The Closed Limit Point Compactness

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Abstract **In this paper, we gave a new topological concept and we called it The closed limit point compactness. This concept is stronger than the concept of a limit point compactness, that is, every a closed limit point compact space is a limit point compact space but the converse is not true. We have proved that the property of being a closed limit point compact is a topological property but not a hereditary property but it inherits to the closed subspace. We have shown that the continuous image of a closed limit point compact need not be a closed limit point compact. Also, we have shown that the quotient space of a closed limit point compact need not be a closed limit point compact. Finally, we have shown that if $X \times Y$ is a closed limit point compact and Y is a T_1-space, then X is a closed limit point compact.**

***Keywords*— compact space, limit point, limit point compact space.**

I. Introduction

In 1906, Frechet used the "Bolzano-Weierstrass Theorem"; every bounded infinite set of real numbers has a limit point ;when he generalized Weierstrass's Theorem to topological spaces. We now know this property as the Bolzano-Weierstrass property, or the limit point compactness [4].

This paper gave a new topological concept which is the closed limit point compactness.

The section one included the fundamental topological concepts such as homeomorphism, topological property, hereditary property, quotient topology, limit point, and limit point compactness.

In section two, we gave the main results of the closed limit point compact . $\acute{A}$ denotes the set of all limit points of A.

II. FUNDAMENTAL CONCEPTS

2.1 Definition[1].

A property of a topological space $(X,\mathcal{T})$ is said to be a topological property if any a topological space $(Y,\mathcal{S})$ which is homeomorphic to $(X,\mathcal{T})$ has that property.

2.2 Definition[3].

A property of a topological space $(X,\mathcal{T})$ is said to be a hereditary property if any subspace of $(X,\mathcal{T})$ has that property.

2.3 Definition[2].

Let $(X,\mathcal{T})$ be space and R be an equivalence relation on X. Take the quotient function,

$q: X \longrightarrow X / R$ which is define by

$q(x) = [x], \forall x \in X$.

Put $\mathcal{T}_q = \{w \subseteq X/R : q^{-1}(w) \in \mathcal{T}\}$, $\mathcal{T}_q$ is a topology on X/R which is called the quotient topology on X/R and the pair $(X/R, \mathcal{T}_q)$ is called quotient space of $(X,\mathcal{T})$.

2.4 Example.

Consider the topological space $(X,\mathcal{T})$ on the set X, where $X = \{3,6,7,8,10\}$ and

$\mathcal{T} = \{X, \emptyset, \{3\}, \{3,6\}, \{3,6,7\}, \{3,6,7,8\}\}$.

Define a relation R on X by $x\ R\ y$ if, and only if, $x - y = 3n$ $\forall x, y \in X$ *and* $n \in \mathbb{N}$. R is an equivalence relation on X.

$q(x) = [x]$, $\forall x, y \in X$.

$\Rightarrow [3] = [6] = \{3,6\}$,

$[7] = [10] = \{7,10\}$,

$[8] = \{8\}$,

Consequently, $X/R = \{\{[3]\}, \{[7]\}, \{[8]\}\}$.

Then $\mathcal{T}_q = \{X/R, \emptyset, \{[3]\}\}$.

2.5 Definition[1].

A space X is said to be compact if every open cover for X has a finite subcover.

2.6 Definition[2].

A space X is said to be limit point compact if every infinite subset of X has a limit point.

2.7 Example.

The real numbers R with the cofinite topology is a limit point compact.

2.8 Theorem[1].

A subset of a topological space is closed if, and only if, it contains all its limit points.

2.9 Theorem[5].

Let X *and* Y be spaces .If $y_o \in Y$, then $X \times \{y_0\} \cong X$.

III. Main results

3.1 Definition.

A topological space $(X,\mathcal{T})$ is said to be a closed limit point compact space if, and only if, any non-empty proper closed set of X has a limit point. A subset A of X is said to be a closed limit point compact if, and only if, any non-empty proper closed set of A has a limit point in A.

3.2 Examples:

(i) Consider the topological space $(X,\mathcal{T})$, where $X = \{a,b,c,d\}$

and $\mathcal{T} = \{X, \emptyset, \{a,b\}, \{c,d\}\}$. The sets $F_1 = \{a,b\}$ and $F_2 = \{c,d\}$ are closed in X and has limit point i.e. $\acute{F_1} = F_1$ and $\acute{F_2} = F_2$. Hence X is a closed limit point compact.

(ii) The real numbers with the usual topology $(R, \mathcal{T}_u)$ is not a closed limit point compact.

$\{3\}$ is non-empty proper closed set in R which has no a limit point compact.

(iii) A discrete topological space $(X, \mathfrak{D})$ of more than one point, is not a closed limit point compact. For every closed set in X has no limit point.

3.3 Theorem.

A closed limit point compact space is a limit point compact space.

Proof.

Let $(X, \mathcal{T})$ be a closed limit point compact space.

Suppose that $(X, \mathcal{T})$ is not a limit point compact.

This means that there exists an infinite set F of X such that $\acute{F} = \emptyset$

$\Longrightarrow F$ is closed (Theorem 2.8)

$\Longrightarrow$ X is not a closed limit point compact which is a contradiction for $(X, \mathcal{T})$ is a closed limit point compact.

Thus $(X, \mathcal{T})$ must be a limit point compact.

3.4 Remark.

The converse of the above theorem is not true as we shall show now:

Consider the space X with the cofinite topology where X is an infinite. It is a limit point compact.

Let $x_o, x_1 \in X$ and $F = \{x_o, x_1\}$. F is a closed set in X, and $X \setminus \{x_o\}$ is an open set in X containing x_1,

$((X \setminus \{x_o\}) \setminus \{x_1\}) \cap F = \phi$

So X_1 is not a limit point of F

Now, $X \setminus \{x_1\}$ is an open set in X containing x_o

$((X \setminus \{x_1\}) \setminus \{x_o\}) \cap F = \phi$

So x_o is not a limit point of F

Then $\acute{F} = \emptyset$.

Hence $(X, \mathcal{T}_c)$ is not a closed limit point compact.

3.5 Theorem.

The property of being of a closed limit point compact space is a topological property.

Proof.

Let $(X, \mathcal{T})$ be a closed limit point compact space and let $(Y, \mathcal{S})$ be a topological space which is a homeomorphic to $(X, \mathcal{T})$.

There exists a homomorphism $f: X \longrightarrow Y$. Let F be non-empty proper closed set of Y.

Then $f^{-1}(F)$ is non-empty proper closed subset of X.

Since $(X, \mathcal{T})$ be a closed limit point compact, then $f^{-1}(F)$ has a limit point, say x_0.

We claim that $f(x_0)$ is a limit point of F:

Suppose that $f(x_0)$ is not a limit point of F. Then there exists an open set V containing $f(x_0)$ such that

$(V \setminus \{f(x_0)\}) \cap F = \emptyset$

$\Longrightarrow f^{-1}((V \setminus \{f(x_0)\}) \cap F) = \emptyset$

$\Longrightarrow f^{-1}(V \setminus \{f(x_0)\}) \cap f^{-1}(F) = \emptyset$

$\Longrightarrow (f^{-1}(V) \setminus f^{-1}(\{f(x_0)\})) \cap f^{-1}(F) = \emptyset$

$\Longrightarrow (f^{-1}(V) \setminus \{x_0\}) \cap f^{-1}(F) = \emptyset$.

This means that x_0 is not a limit point of $f^{-1}(F)$ which is a contradiction.

So $f(x_0)$ is a limit point of F.

Hence Y is a closed limit point compact.

3.6 Remark.

A closed limit point compact is not a hereditary property:

Consider the topological space $(X, \mathcal{T})$, where $X = \{a,b,c,d,e\}$ and $\mathcal{T} = \{X, \emptyset, \{a,b,c\}, \{d,e\}\}$.

The sets $F_1 = \{a,b,c\}$ and $F_2 = \{d,e\}$ are closed and have limit points. $\acute{F_1} = F_1$ and $\acute{F_2} = F_2$.

$\Longrightarrow X$ is a closed limit point compact but let $A = \{a,e\} \subset X$ where $\mathcal{T}_A = \{A, \emptyset, \{a\}, \{e\}\}$ is not a closed limit point compact.

For the set $U = \{a\}$ is closed but has no a limit point.

3.7 Theorem.

Every a closed subspace of a closed limit point compact space is a closed limit point compact.

Proof.

Let $(X, \mathcal{T})$ be a closed limit point compact space and let Y be a non-empty proper closed set in X. Let A be non-empty proper closed set of Y.

$\Longrightarrow A = Y \cap F$, where F is non-empty proper closed in X, so A is a non-empty proper closed in X, then it has a limit point a for X is a limit point compact space.

Since A is closed, then $a \in A$ and so $a \in Y$. Hence Y is a closed limit point compact.

3.8 Remark.

The continuous image of a closed limit point compact space need not be a closed limit point compact:

Let $X = \{a,b,c,d\}$, $\mathcal{T} = \{X, \emptyset, \{a,b\}, \{c,d\}\}$ and Let $Y = \{0,1\}$ with the discrete topology.

Define $f: X \to Y$ by

$f(a) = f(b) = 0$,

$f(c) = f(d) = 1$.

f is continuous and X is a closed limit point compact but Y is not a closed limit point compact see Example (3.2(iii)).

3.9 Remark.

The quotient space of a closed limit point compact space need not be a closed limit point compact:

Consider the topological space, where

$X = \{3,6,7,8,10,11\}$, and

$\mathcal{T} =$

$\{X, \emptyset, \{3,6\}, \{7,10\}, \{8,11\}, \{3,6,7,10\}, \{3,6,8,11\}, \{7,10,8,11\}\}$.

It is clear that X is a closed limit point compact. For every a closed set in X has a limit point.

Define a relation R on X by xRy iff

$x - y = 3n$, $n \in \mathbb{Z}^+$, R is an equivalence relation on X.

$q: X \longrightarrow X/\mathrm{R}$
$q(x) = [x], \forall\, x, y \in X,$
$\Longrightarrow [3] = [6] = \{3, 6\},$
$[7] = [10] = \{7, 10\},$
$[8] = [11] = \{8, 11\}.$
$\Longrightarrow\ X/\mathrm{R} = \{\{[3]\}, \{[7]\}, \{[8]\}\}$
Consequently,
$\mathcal{T}_q = \{G \in X/\mathrm{R} :\ q^{-1} \in \mathcal{T}\} = \mathfrak{D}$. which is not a closed limit point compact. See Example (3.2(iii)).

3.10 Theorem.

If $X \times Y$ is a closed limit point compact and Y is a T_1-space, then X is a closed limit point compact.

Proof:

Let $y_0 \in Y$, then $\{y_0\}$ is a closed set in Y.
So $X \times \{y_0\}$ is a closed set in $X \times Y$. Since $X \times Y$ is a closed limit point compact, then $X \times \{y_0\}$ is a closed limit point compact (Theorem 3.7).
But by (Theorem 2.9), $X \times \{y_0\} \cong X$.
Then X is a closed limit point compact (Theorem 3.5).

IV. CONCLUSIONS

Closed limit point compactness is a new topological concept. Continuous image of a closed limit point compact space need not be a closed limit point compact .It is a topological property , but it is not a hereditary property.
It inherent to closed subspaces. Closed limit point compact spaces implies limit point compact spaces.

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