

Let $d > 0$, $c > -p$ and $f(v)$ belong to the class $K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. Then the function $I(v)$ defined by (8.3) is p-valent in the disc $|v| < R_2$, where

$$R_2 = \inf_k\left\{\frac{p[(1+\beta)(k-p)+(1-\lambda)|\sigma|]\Gamma(d+c+k)\Gamma(c+p)g(k)}{k(1-\lambda)|\sigma|\Gamma(d+c+p)\Gamma(c+k)}\right\}^{\frac{1}{k-p}} \quad (6.10)$$

The result is sharp for the function $f(v)$ given by

$$f(v) = v^p - \sum_{k=p+n}^{\infty}\frac{(1-\lambda)|\sigma|\Gamma(d+c+p)\Gamma(c+k)}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]\Gamma(d+c+k)\Gamma(c+p)g(k)}v^{p+n}, \quad n \in N.$$

Proof: Notice that $I(z) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$and has the form

$$I(v) = v^p - \sum_{k=p+n}^{\infty}\frac{\Gamma(d+c+p)\Gamma(c+k)}{\Gamma(d+c+k)\Gamma(c+p)}a_k v^{p+n} \quad (6.11)$$

In view of the arguments similar to Theorem8.1 and relation (6.11), we get

$$|v| = \left\{\frac{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]\Gamma(d+c+k)\Gamma(c+p)g(k)}{(1-\lambda)|\sigma|\Gamma(d+c+p)\Gamma(c+k)}\right\}^{\frac{1}{k-p}} \quad (6.11)$$

For $k \geq p+n, n \in N$.

VII. RADIUS OF UNIFORM STARLIKENESS, CONVEXITY AND CLOSE-TO-CONVEXITY:

A. *Theorem 7.1:*

Let the function $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$ f is p-valently starlike of order s, $(0 \leq s < p)$ in the disk $|v| \leq R_3$, where

$$R_3 = \inf_k \left[\frac{(p-s)[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}{(k-s)(1-\lambda)|\sigma|}\right]^{1/k-p} \quad (7.1)$$

The result sharp for the extremal function given by (2.4)

Proof: For $0 \leq s < p$, we need to show that

$$\left|\frac{zf'(v)}{f(v)} - p\right| \leq p - s\,, \quad (7.2)$$

And $|z| < R_3$.

$$\left|\frac{1}{\sigma}\left(\frac{v\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)'}{\left(M_{0,v}^{\mu,\gamma,\eta,a,b,c}f(v)\right)} - p\right)\right| = \left|\frac{1}{\sigma}\left(-\frac{\sum_{k=p+n}^{\infty}(k-p)g(k)a_k v^{k-p}}{1-\sum_{k=p+n}^{\infty}g(k)a_k v^{k-p}}\right)\right| \leq p - s$$

$$|v|^{k-p} \leq \frac{(p-s)[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}{[(k-s)+(|\sigma|-1)p](1-\lambda)}$$

B. *Theorem 7.2:*

Let the function $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. Then f is p-valently convex of order s, $(0 \leq s < p)$ in the disk $|v| \leq R_4$, where

$$R_4 = \inf_k \left[\frac{p(p-s)[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}{k(k-s)(1-\lambda)|\sigma|}\right]^{1/k-p} \quad (7.3)$$

The result sharp for the extremal function given by (2.4)

C. *Theorem 7.3:*

Let the function $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. Then $f(v)$ is p-valently close-to-convex of order s, $(0 \leq s < p)$ in the disk $|v| \leq R_5$, where

$$R_5 = \inf_k \left[\frac{(p-s)[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)}{k(k-s)(1-\lambda)|\sigma|}\right]^{1/k-p} \quad (7.4)$$

VIII. GROWTH AND DISTORTION BOUND:

A. *Theorem 8.1:*

Let the function $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. Then for $|v| \leq r$, we have

$$|f(v)| \geq r^p - \frac{(1-\lambda)|\sigma|}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)} r^{p+n} \quad (8.1)$$

$$|f(v)| \leq r^p + \frac{(1-\lambda)|\sigma|}{[(1+\beta)(k-p)+(1-\lambda)|\sigma|]g(k)} r^{p+n} \quad (8.2)$$

$$|f(v)| = v^p - \frac{(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} v^{p+n}$$

Proof: Given that $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. From the equation (2.1) we have

$$[(1+\beta)+(1-\lambda)|\sigma|]g(p+n) \leq \sum_{k=p+n}^{\infty}[(1+\beta)(k-p) + (1-\lambda)|\sigma|]g(k)a_k \leq (1-\lambda)|\sigma|$$

which is equivalent to

$$\sum_{k=p+n}^{\infty} a_k \leq \frac{(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} \quad (8.3)$$

Using (1.1) and (8.3), we obtain

$$|f(v)| = |v|^p + \sum_{k=p+n}^{\infty} a_k|v|^k \leq f(v) = r^p + \sum_{k=p+n}^{\infty} a_k r^k,$$

$$f(v) = r^p + r^{p+n}\sum_{k=p+n}^{\infty} a_k \leq r^p + \frac{(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} r^{p+n}$$

Similarly

$$f(z) \geq r^p - \frac{(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} r^{p+n}$$

This complete the proof of the theorem 8.1

B. *Theorem 8.2:*

Let the function $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. Then for $|z| \leq r$, we have

$$|f'(v)| \geq pr^{p-1} - \frac{(p+n)(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} r^{p+n-1} \quad (8.4)$$

$$|f'(v)| \leq pr^{p-1} + \frac{(p+n)(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} r^{p+n-1} \quad (8.5)$$

And results are sharp for f given by

$$f(v) = v^p - \frac{(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} v^{p+n}$$

Proof: Given that $f(v) \in K(\mu,\gamma,\eta,a,b,c,\sigma,\lambda,\beta)$. From the equation (2.1) and (8.3) we have

$$f'(v) = pv^{p-1} + \sum_{k=p+n}^{\infty} ka_k v^{k-1},$$

$$|f'(v)| = p|v|^{p-1} + \sum_{k=p+n}^{\infty} ka_k |v|^{k-1}$$

$$= pr^{p-1} + \sum_{k=p+n}^{\infty} ka_k r^{k-1},$$

$$|f'(v)| \leq pr^{p-1} + (p+n)r^{p+n-1} \sum_{k=p+n}^{\infty} a_k$$

$$\leq pr^{p-1} + \frac{(p+n)(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} r^{p+n-1}$$

Similarly

$$|f'(v)| \geq pr^{p-1} - \frac{(p+n)(1-\lambda)|\sigma|}{[(1+\beta)+(1-\lambda)|\sigma|]g(p+n)} r^{p+n-1}$$

This complete the proof of the theorem 8.2

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