Optimal Fourth-Order Iterative Methods for Solving Nonlinear Equations

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Abstract—**This paper presents a new optimal fourth-order iterative methods for solving nonlinear equations $f(x) = 0$. The proposed iterative methods are obtained by composing the fourth-order iterative family method proposed by King 1973 with the ellipse method described by Gupta et al.(1998), which is always defined even if the first directive of the function at a point is zero. The convergence analysis of the proposed iterative methods was performed. Several numerical examples are also considered to illustrate the efficiency and the accuracy of the proposed iterative methods.**

Keywords— **Nonlinear Equations, Optimal Iterative Methods, Order of Convergence, Newton-Type Methed, Ellipse Method.**

I. INTRODUCTION

One of the most investigated topics in applied mathematics is to approximate the solutions of the nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f: D \to R$ is a scalar function on an open interval D. Therefore, the design of the iterative methods for solving (1) is an interesting and important task in numerical analysis.

Assume that equation (1) has a simple root α, which is to be found and it is sufficiently smooth the neighborhood of α, and let x_0 be the initial value to this root.

Definition 1: [5] Let the nonlinear equation (1) be a real function with a simple root α, and let x_n be a sequence of real numbers that converge towards α. The order of convergence $m \in R$ is given by

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^m} = \lambda, \tag{2}$$

where λ is the asymptotic error constant.

To solve equation (1), one can use iterative methods such as Newton's method [22, 30] or its variants. Newton's method is the most famous basic iterative method in the literature for the computation of the root of the nonlinear equation by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3}$$

We consider the definition of the efficiency index as $EI = m^{1/w}$, where m is the order of the method and w is the number of function- and derivative-evaluation per cycle required by the method [30]. The Newton method requires $w = 2$ function- and one derivative-evaluation per cycle and of order $m = 2^{w-1} = 2$; thus, the efficiency index of Newton's method is $EI = 2^{\frac{w-1}{w}} = 1.4142$, which supports the Kung-Traub conjecture on the upper bound of the efficiency index (see [18]). Methods with this property are called optimal methods in this paper.

A number of methods have been considered by many researchers to improve the local order convergence of Newton's method by the expense of additional evaluations of the functions, derivatives and changes in the points of iterations; see, for example [1 - 4, 6 - 9, 11 - 15, 18 - 20, 24, 26 - 28, 30]. One of Newton-type method was considered by King [14]

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}, \beta \in R, \end{aligned} \tag{4}$$

for solving equation (1). The error equation corresponding to the above method is

$$e_{n+1} = (-c_2c_3 + 3c_2^3)e_n^4 + O(e_n^5).$$

This is the famous King formula; for example, see [3 - 9, 14, 23 - 25, 27, 28], which is fourth-order formula. This method requires one evaluation of the function and two evaluations of its derivative per iteration, $w = 3$. Thus, the method has efficiency indices $EI = 2^{\frac{2}{3}} = 1.587401$. Based on King's optimal-two-point methods (4), Bi et al. [4] used the method in the third step and constructed the optimal family of three-point methods of order eight. In additition, King's method (??) makes the first two steps of the optimal eighth order of Thurkal and Petkovi´c [29]. Some special cases of King's method have been considered after, for example, Ostrowski's method for $\beta = 0$ [22], Kou et al. [16] ($\beta = 1$), Chun [6] ($\beta = 2$), Chun and Ham [7], Kou et al. [15].

All of these multipoint iterative methods mentioned above are variants of the Newton-type method. In addition, they require a sufficiently good initial approximation, and the requirement of $f'(x) \neq 0$ is an essential condition for the convergence of methods such as Newton's method. Therefore, Newton's method was modified by Steffensen [11], who replaced the first derivative $f'(x_n)$ by the forward difference approximation. Steffensen's methods are of optimal second order convergence and free from any derivative of the function.

Recently, Kanwar and Tomar [12, 13] proposed an alternative to the failure situation of Newton's method and its various variants. The various families introduced by Kanwar and Tomar [13] produces only multipoint iterative methods of order three. In addition, Mir et al. [21] have proposed a new predictor-corrector method, not optimal third-order of convergence, designated as the Simpson-Mamta method [17].

More recently, a family of ellipse methods were developed by Gupta et al. [10] given by

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}, \tag{5}$$

where the parameter p to be freely chosen in $R-\{0\}$, in which $f'(x) = 0$ is permitted at some points in the vicinity of the root. This method converges quadratically and moreover, has the same error equation as Newton's method. Therefore, this method is an efficient alternative to Newton's method.

In this paper, we present a family of predictor-corrector iterative methods based on quadratically convergent ellipse family methods (5) and the King family methods (4). The following sections of the paper are organized as follows. In Section 2, we describe a new family of methods, and the order of convergence of these methods is also presented. In Section 3, a different numerical test conformed the theoretical results and allowed us to compare this family with the other known methods mentioned in this section.

II. Development of Methods

Using ellipse method (5) in (??), we have

$$\begin{aligned} y_n &= x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}} \\ x_{n+1} &= y_n \pm \frac{f(y_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}\frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}, \end{aligned} \tag{6}$$

the correction factor $\frac{f(x_n)}{f'(x_n)}$ that appears in the multi-point iterative schemes (5) is now modified by

$$\pm\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}$$

where $p \neq 0 \in R$. The new iterative method (6) is always well defined, even if $f'(x_n) = 0$.

Theorem 1: Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f: D \subseteq R \to R$ for an open interval D. If x_0 is sufficiently close to α, then the family of methods (6) has fourth-order of convergence.

Proof: Let α be a simple root of $f(x) = 0$, that is, $f(\alpha) = 0$, and the error is expressed as

$$x_n = \alpha + e_n. \tag{7}$$

Using the Taylor expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)], \tag{8}$$

$$f^2(x_n) = f'^2(\alpha)(e_n^2 + 2c_2e_n^3 + (2c_3 + c_2^2)e_n^4 + O(e_n^5). \tag{9}$$

Furthermore, we have

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)], \tag{10}$$

and

$$f'^2(x_n) = f'(\alpha)[1 + 4c_2e_n + (6c_3 + 4c_2^2)e_n^2 + (12c_2c_3 + 8c_4)e_n^3 + (9c_3^2 + 16c_2c_4 + 10c_5)e_n^4 + O(e_n^5)]. \tag{11}$$

Using (8), (9), and (10) we obtain

$$\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}} = e_n - c_2e_n^2 + (-2c_3 + 2c_2^2 - \tfrac{1}{2}p^2)e_n^3 + (\tfrac{3}{2}c_2p^2 - 3c_4 + 7c_2c_3 - 4c_2^3)e_n^4 + O(e_n^5). \tag{12}$$

Accordingly, for the first step of (6), we obtain

$$y_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2 + \frac{1}{2}p^2)e_n^3 + (\frac{-3}{2}c_2 p^2 + 3c_4 - 7c_2 c_3 + 4c_2^3)e_n^4 + O(e_n^5). \quad (13)$$

Expanding $f(y)$ about α, we have

$$f(y_n) = f'(\alpha)[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 5c_2^3)e_n^4 + O(e_n^5)]. \quad (14)$$

Using (8), (9), and (14) we obtain

$$\frac{f(y_n)}{\sqrt{f'^2(x_n) + p^2 f^2(x_n)}} = c_2 e_n^2 + (2c_3 - 4c_2^2 + 1/2p^2)e_n^3 + (-3c_2 p^2 + 3c_4 - 14c_2 c_3 + 13c_2^3)e_n^4 + O(e_n^5). \quad (15)$$

Using (8) and (15), we have

$$\frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} = 1 + 2c_2 e_n + (-2c_2^2\beta + 4c_3 - 2c_2^2 + p^2)e_n^2 + (-2c_2\beta p^2 - 8c_2\beta c_3 + 4c_2^3\beta + 6c_4 - 4c_2 c_3$$
$$+2c_2^3\beta^2)e_n^3 + (-4c_3\beta p^2 + 2c_2^2\beta p^2 - 12c_2\beta c_4 + 16c_2^2\beta c_3 + \frac{1}{4}p^4 + 8c_5 + 4c_2^4 + c_3 p^2 - c_2^2 p^2 - 4c_2 c_4 - 6c_2^2 c_3$$
$$-\frac{1}{2}p^4\beta - 8\beta c_3^2 - 2c_2^4\beta + 12c_2^2\beta^2 c_3 + 3c_2^2\beta^2 p^2 - 6c_2^4\beta^2 - 2c_2^4\beta^3)e_n^4 + O(e_n^5)]. \quad (16)$$

Substituting (13), (15), and (16) into the second step x_{n+1} of (6), we have

$$x_{n+1} = \alpha + (-c_2 c_3 + c_2^3 - \frac{1}{2}c_2 p^2 + 2c_2^3\beta)e_n^4 + O(e_n^5). \quad (17)$$

Therefore,

$$e_{n+1} = (-c_2 c_3 + c_2^3 - \frac{1}{2}c_2 p^2 + 2c_2^3\beta)e_n^4 + O(e_n^5).$$

This shows that the order of convergence of the family (6) is four, which also ends the proof.

This is the modification over the formula (4) of King [14], one one hand, which will not fail, as did (4), if $f'(x_n) = 0$, and on the other hand, is a modification of the second-order ellipse's family methods into a parameter family iterative methods of forth-order of convergence.

In terms of the computational cost, this method (6) requires the same evaluation of functions, $w = 3$, as King's method (4) per step. As a result, it has an efficiency index $EI = 2^{\frac{2}{3}} = 1.587401$, which is optimal and higher than those of Newton and ellipse methods. In addition, the method is defined in the calculations. Several numerical examples are considered to illustrate the efficiency and the accuracy of the proposed iterative methods in the next section.

III. NUMERICAL RESULTS

The present family of fourth-order methods given by (6), (KEM), is used to solve nonlinear equations. To check the validity of the theoretical results, we present eight numerical test results for various iterative schemes in Table 1. The parameters $\beta = 1$ and $p = 0.5$ have been considered for method (KM), (4) or (KEM), (6). Furthermore, the performance is compared with some closed competitor methods, for example, the Jarratt's method [2] (JM) defined by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}[1 - \frac{2}{3}\frac{(f'(y_n) - f'(x_n))}{(3f'(y_n) - f'(x_n))}],$$

the method of Traub-Ostrowski [30] (TOM) defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}[\frac{f^2(x_n)}{f^2(x_n) - 2f(x_n)f(y_n)}],$$

the method of Siyyam et.al. [28] (SSM) defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = y_n - \frac{2f(x_n)G(u_n) + (y_n - x_n)(1 + G(u_n))f'(x_n)}{2f'(x_n)},$$

where $u_n = \frac{f(y_n)}{f(x_n)}$ and $G(u_n) = 1 + 2u_n + 4u_n^2 + u_n^3$, and the Lotfi method [20] (LM) defined by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - g(s_n)\frac{2f(x_n)}{f'(x_n) + f'(y_n)}$$

where $s_n = \frac{f'(y_n)}{f'(x_n)}$ and $g(s_n) = 2 - \frac{7}{4}s + \frac{3}{4}s^2$. Moreover, Newton's method (3) and the ellipse method (5) also been applied for the test function.

The following test functions and display the approximate zero α found up to the 15th decimal places

$f_1(x) = e^{(x^2+7x-30)} - 1, \alpha = 3.0,$
$f_2(x) = e^{-x} - \sin(x), \alpha = 6.285049273382587,$
$f_3(x) = (x-1)^4 - 1, \alpha = 2.0,$
$f_4(x) = 4x^4 - 4x^2, \alpha = -1.0,$
$f_5(x) = \sqrt{x^2+2}\sin(\frac{\pi}{x^2}) + \frac{1}{x^4+1} - \sqrt{3} - \frac{1}{17}, \alpha = -2.0,$
$f_6(x) = x^3 - 10, \alpha = 0.0,$
$f_7(x) = \tan^{-1}(x), \alpha = 0.0,$
$f_8(x) = 10xe^{-x^2}, \alpha = 0.0.$

All numerical computations listed in the (Table 1) and (Table 2) were performed using the MATLAB 7.1 package. The tables present the initial approximation x_0, and the number of iterations (IT). Further, the number of function evaluations (NFE) such that the stopping criteria are $|f(x_n)|$ and $|x_{n+1} - x_n|$ are less than 10^{-14}. In addition, the values of x_n and $|f(x_n)|$ are shown after the required stopping criteria. Also, presented in tables 1 and 2 are the values of the computational order of convergence (COC), which is approximated using the formula [26]

$$COC = \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}.$$

Formulas (5) were found to provide a good approximation to the root when $0 < p < 1$. This capability is a result of the ellipse shrinking in the vertical direction and extending along the horizontal direction. In other words, the next approximation will move faster towards the root. However, the formulas work if the initial guess is very close to the required root and $p > 1$ but not very large [10, 17]. As a result, $p = 0.5$ has been considered in (5) for the numerical examples.

IV. CONCLUSIONS

An optimal family of fourth-order iterative methods of convergence was developed. These proposed iterative methods are based on the combination of the fourth-order iterative method of King [14] with the ellipse method [10]. To confirm our theoretical results and to illustrate the efficiency and the accuracy of the developed iterative methods, we presented several numerical examples and compared the result with the other iterative methods, we found that the new results are satisfactory and compatible with the other iterative methods. In addition, the new family of methods can determine the required root, which is nearest to the initial value if the sign is chosen suitably; meanwhile, the Newton-type methods reproduce a far root in some cases. Moreover, the new iterative family did not fail if the first derivative at a point is near or equal to zero.

Table 1: Comparison of the Methods

	IT	NFE	x_n	$\|f(x_n)\|$	$\|x_{n+1} - x_n\|$	COC
$f_1(x) = e^{(x^2+7x-30)} - 1, x_0 = 3.5$						
NM	13	26	3.000	0.152e-46	0.421e-24	2.00
EM	21	42	3.000	0.719e-38	0.917e-20	2.00
JM	7	21	3.000	0.466e-226	0.431e-57	4.00
TOM	6	18	3.000	0.278e-61	0.693e-16	4.00
SSM	8	24	3.000	0.10e-113	0.290e-29	4.00
LM	8	24	3.000	0.62e-142	0.272e-36	4.00
KM	8	24	3.000	0.22e-190	0.226e-48	4.00
KEM	8	24	3.000	0.27e-190	0.237e-48	4.00
$f_2(x) = e^{-x} - \sin(x), x_0 = 5$						
NM	7	14	9.424	0.552e-38	0.826e-17	2.00
EM	5	10	6.285	0.40e-33	0.463e-15	2.00
JM	5	15	6.285	0.14e-105	0.261e-25	4.00
TOM	5	15	6.285	0.28e-119	0.982e-29	4.00
SSM	4	12	9.424	0.13e-101	0.563e-24	4.00
LM	5	15	9.424	0.103e-59	0.166e-13	4.00
KM	-	-	Fails	-	-	-
KEM	4	12	6.285	0.41e-122	0.271e-29	4.00
$f_3(x) = (x-1)^4 - 1, x_0 = 1.5$						
NM	10	20	2.000	0.196e-46	0.180e-23	2.00
EM	9	18	2.000	0.736e-41	0.110e-20	2.00
JM	5	15	2.000	0.115e-96	0.351e-24	4.00
TOM	5	15	2.000	0.356e-97	0.262e-24	4.00

SSM	-	-	Div	-	-	-
LM	14	42	2.000	0.284e-79	0.505e-20	4.00
KM	13	39	2.000	0.45e-168	0.339e-42	4.00
KEM	10	30	2.000	0.100e-95	0.415e-24	4.00
$f_4(x) = 4x^4 - 4x^2, x_0 = -\frac{\sqrt{(2)}}{2}$						
NM	749	1498	1.000	0.689e-51	0.587e-26	2.00
EM	13	26	-1.000	0.107e-51	0.231e-26	2.00
JM	-	-	Div.	-	-	-
TOM	-	-	Fails	-	-	-
SSM	9	27	-1.000	0.37e-157	0.919e-40	4.00
LM	12	36	1.000	0.46e-223	0.323e-56	4.00
KM	-	-	Fails	-	-	-
KEM	12	36	-1.000	0.29e-159	0.306e-40	4.00
$f_5(x) = \sqrt{x^2 + 2}\sin(\frac{\pi}{x^2}) + \frac{1}{x^4+1} - \sqrt{3} - \frac{1}{17}, x_0 = -4.25$						
NM	8	16	-2.000	0.180e-53	0.295e-26	2.00
EM	6	12	-2.000	0.826e-31	0.633e-15	2.00
JM	4	12	-2.000	0.185e-61	0.105e-14	4.00
TOM	4	12	-2.000	0.718e-58	0.563e-14	4.00
SSM	-	-	Div	-	-	-
LM	-	-	Div	-	-	-
KM	-	-	Fails	-	-	-
KEM	5	10	-2.000	0.44e-200	0.208e-49	4.00
$f_6(x) = \sin(x), x_0 = 1.53$						
NM	7	14	-21.991	0.88e-126	0.138e-41	2.00
EM	5	10	-0.60e-488	0.113e-53	0.176e-17	3.00
JM	-	-	Div	-	-	-
TOM	-	-	Div	-	-	-
SSM	-	-	Div	-	-	-
LM	-	-	Div	-	-	-
KM	-	-	Div	-	-	-
KEM	4	8	0.18e-982	0.73e-196	0.153e-38	5.00
$f_7(x) = \tan^{-1}(x), x_0 = -2$						
NM	-	-	Fails	-	-	-
EM	5	10	0.63e-762	0.22e-85	0.347e-28	3.00
JM	-	-	Fails	-	-	-
TOM	6	18	0.208e-989	0.15e-197	0.371e-39	5.00
SSM	-	-	Fails	-	-	-
LM	-	-	Fails	-	-	-
KM	-	-	Fails	-	-	-
KEM	4	12	0.71e-700	0.14e-139	0.166e-27	5.00
$f_8(x) = 10xe^{-x^2}, x_0 = 1$						
NM	-	-	Div	-	-	-
EM	5	10	0.409e-610	0.114e-66	0.182e-22	3.00
JM	-	-	Div	-	-	-
TOM	-	-	Div	-	-	-
SSM	-	-	Div	-	-	-
LM	-	-	Div	-	-	-
KM	-	-	Div	-	-	-
KEM	6	12	0.703e-1532	0.336e-305	0.459e-61	5.00

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