Some Common Fixed Point Theorems for Sequence of Mappings in Two Metric Spaces

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ABSTRACT:

In this paper we prove some common fixed point theorems for sequence of mappings in two complete metric spaces.

Key words and Phrases : fixed point, common fixed point, sequence of maps and complete metric space.

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1. INTRODUCTION.

Fixed point theory and common fixed point theory have basic roles in the application of some branches of mathematics. There are many articles about common fixed point theorems in metric spaces([3]-[5]). In [6] and [7], B.Fisher proved some theorems in two complete metric spaces. Later some authors proved some kind of fixed and common fixed point theorems in two metric spaces ([1], [2], [8]-[10]. In this paper we prove some common fixed point theorems for sequence of mappings in two complete metric spaces. The following definitions are necessary for the present study.

Definition1.1 A sequence $\{x_n\}$ in a metric space (X, d) is said to be convergent to a point $x \in X$ if given $\in > 0$ there exists a positive integer n_0 such that $d(x_n,x) < \in$ for all $n \geq n_0$.

Definition1.2. A sequence $\{x_n\}$ in a metric space (X,d) is said to be a Cauchy sequence in X if given $\in>0$ there exists a positive integer n_0 such that $d(x_m,x_n) < \in$ for all $m,n \geq n_0$.

Definition1.3 A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition1.4 Let X be a non-empty set and $f : X \to X$ be a map. An element x in X is called a fixed point of X if f(x) = x .

Definition1.5. Let X be a non-empty set and $f , g : X \to X$ be two maps. An element x in X is called a common fixed point of f and g if f(x) = g(x) = x.

Definition1.6. Let X be a non-empty set and a point x in X is said to be a common fixed point of sequence of maps $T_n : X \to X$ if $T_n(x) = x$ for all n.

2.MAIN RESULTS

Theorem 2.1: Let (X, d) and (Y, e) be complete metric spaces. Let $\{A_n\}$, $\{B_n\}$ ($n \in \mathbb{N}$) be sequence of mappings of X into Y and $\{S_n\}$, $\{T_n\}$ ($n \in \mathbb{N}$) be sequence of mappings of Y into X satisfying the inequalities.

$$d(S_pA_ix, T_qB_jx') \leq c_1. \max\{d(x,x'), d(x, S_pA_ix), d(x',T_qB_jx'),e(A_ix,B_jx'), d(x,T_qB_jx').d(S_pA_ix,x')\} \text{----- (2.1.1)}$$

$$e(B_jS_py, A_iT_qy') \leq c_2. \max\{ e(y,y'), e(y,B_jS_py), e(y',A_iT_qy'), d(S_py,T_qy'), e(y, A_iT_qy').e(B_jS_py ,y')\} \text{------ (2.1.2)}$$

for all $i \neq j \neq p \neq q$, x, x′ in X and y,y′ in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$. If one of the mappings $\{A_n\}$,$\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous , then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a common fixed point z in X and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a common fixed point w in Y. Further, $\{A_n\}z = \{B_n\}z = w$ and $\{S_n\}w =\{ T_n \}w = z$.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$$A_n x_{2n-2} = y_{2n-1}, S_n y_{2n-1} = x_{2n-1}, B_n x_{2n-1} = y_{2n};$$

$$T_n y_{2n} = x_{2n} \quad \text{for } n = 1, 2, 3 \ldots\ldots$$

Now using inequality (2.1.1) we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(S_nA_nx_{2n}, T_nB_nx_{2n-1}) \\ &\le c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n},S_nA_nx_{2n}), \\ &\qquad d(x_{2n-1}, T_nB_nx_{2n-1}), e(A_nx_{2n},B_nx_{2n-1}), \\ &\qquad d(x_{2n}, T_nB_nx_{2n-1}).d(S_nA_nx_{2n}, x_{2n-1})\} \\ &= c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \\ &\qquad e(y_{2n+1},y_{2n}), d(x_{2n}, x_{2n}). d(x_{2n-1}, x_{2n+1})\} \\ &\qquad e(y_{2n+1},y_{2n}), 0\} \\ &\le c_1.\max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n})\} \text{ ------- (2.1.3)} \end{aligned}$$

Now using inequality (2.1.2) we have

$$\begin{aligned} e(y_{2n}, y_{2n+1}) &= e(B_nS_ny_{2n-1}, A_nT_ny_{2n}) \\ &\le c_2.\max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_nS_ny_{2n-1}), \\ &\qquad e(y_{2n}, A_nT_ny_{2n}), d(S_ny_{2n-1}, T_ny_{2n}), \\ &\qquad e(y_{2n-1}, A_nT_ny_{2n}). e(B_nS_ny_{2n-1}, y_{2n})\} \\ &= c_2.\max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), \\ &\qquad d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n+1}).e(y_{2n}, y_{2n})\} \\ &= c_2.\max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), \\ &\qquad d(x_{2n-1}, x_{2n}), 0\} \\ &\le c_2.\max\{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\}\text{------- (2.1.4)} \end{aligned}$$

Again using inequality (2.1.1) we have

$$\begin{aligned} d(x_{2n}, x_{2n-1}) &= d(x_{2n-1}, x_{2n}) \\ &= d(S_nA_nx_{2n-2}, T_nB_nx_{2n-1}) \\ &\le c_1.\max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2},S_nA_nx_{2n-2}), \\ &\qquad d(x_{2n-1}, T_nB_nx_{2n-1}), e(A_nx_{2n-2}, B_nx_{2n-1}) \\ &\qquad d(x_{2n-2}, T_nB_nx_{2n-1}). d(S_nA_nx_{2n-2}, x_{2n-1})\} \\ &= c_1.\max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), \\ &\qquad d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \\ &\qquad d(x_{2n-2},x_{2n}). d(x_{2n-1}, x_{2n-1})\} \\ &= c_1.\max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), \\ &\qquad d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), 0\} \\ &\le c_1.\max\{ d(x_{2n-2},x_{2n-1}), e(y_{2n-1},y_{2n})\} \text{ ------ (2.1.5)} \end{aligned}$$

Now using inequality (2.1.2)

$$\begin{aligned} e(y_{2n},y_{2n-1}) &= e(B_nS_ny_{2n-1}, A_nT_ny_{2n-2}) \\ &\le c_2.\max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, B_nS_ny_{2n-1}), \\ &\qquad e(y_{2n-2}, A_nT_ny_{2n-2}), d(S_ny_{2n-1}, T_ny_{2n-2}), \\ &\qquad e(y_{2n-1}, A_nT_ny_{2n-2}).e(B_nS_ny_{2n-1},y_{2n-2})\} \\ &= c_2.\max\{ e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), \\ &\qquad e(y_{2n-1}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}) \\ &\qquad e(y_{2n-1}, y_{2n-1}). e(y_{2n}, y_{2n-2})\} \\ &= c_2.\max\{ e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), \\ &\qquad e(y_{2n-1}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), 0\} \\ &\le c_2.\max\{ e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\}\text{--.- (2.1.6)} \end{aligned}$$

from inequalities (2.1.3), (2.1.4), (2.1.5)and (2.1.6), we have

$$d(x_{n+1},x_n) \le c_1^{\,n}c_2^{\,n-1}.\max\{ d(x_1,x_0), e(y_1, y_2)\} \to 0 \text{ as } n\to\infty$$

Thus $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, $\{x_n\}$ converges to a point z in X. Similarly using inequalities (2.1.3), (2.1.4), (2.1.5)and (2.1.6), we prove $\{y_n\}$ is a Cauchy sequence in (Y,e) with the limit w in Y.

Suppose $\{A_n\}$ is continuous, then

$$\lim_{n\to\infty} A_nx_{2n} = A_nz = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove $S_nA_nz = z$. .

Suppose $S_nA_nz \ne z$.

We have

$$\begin{aligned} d(S_nA_nz, z) &= \lim_{n\to\infty} d(S_nA_nz, T_nB_nx_{2n-1}) \\ &\le \lim_{n\to\infty} c_1.\max\{ d(z, x_{2n-1}), d(z, S_nA_nz), \\ &\qquad d(x_{2n-1},T_nB_n x_{2n-1}), e(A_nz, B_nx_{2n-1}), \\ &\qquad d(z, T_nB_nx_{2n-1}).d(S_nA_nz , x_{2n-1})\} \\ &\le \lim_{n\to\infty} c_1.\max\{ d(z, x_{2n-1}), d(z,S_nA_nz), \\ &\qquad d(x_{2n-1},T_nB_n x_{2n-1}), e(A_nz, B_nx_{2n-1}), \\ &\qquad d(z, x_{2n}).d(S_nA_nz , x_{2n-1})\} \\ &= c_1.\max\{ d(z,z), d(z, S_nA_nz), d(z,z), e(w,w), \\ &\qquad d(z, z).d(S_nA_nz, z)\} \\ &= c_1.\max\{ 0, d(z,SAz), 0, 0, 0\} \\ &\le c_1.d(z, S_nA_nz) \\ &< d(z, S_nA_nz) \quad (\text{Since } 0 \le c_1 < 1) \end{aligned}$$

Which is a contradiction.

Thus $S_nA_nz = z$.

Hence $S_nw = z$. (Since $A_nz = w$)

Now we prove $B_nS_nw = w$.

Suppose $B_nS_nw \ne w$.

We have

$$e(B_nS_nw, w) = \lim_{n\to\infty} e(B_nS_nw, y_{2n+1})$$

$$= \lim_{n\to\infty} e(B_nS_nw, A_nT_ny_{2n})$$

$$\le \lim_{n\to\infty} c_2 \max\{ e(w, y_{2n}), e(w, B_nS_nw), e(y_{2n}, A_nT_ny_{2n}), d(S_nw, T_ny_{2n}), e(w, A_nT_ny_{2n}). e(B_nS_nw, y_{2n})\}$$

$$= c_2. \max\{ e(w, w), e(w, B_nS_nw), e(w, w), d(z, z), e(w,w).e(B_nS_nw,w)]\}$$

$$< e(w, B_nS_nw) \quad (\text{Since } 0 \le c_2 < 1)$$

Which is a contradiction.

Thus $B_nS_nw = w$.

Hence $B_nz = w$. (Since $S_nw = z$)

Now we prove $T_nB_nz = z$.

Suppose $T_nB_nz \ne z$.

$$d(z, T_nB_nz) = \lim_{n\to\infty} d(x_{2n+1}, T_nB_nz)$$

$$= \lim_{n\to\infty} d(S_nA_nx_{2n}, T_nB_nz)$$

$$\le \lim_{n\to\infty} c_1. \max\{ d(x_{2n}, z), d(x_{2n}, S_nA_nx_{2n}), d(z, T_nB_nz), e(A_nx_{2n}, B_nz), d(x_{2n}, T_nB_nz).d(SAx_{2n}, z)\}$$

$$= c_1. \max\{d(z, z), d(z, z), d(z, T_nB_nz), e(w, B_nz), d(z, T_nB_nz).d(z,z)\}$$

$$= c_1. \max\{0, 0, d(z, T_nB_nz), 0, 0\}$$

$$< d(z, T_nB_nz) \quad (\text{Since } 0 \le c_1 < 1)$$

Which is a contradiction.

Thus $T_nB_nz = z$.

Hence $T_nw = z$. (Since $B_nz = w$)

Now we prove $A_nT_nw = w$.

Suppose $A_nT_nw \ne w$.

$$e(w, A_nT_nw) = \lim_{n\to\infty} e(y_{2n}, A_nT_nw)$$

$$= \lim_{n\to\infty} e(B_nS_ny_{2n-1}, A_nT_nw)$$

$$\le \lim_{n\to\infty} c_2. \max\{ e(y_{2n-1}, w), e(y_{2n-1}, B_nS_ny_{2n-1}), e(w, A_nT_nw), d(S_ny_{2n-1}, T_nw), e(y_{2n-1}, A_nT_n w). e(B_nS_ny_{2n-1},w)\}$$

$$= c_2. \max\{ e(w, w), e(w, w), e(w, A_nT_nw), d(z,z), e(w, A_nT_nw).e(w,w)\}$$

$$< e(w, A_nT_nw) \quad (\text{Since } 0 \le c_2 < 1)$$

Which is a contradiction.

Thus $A_nT_nw = w$.

The same results hold if one of the mappings $\{B_n\}$,$\{S_n\}$ and $\{T_n\}$ is continuous.

So the point z is the common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove w is a common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

Remark :2.2 : If we put $A_i = A$, $B_j = B$, $S_p = S$ and $T_q = T$ in the above theorem 2.1, we get the following corollary.

Corollary 2.3: Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S , T be mappings of Y into X satisfying the inequalities.

$$d(SAx, TBx') \le c_1. \max\{d(x,x'), d(x, SAx), d(x',TBx'), e(Ax,Bx'), d(x,TBx').d(SAx,x')\}$$

$$e(BSy, ATy') \le c_2. \max\{ e(y,y'), e(y,BSy), e(y',ATy'), d(Sy,Ty'), e(y, ATy').e(BSy ,y')\}\text{-}$$

for all x, x′ in X and y,y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A, B, S and T is continuous , then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

Theorem 2.4: Let (X, d) and (Y, e) be complete metric spaces. Let $\{A_n\}$, $\{B_n\}$ ($n \in \mathbb{N}$) be sequence of mappings of X into Y and $\{S_n\}$, $\{T_n\}$, ($n \in \mathbb{N}$) be sequence of mappings of Y into X satisfying the inequalities.

$$d(S_pA_ix, T_qB_jx') \le c_1. \max\{ d(x, x'), d(x,S_pA_ix), d(x',T_qB_jx'), e(A_ix,B_jx'), \frac{d(x,T_qB_jx')}{2}, \frac{d(S_pA_ix,x')}{2}\} \text{---(2.4.1)}$$

$$e(B_jS_py, A_iT_qy') \le c_2. \max\{ e(y, y'), e(y,B_jS_py), e(y',A_iT_qy'), d(S_py,T_qy'), \frac{e(y,A_iT_qy')}{2}, \frac{e(B_jS_py,y')}{2}\} \text{---(2.4.2)}$$

for all $i \neq j \neq p \neq q$, x, x′ in X and y,y′ in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$. If one of the mappings $\{A_n\}$,$\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous , then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a unique common fixed point z in X and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a unique common fixed point w in Y. Further, $\{A_n\}z = \{B_n\}z = w$ and $\{S_n\}w = \{T_n\}w = z$.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$$A_n x_{2n-2} = y_{2n-1}, S_ny_{2n-1} = x_{2n-1}, B_nx_{2n-1} = y_{2n}; T_ny_{2n} = x_{2n}$$

for n = 1, 2, 3 . . . . .

Now we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(S_nA_nx_{2n}, T_nB_nx_{2n-1}) \\ &\leq c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, S_nA_n x_{2n}), \\ &\quad d(x_{2n-1}, T_nB_nx_{2n-1}), e(A_nx_{2n},B_nx_{2n-1}), \\ &\quad \frac{d(x_{2n},T_nB_nx_{2n-1})}{2}, \frac{d(S_nA_nx_{2n},x_{2n-1})}{2} \} \\ &= c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \\ &\quad e(y_{2n+1},y_{2n}), \frac{d(x_{2n},x_{2n})}{2}, \frac{d(x_{2n+1},x_{2n-1})}{2} \} \\ &\leq c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \\ &\quad e(y_{2n+1},y_{2n}), 0, \frac{d(x_{2n+1},x_{2n})+d(x_{2n},x_{2n-1})}{2} \} \\ &\leq c_1.\max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n})\} \text{-------2.4.3)} \end{aligned}$$

Now

$$\begin{aligned} e(y_{2n}, y_{2n+1}) &= e(B_nS_ny_{2n-1}, A_nT_n y_{2n}) \\ &\leq c_2.\max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_nS_ny_{2n-1}), \\ &\quad e(y_{2n}, A_nT_ny_{2n}), d(S_ny_{2n-1}, T_ny_{2n}), \\ &\quad \frac{e(y_{2n-1},A_nT_ny_{2n})}{2}, \frac{e(B_nS_ny_{2n-1},y_{2n})}{2} \} \\ &= c_2.\max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), \\ &\quad d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1},y_{2n+1})}{2}, \frac{e(y_{2n},y_{2n})}{2} \} \\ &= c_2.\max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), \\ &\quad d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1},y_{2n})+e(y_{2n},y_{2n+1})}{2}, 0 \} \\ &\leq c_2.\max\{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \text{------ (2.4.4)} \end{aligned}$$

we have

$$\begin{aligned} d(x_{2n}, x_{2n-1}) &= d(x_{2n-1}, x_{2n}) \\ &= d(S_nA_nx_{2n-2}, T_nB_nx_{2n-1}) \\ &\leq c_1.\max\{ d(x_{2n-2}, x_{2n-1}), d(x_{2n-2},S_nA_nx_{2n-2}), \\ &\quad d(x_{2n-1}, T_nB_nx_{2n-1}), e(A_nx_{2n-2}, B_nx_{2n-1}), \\ &\quad \frac{d(x_{2n-2},T_nB_nx_{2n-1})}{2}, \frac{d(S_nA_nx_{2n-2},x_{2n-1})}{2} \} \\ &= c_1.\max\{ d(x_{2n-2}, x_{2n-1}), d(x_{2n-2}, x_{2n-1}), \\ &\quad d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \\ &\quad \frac{d(x_{2n-2},x_{2n})}{2}, \frac{d(x_{2n-1},x_{2n-1})}{2} \} \\ &= c_1.\max\{ d(x_{2n-2}, x_{2n-1}), d(x_{2n-2}, x_{2n-1}), \\ &\quad d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \\ &\quad \frac{d(x_{2n-2},x_{2n-1})+d(x_{2n-1},x_{2n})}{2}, 0 \} \\ &\leq c_1.\max\{ d(x_{2n-2},x_{2n-1}), e(y_{2n-1},y_{2n})\} \text{-------(2.4.5)} \end{aligned}$$

Now

$$\begin{aligned} e(y_{2n},y_{2n-1}) &= e(B_nS_ny_{2n-1}, A_nT_ny_{2n-2}) \\ &\leq c_2.\max\{ e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, B_nS_ny_{2n-1}), \\ &\quad e(y_{2n-2}, A_nT_ny_{2n-2}), d(S_ny_{2n-1}, T_ny_{2n-2}), \\ &\quad \frac{e(y_{2n-1},A_nT_ny_{2n-2})}{2}, \frac{e(B_nS_ny_{2n-1},y_{2n-2})}{2} \} \\ &= c_2.\max\{ e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), \\ &\quad e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), \\ &\quad \frac{e(y_{2n-1},y_{2n-1})}{2}, \frac{e(y_{2n},y_{2n-2})}{2} \} \\ &\leq c_2.\max\{ e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), \\ &\quad e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), 0, \\ &\quad \frac{e(y_{2n},y_{2n-1})+e(y_{2n-1},y_{2n-2})}{2} \} \\ &\leq c_2.\max\{ e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\} \text{------2.4.6)} \end{aligned}$$

from inequalities (2.4.3), (2.4.4), (2.4.5)and (2.4.6), we have

$$d(x_{n+1},x_n) \leq c_1{}^n c_2{}^{n-1}.\max\{ d(x_1,x_0), e(y_1, y_2)\} \rightarrow 0 \text{ as } n\rightarrow\infty$$

Thus $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, $\{x_n\}$ converges to a point z in X. Similarly using inequalities (2.4.3), (2.4.4), (2.4.5) and (2.4.6), we prove $\{y_n\}$ is a Cauchy sequence in (Y,e) with the limit w in Y.

from inequalities (2.4.3) and (2.4.5), we have

$$d(x_{n+1},x_n) \leq c_1(c_2)^n.\max\{ d(x_1,x_0), e(y_1, y_2)\} \rightarrow 0 \text{ as } n\rightarrow\infty$$

Thus $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, $\{x_n\}$ converges to a point z in X. Similarly using inequalities (2.4.4) and (2.4.6), we prove $\{y_n\}$ is a Cauchy sequence in (Y,e) with the limit w in Y.

Suppose A is continuous, then

$$\lim_{n\to\infty} A_n x_{2n} = A_n z = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove $S_nA_nz = z$.

Suppose $S_nA_nz \neq z$.

We have

$$d(S_nA_nz, z) = \lim_{n\to\infty} d(S_nA_nz, T_nB_nx_{2n-1})$$

$$\leq \lim_{n\to\infty} c_1.\max\{ d(z, x_{2n-1}), d(z,S_nA_nz), d(x_{2n-1},T_nB_n x_{2n-1}), e(A_nz, B_nx_{2n-1}), \frac{d(z, T_nB_nx_{2n-1})}{2}, \frac{d(S_nA_nz, x_{2n-1})}{2} \}$$

$$= \lim_{n\to\infty} c_1.\max\{ d(z, x_{2n-1}), d(z,S_nA_nz), d(x_{2n-1}, x_{2n}), e(A_nz, y_{2n}), \frac{d(z, x_{2n})}{2}, \frac{d(S_nA_nz, x_{2n-1})}{2} \}$$

$$= c_1.\max\{ d(z, x_{2n-1}), d(z,S_nA_nz), d(z,z), e(w,w), \frac{d(z, z)}{2}, \frac{d(S_nA_nz, z)}{2} \}$$

$$= c_1.\max\{ d(z, x_{2n-1}), d(z,S_nA_nz), 0, 0, 0, \frac{d(S_nA_nz, z)}{2} \}$$

$$\leq c_1.\, d(z,S_nA_nz)$$

$$< d(z,S_nA_nz) \quad (\text{Since } 0 \leq c_1 < 1)$$

Which is a contradiction.

Thus $S_nA_nz = z$.

Hence $S_nw = z$. (Since $A_nz = w$)

Now we prove $B_nS_nw = w$.

Suppose $B_nS_nw \neq w$.

We have

$$e(B_nS_nw, w) = \lim_{n\to\infty} e(B_nS_nw, y_{2n+1})$$

$$= \lim_{n\to\infty} e(B_nS_nw, A_nT_ny_{2n})$$

$$\leq \lim_{n\to\infty} c_2 \max\{ e(w, y_{2n}), e(w, B_nS_nw), e(y_{2n}, A_nT_ny_{2n}), d(S_nw, T_ny_{2n}), \frac{e(w, ATy_{2n})}{2}, \frac{e(BSw, y_{2n})}{2} \}$$

$$= c_2.\max\{ e(w, y_{2n}), (w, B_nS_nw), e(w, w), d(z, z), \frac{e(w, w)}{2}, \frac{e(BSw, w)}{2} \}$$

$$< e(w, B_nS_nw) \quad (\text{Since } 0 \leq c_2 < 1)$$

Which is a contradiction.

Thus $B_nS_nw = w$.

Hence $B_nz = w$. (Since $S_nw = z$)

Now we prove $T_nB_nz = z$.

Suppose $T_nB_nz \neq z$.

$$d(z,TBz) = \lim_{n\to\infty} d(x_{2n+1}, T_nB_nz)$$

$$= \lim_{n\to\infty} d(S_nA_nx_{2n}, T_nB_nz)$$

$$\leq \lim_{n\to\infty} c_1.\max\{ d(x_{2n},z), d(x_{2n},S_nA_nx_{2n}), d(z, T_nB_nz), e(Ax_{2n}, B_nz), \frac{d(x_{2n}, TBz)}{2}, \frac{d(SAx_{2n}, z)}{2} \}$$

$$= c_1.\max\{ d(z, z), d(z, z), d(z, T_nB_nz), e(w, B_nz), \frac{d(z, TBz)}{2}, \frac{d(z, z)}{2} \}$$

$$= c_1.\max\{ 0,0, d(z, T_nB_nz), 0, \frac{d(z, TBz)}{2}, 0\}$$

$$< d(z, T_nB_nz) \quad (\text{Since } 0 \leq c_1 < 1)$$

Which is a contradiction.

Thus $T_nB_nz = z$.

Hence $T_nw = z$. (Since $B_nz = w$)

Now we prove $A_nT_nw = w$.

Suppose $A_nT_nw \neq w$.

$$e(w, A_nT_nw) = \lim_{n\to\infty} e(y_{2n}, A_nT_nw)$$

$$= \lim_{n\to\infty} e(B_nS_ny_{2n-1}, A_nT_nw)$$

$\leq \lim_{n\to\infty} c_2.\ \max\{\ e(y_{2n-1}, w), e(y_{2n-1}, B_nS_ny_{2n-1}),$

$e(w, A_nT_nw),\ \ d(S_ny_{2n-1}, T_nw),$

$\frac{e(y_{2n-1}, A_nT_nw)}{2}, \frac{e(B_nS_ny_{2n-1}, w)}{2}\}$

$= c_2.\ \max\{\ e(w, w), e(w, w), e(w, A_nT_nw),$

$d(z,z), \frac{e(w, A_nT_nw)}{2}, \frac{e(w, w)}{2}\}$

$= c_2.\ \max\{\ 0, 0, e(w, A_nT_nw), 0,$

$\frac{e(w, A_nT_nw)}{2}, 0\}$

$< e(w, A_nT_nw)$ (Since $0 \leq c_2 < 1$)

Which is a contradiction.

Thus $A_nT_nw = w$.

The same results hold if one of the mappings $\{B_n\}$,$\{S_n\}$ and $\{T_n\}$ is continuous.

Uniqueness: Let z′ be another common fixed point of S_nA_n and T_nB_n in X, w′ be another common fixed point of B_nS_n and A_nT_n in Y.

We have $d(z, z') = d(S_nA_nz, T_nB_nz')$

$\leq c_1.\ \max\{\ d(z, z'), d(z, S_nA_nz),$

$d(z',T_nB_nz'), e(A_nz,B_nz'),$

$\frac{d(z, T_nB_nz')}{2}, \frac{d(S_nA_nz, z')}{2}\}$

$= c_1.\ \max\{\ d(z, z'), d(z, z),\ d(z',z'),$

$e(w,w'), \frac{d(z, z')}{2}, \frac{d(z, z')}{2}\}$

$= c_1.\ \max\{\ d(z, z'), d(z, z),\ d(z', z'),$

$e(w,w'), \frac{d(z, z')}{2}\}$

$= c_1.\ \max\{\ d(z, z'), 0, 0, e(w,w'), \frac{d(z, z')}{2}\}$

$< e(w, w')$ (Since $c_1 < 1$)

$e(w, w') = e(B_nS_nw, A_nT_nw')$

$\leq c_2.\max\{\ e(w, w'), e(w, B_nS_nw),$

$e(w',A_nT_nw'), d(S_nw,T_nw'),$

$\frac{e(w, A_nT_nw')}{2}, \frac{e(B_nS_nw, w')}{2}\}$

$= c_2.\ \max\{\ e(w, w'), e(w,w), e(w', w'),$

$d(z, z'), \frac{e(w, w')}{2}, \frac{e(w, w')}{2}\}$

$= c_2.\ \max\{\ e(w, w'), 0, 0, d(z, z'), \frac{e(w, w')}{2}\}$

$< d(z, z')$

Hence $d(z, z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus z = z′.

So the point z is the unique common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove w is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

Remark :2.5 : If we put $A_i = A$, $B_j = B$, $S_p = S$ and $T_q = T$ in the above theorem 2.4, we get the following corollary.

Corollory 2.6: Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S , T be mappings of Y into X satisfying the inequalities.

$d(SAx, TBx') \leq c_1.\ \max\{\ d(x, x'), d(x,SAx), d(x',TBx'),$

$e(Ax,Bx'), \frac{d(x,TBx')}{2}\ \frac{d(SAx,x')}{2}\}$

$e(BSy, ATy') \leq c_2.\ \max\{\ e(y, y'), e(y,BSy), e(y',ATy'),$

$d(Sy,Ty'), \frac{e(y,ATy')}{2}, \frac{e(BSy,y')}{2}\}$

for all x, x′ in X and y,y′ in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$.

If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

Theorem2.7:Let (X, d) and (Y, e) be complete metric spaces. Let $\{A_n\}$, $\{B_n\}$ ($n \in \mathbb{N}$) be sequence of mappings of X into Y and $\{S_n\}$, $\{T_n\}$, ($n \in \mathbb{N}$) be sequence of mappings of Y into X satisfying the inequalities.

$d(S_pA_ix, T_qB_jx') \leq c_1.\ \max\{d(x,x'), d(x, S_pA_ix), e(A_ix,B_jx'),$

$d(x,T_qB_jx')/2, d(S_pA_ix,x')/2,$

$d(x, S_pA_ix).d(x',T_qB_jx')/ d(x,x')\}$----- (2.7.1)

$e(B_jS_py, A_iT_qy') \leq c_2.\ \max\{\ e(y,y'), e(y,B_jS_py), d(S_py,T_qy'),$

$e(y, A_iT_qy')/2, e(B_jS_py ,y')/2$

$e(y,B_jS_py).e(y',A_iT_qy') / e(y,y')\}$------ (2.7.2) for all $i \neq j \neq p \neq q$, x, x′ in X and y,y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings $\{A_n\}$,$\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous , then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a common fixed point z in X and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a common fixed point w in Y. Further, $\{A_n\}z = \{B_n\}z = w$ and $\{S_n\}w = \{T_n\}w = z$.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$$A_n x_{2n-2} = y_{2n-1}, S_ny_{2n-1} = x_{2n-1}, B_nx_{2n-1} = y_{2n}; T_ny_{2n} = x_{2n}$$

for n = 1, 2, 3 . . . . .

Now we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n}) &= d(S_nA_nx_{2n}, T_nB_nx_{2n-1}) \\
&\le c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n},S_nA_nx_{2n}), \\
&\quad e(A_nx_{2n},B_nx_{2n-1}), d(x_{2n}, T_nB_nx_{2n-1})/2, \\
&\quad d(S_nA_nx_{2n} ,x_{2n-1})/2, \\
&\quad d(x_{2n},S_nA_nx_{2n}). d(x_{2n-1}, T_nB_nx_{2n-1})/ d(x_{2n}, x_{2n-1})\} \\
&= c_1. \max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), e(y_{2n+1},y_{2n}), \\
&\quad d(x_{2n}, x_{2n})/2, d(x_{2n-1}, x_{2n+1})/2, \\
&\quad d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n})/ d(x_{2n}, x_{2n-1})\} \\
&= c_1. \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), e(y_{2n+1},y_{2n}) ,0, \\
&\quad [d(x_{2n-1}, x_{2n})+ d(x_{2n}, x_{2n+1})]/2, d(x_{2n}, x_{2n+1})\} \\
&\le c_1. \max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n})\})\} \text{----- (2.7.3)}
\end{aligned}$$

Now

$$\begin{aligned}
e(y_{2n}, y_{2n+1}) &= e(B_nS_ny_{2n-1}, A_nT_ny_{2n}) \\
&\le c_2. \max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_nS_ny_{2n-1}), \\
&\quad d(S_ny_{2n-1}, T_ny_{2n}), e(y_{2n-1}, A_nT_ny_{2n})/2, \\
&\quad e(B_nS_ny_{2n-1}, y_{2n})/2, \\
&\quad e(y_{2n-1}, B_nS_ny_{2n-1}).e(y_{2n}, A_nT_ny_{2n})/ e(y_{2n-1}, y_{2n})\} \\
&= c_2.\max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}), \\
&\quad e(y_{2n-1}, y_{2n+1}) / 2, e(y_{2n}, y_{2n}) / 2, \\
&\quad e(y_{2n-1}, y_{2n}).e(y_{2n}, y_{2n+1}) / e(y_{2n-1}, y_{2n}) \} \\
&\le c_2. \max\{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \text{------ (2.7.4)}
\end{aligned}$$

Similarly,

$$d(x_{2n}, x_{2n-1}) \le c_1. \max\{ d(x_{2n-2},x_{2n-1}), e(y_{2n-1},y_{2n})\} \text{ ----(2.7.5)}$$

$$e(y_{2n},y_{2n-1}) \le c_2. \max\{ e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\} \text{---2.7.6)}$$

from inequalities (2.7.3), (2.7.4), (2.7.5)and (2.7.6), we have

$$d(x_{n+1},x_n) \le c_1^{\,n}c_2^{\,n-1}. \max\{ d(x_1,x_0), e(y_1, y_2)\} \to 0 \text{ as } n\to\infty$$

Thus $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, $\{x_n\}$ converges to a point z in X. Similarly using inequalities (2.7.3), (2.7.4), (2.7.5)and (2.7.6), we prove $\{y_n\}$ is a Cauchy sequence in (Y,e) with the limit w in Y.

Suppose $\{A_n\}$ is continuous, then

$$\lim_{n\to\infty} A_nx_{2n} = A_nz = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove $S_nA_nz = z$. .

Suppose $S_nA_nz \neq z$.

We have

$$\begin{aligned}
d(S_nA_nz, z) &= \lim_{n\to\infty} d(S_nA_nz, T_nB_nx_{2n-1}) \\
&\le \lim_{n\to\infty} c_1. \max\{ d(z, x_{2n-1}), d(z, S_nA_nz), \\
&\quad e(A_nz, B_nx_{2n-1}), d(z, T_nB_nx_{2n-1}) / 2, \\
&\quad d(S_nA_nz , x_{2n-1}) / 2, \\
&\quad d(z, S_nA_nz).d(x_{2n-1},T_nB_n x_{2n-1}) / d(z, x_{2n-1})\} \\
&\le \lim_{n\to\infty} c_1. \max\{ d(z, x_{2n-1}), d(z,S_nA_nz), \\
&\quad e(A_nz, y_{2n}), d(z, x_{2n}) / 2\ d(S_nA_nz , x_{2n-1}) / 2, \\
&\quad d(z, S_nA_nz).d(x_{2n-1}, x_{2n}) / d(z, x_{2n-1})\} \\
&= c_1. \max\{ d(z,z), d(z, S_nA_nz), e(w,w), d(z,z)/2, \\
&\quad d(S_nA_nz ,z) / 2, d(z, z).d(S_nA_nz, z) / d(z, z)\} \\
&\le c_1. d(z, S_nA_nz) \\
&< d(z, S_nA_nz) \quad (\text{Since } 0 \le c_1 < 1)
\end{aligned}$$

Which is a contradiction.

Thus $S_nA_nz = z$.

Hence$S_nw = z$. (Since $A_nz = w$)

Now we prove $B_nS_nw = w$.

Suppose $B_nS_nw \neq w$.

We have

$$\begin{aligned}
e(B_nS_nw, w) &= \lim_{n\to\infty} e(B_nS_nw , y_{2n+1}) \\
&= \lim_{n\to\infty} e(B_nS_nw, A_nT_ny_{2n}) \\
&\le \lim_{n\to\infty} c_2 \max\{ e(w, y_{2n}), e(w, B_nS_nw), \\
&\quad d(S_nw, T_ny_{2n}), e(w, A_nT_ny_{2n})/ 2, e(B_nS_nw, y_{2n}) /2, \\
&\quad e(w, B_nS_nw). e(y_{2n}, A_nT_ny_{2n}) / e(w, y_{2n})\} \\
&< e(w, B_nS_nw) \quad (\text{Since } 0 \le c_2 < 1)
\end{aligned}$$

Which is a contradiction.

Thus $B_nS_nw = w$.

Hence $B_nz = w$. (Since $S_nw = z$)

Now we prove $T_nB_nz = z$.

Suppose $T_nB_nz \neq z$.

$$d(z, T_nB_nz) = \lim_{n\to\infty} d(x_{2n+1}, T_nB_nz)$$

$$= \lim_{n\to\infty} d(S_nA_nx_{2n}, T_nB_nz)$$

$$\leq \lim_{n\to\infty} c_1.\max\{ d(x_{2n},z), d(x_{2n}, S_nA_nx_{2n}), e(A_nx_{2n}, B_nz), d(x_{2n}, T_nB_nz)/2,$$

$$d(SAx_{2n}, z)/2, d(x_{2n}, S_nA_nx_{2n}).d(z, T_nB_nz)/d(x_{2n},z)\}$$

$$< d(z, T_nB_nz) \quad (\text{Since } 0 \leq c_1 < 1)$$

Which is a contradiction.

Thus $T_nB_nz = z$.

Hence $T_nw = z$. (Since $B_nz = w$)

Now we prove $A_nT_nw = w$.

Suppose $A_nT_nw \neq w$.

$$e(w, A_nT_nw) = \lim_{n\to\infty} e(y_{2n}, A_nT_nw)$$

$$= \lim_{n\to\infty} e(B_nS_ny_{2n-1}, A_nT_nw)$$

$$\leq \lim_{n\to\infty} c_2.\max\{e(y_{2n-1}, w), e(y_{2n-1}, B_nS_ny_{2n-1}),$$

$$d(S_ny_{2n-1}, T_nw), e(y_{2n-1}, A_nT_n w)/2, e(B_nS_ny_{2n-1},w)/2,$$

$$e(y_{2n-1}, B_nS_ny_{2n-1}).e(w, A_nT_nw)/e(y_{2n-1}, w)\}$$

$$< e(w, A_nT_nw) \quad (\text{Since } 0 \leq c_2 < 1)$$

Which is a contradiction.

Thus $A_nT_nw = w$.

The same results hold if one of the mappings $\{B_n\}$,$\{S_n\}$ and $\{T_n\}$ is continuous.

So the point z is the common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove w is a common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

Uniqueness: Let z' be another common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in X, w' be another common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in Y.

We have $d(z, z') = d(S_nA_nz, T_nB_nz')$

$$\leq c_1.\max\{ d(z, z'), d(z, S_nA_nz), e(A_nz,B_nz'), d(z, T_nB_nz')/2, d(S_nA_nz, z')/2, d(z, S_nA_nz).d(z',T_nB_nz')/d(z, z')\}$$

$$= c_1.\max\{ d(z, z'), d(z, z), e(w,w'), d(z, z')/2, d(z, z')/2, d(z, z).d(z',z')/d(z, z')\}$$

$$< e(w, w')$$

$$e(w, w') = e(B_nS_nw, A_nT_nw')$$

$$\leq c_2.\max\{ e(w, w'), e(w, B_nS_nw), d(S_nw,T_nw'), e(w, A_nT_nw')/2, e(B_nS_nw, w')/2, e(w, B_nS_nw).e(w',A_nT_nw')/e(w, w')\}$$

$$< d(z, z')$$

Hence $d(z, z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus $z = z'$.

So the point z is the unique common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove w is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

Remark :2.8 : If we put $A_i = A$, $B_j = B$, $S_p = S$ and $T_q = T$ in the above theorem 2.7, we get the following corollary.

Corollory 2.9: Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S , T be mappings of Y into X satisfying the inequalities.

$$d(SAx, TBx') \leq c_1.\max\{d(x,x'), d(x, SAx), e(Ax,Bx'), d(x,TBx')/2, d(SAx,x')/2, d(x, SAx).d(x',TBx')/d(x,x')\}$$

$$e(BSy, ATy') \leq c_2.\max\{ e(y,y'), e(y,BSy), d(Sy,Ty'), e(y, ATy')/2, e(BSy ,y')/2, e(y,BSy).e(y',ATy')/e(y,y')\}$$

for all x, x′ in X and y,y′ in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$.

If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

Theorem 2.10: Let (X, d) and (Y, e) be complete metric spaces. Let $\{A_n\}$, $\{B_n\}$ $(n \in \mathbb{N})$ be sequence of mappings of X into Y and $\{S_n\}$, $\{T_n\}$, $(n \in \mathbb{N})$ be sequence of mappings of Y into X satisfying the inequalities.

$$d(S_pA_ix, T_qB_jx') \leq c_1.\max\{d(x,x'), d(x, S_pA_ix), e(A_ix,B_jx'), [d(x,T_qB_jx')+ d(S_pA_ix,x')]/2, d(x, S_pA_ix).d(x',T_qB_jx')/d(x,x')\} \text{-----} \ (\mathbf{2.10}.1)$$

$$e(B_jS_py, A_iT_qy') \leq c_2.\max\{ e(y,y'), e(y,B_jS_py),$$

$d(S_py, T_qy'), [e(y, A_iT_qy') + e(B_jS_py, y')]/2$

$e(y, B_jS_py).e(y', A_iT_qy') / e(y,y')\}$----- (**2.10.**2)

for all $i \neq j \neq p \neq q$, x, x′ in X and y,y′ in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$. If one of the mappings $\{A_n\}$,$\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous , then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a common fixed point z in X and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a common fixed point w in Y. Further, $\{A_n\}z = \{B_n\}z = w$ and $\{S_n\}w = \{T_n\}w = z$.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$A_n x_{2n-2} = y_{2n-1}, S_ny_{2n-1} = x_{2n-1}, B_nx_{2n-1} = y_{2n}; T_ny_{2n} = x_{2n}$

for n = 1, 2, 3 . . . . .

Now we have

$d(x_{2n+1}, x_{2n}) = d(S_nA_nx_{2n}, T_nB_nx_{2n-1})$

$\leq c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, S_nA_nx_{2n}),$

$e(A_nx_{2n}, B_nx_{2n-1}), [d(x_{2n}, T_nB_nx_{2n-1}) + d(S_nA_nx_{2n}, x_{2n-1})]/2,$

$d(x_{2n}, S_nA_nx_{2n}). d(x_{2n-1}, T_nB_nx_{2n-1})/ d(x_{2n}, x_{2n-1})\}$

$= c_1. \max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), e(y_{2n+1}, y_{2n}),$

$[d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})] / 2,$

$d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n})/ d(x_{2n}, x_{2n-1})\}$

$= c_1. \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), e(y_{2n+1}, y_{2n}),$

$[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]/2, d(x_{2n}, x_{2n+1})\}$

$\leq c_1. \max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1}, y_{2n})\}$------- (2.10.3)

Now

$e(y_{2n}, y_{2n+1}) = e(B_nS_ny_{2n-1}, A_nT_ny_{2n})$

$\leq c_2. \max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, B_nS_ny_{2n-1}),$

$d(S_ny_{2n-1}, T_ny_{2n}),$

$[e(y_{2n-1}, A_nT_ny_{2n}) + e(B_nS_ny_{2n-1}, y_{2n})]/2,$

$e(y_{2n-1}, B_nS_ny_{2n-1}).e(y_{2n}, A_nT_ny_{2n})/ e(y_{2n-1}, y_{2n})\}$

$= c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}),$

$[e(y_{2n-1}, y_{2n+1}) + e(y_{2n}, y_{2n})] / 2,$

$e(y_{2n-1}, y_{2n}).e(y_{2n}, y_{2n+1}) / e(y_{2n-1}, y_{2n}) \}$

$\leq c_2. \max\{ e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\}$-----(2.10.4)

Similarly,

$d(x_{2n}, x_{2n-1}) \leq c_1. \max\{ d(x_{2n-2}, x_{2n-1}), e(y_{2n-1}, y_{2n})\}$ ---(2.10.5)

$e(y_{2n}, y_{2n-1}) \leq c_2.\max\{ e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\}$---(2.10.6)

from inequalities (2.10.3), (2.10.4), (2.10.5)and (2.10.6), we have

$d(x_{n+1}, x_n) \leq c_1^{\,n} c_2^{\,n-1}. \max\{ d(x_1, x_0), e(y_1, y_2)\} \to 0$ as $n \to \infty$

Thus $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, $\{x_n\}$ converges to a point z in X. Similarly using inequalities (2.10.3), (2.10.4), (2.10.5)and (2.1.6), we prove $\{y_n\}$ is a Cauchy sequence in (Y,e) with the limit w in Y.

Suppose $\{A_n\}$ is continuous, then

$$\lim_{n\to\infty} A_nx_{2n} = A_nz = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove $S_nA_nz = z$.

Suppose $S_nA_nz \neq z$.

We have

$d(S_nA_nz, z) = \lim_{n\to\infty} d(S_nA_nz, T_nB_nx_{2n-1})$

$\leq \lim_{n\to\infty} c_1. \max\{ d(z, x_{2n-1}), d(z, S_nA_nz), e(A_nz, B_nx_{2n-1}),$

$[d(z, T_nB_nx_{2n-1}) + d(S_nA_nz, x_{2n-1})]/ 2,$

$d(z, S_nA_nz).d(x_{2n-1}, T_nB_n x_{2n-1}) / d(z, x_{2n-1})\}$

$\leq \lim_{n\to\infty} c_1. \max\{d(z, x_{2n-1}), d(z, S_nA_nz), e(A_nz, y_{2n}),$

$[d(z, x_{2n}) + d(S_nA_nz, x_{2n-1})] / 2,$

$d(z, S_nA_nz).d(x_{2n-1}, x_{2n}) / d(z, x_{2n-1})\}$

$= c_1. \max\{ d(z,z), d(z, S_nA_nz), e(w,w),$

$[d(z,z) + d(S_nA_nz, z)] / 2, d(z, z).d(S_nA_nz, z) / d(z, z)\}$

$\leq c_1. d(z, S_nA_nz)$

$< d(z, S_nA_nz)$ (Since $0 \leq c_1 < 1$)

Which is a contradiction.

Thus $S_nA_nz = z$.

Hence $S_nw = z$. (Since $A_nz = w$)

Now we prove $B_nS_nw = w$.

Suppose $B_nS_nw \neq w$.

We have

$e(B_nS_nw, w) = \lim_{n\to\infty} e(B_nS_nw, y_{2n+1})$

$= \lim_{n\to\infty} e(B_nS_nw, A_nT_ny_{2n})$

$\leq \lim_{n\to\infty} c_2 \max\{ e(w, y_{2n}), e(w, B_nS_nw), d(S_nw, T_ny_{2n}),$

$[e(w, A_nT_ny_{2n}) + e(B_nS_nw, y_{2n})] /2,$

$e(w, B_nS_nw). e(y_{2n}, A_nT_ny_{2n}) / e(w, y_{2n})\}$

$< e(w, B_nS_nw)$ (Since $0 \leq c_2 < 1$)

Which is a contradiction.

Thus $B_nS_nw = w$.

Hence $B_nz = w$. (Since $S_nw = z$)

Now we prove $T_nB_nz = z$.

Suppose $T_nB_nz \neq z$.

$$d(z, T_nB_nz) = \lim_{n\to\infty} d(x_{2n+1}, T_nB_nz)$$
$$= \lim_{n\to\infty} d(S_nA_nx_{2n}, T_nB_nz)$$
$$\leq \lim_{n\to\infty} c_1. \max\{ d(x_{2n}, z), d(x_{2n}, S_nA_nx_{2n}), e(A_nx_{2n}, B_nz), [d(x_{2n}, T_nB_nz)+d(SAx_{2n}, z)] / 2, d(x_{2n}, S_nA_nx_{2n}).d(z, T_nB_nz) / d(x_{2n}, z)\}$$
$$< d(z, T_nB_nz) \quad (\text{Since } 0 \leq c_1 < 1)$$

Which is a contradiction.

Thus $T_nB_nz = z$.

Hence $T_nw = z$. (Since $B_nz = w$)

Now we prove $A_nT_nw = w$.

Suppose $A_nT_nw \neq w$.

$$e(w, A_nT_nw) = \lim_{n\to\infty} e(y_{2n}, A_nT_nw)$$
$$= \lim_{n\to\infty} e(B_nS_ny_{2n-1}, A_nT_nw)$$
$$\leq \lim_{n\to\infty} c_2. \max\{e(y_{2n-1}, w), e(y_{2n-1}, B_nS_ny_{2n-1}), d(S_ny_{2n-1}, T_nw), [e(y_{2n-1}, A_nT_n w)+ e(B_nS_ny_{2n-1}, w)] / 2, e(y_{2n-1}, B_nS_ny_{2n-1}).e(w, A_nT_nw) / e(y_{2n-1}, w)\}$$
$$< e(w, A_nT_nw) \quad (\text{Since } 0 \leq c_2 < 1)$$

Which is a contradiction.

Thus $A_nT_nw = w$.

The same results hold if one of the mappings $\{B_n\}$,$\{S_n\}$ and $\{T_n\}$ is continuous.

So the point z is the common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove w is a common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

Uniqueness: Let z′ be another common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in X, w′ be another common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in Y.

We have $d(z, z') = d(S_nA_nz, T_nB_nz')$

$$\leq c_1. \max\{ d(z, z'), d(z, S_nA_nz), e(A_nz,B_nz'), [d(z, T_nB_nz')+d (S_nA_nz, z')] / 2, d(z, S_nA_nz). d(z',T_nB_nz') / d(z, z') \}$$
$$= c_1. \max\{ d(z, z'), d(z, z), e(w,w'), [d(z, z')+ d(z, z')] / 2, d(z, z).d(z',z') / d(z, z') \}$$
$$< e(w, w')$$

$$e(w, w') = e(B_nS_nw, A_nT_nw')$$
$$\leq c_2.\max\{ e(w, w'), e(w, B_nS_nw), d(S_nw,T_nw'), [e(w, A_nT_nw')+ e(B_nS_nw, w')] /2, e(w, B_nS_nw).e(w',A_nT_nw') / e(w, w') \}$$
$$< d(z, z')$$

Hence $d(z, z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus $z = z'$.

So the point z is the unique common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$. Similarly we prove w is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$.

Remark :2.11 : If we put $A_i = A$, $B_j = B$, $S_p = S$ and $T_q = T$ in the above theorem 2.10, we get the following corollary.

Corollory 2.12: Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S , T be mappings of Y into X satisfying the inequalities.

$$d(SAx, TBx') \leq c_1. \max\{d(x,x'), d(x, SAx), e(Ax,Bx'), [d(x,TBx')+ d(SAx,x')]/2, d(x, SAx).d(x',TBx')/ d(x,x')\}$$

$$e(BSy, ATy') \leq c_2. \max\{ e(y,y'), e(y,BSy), d(Sy,Ty'), [e(y, ATy')+ e(BSy ,y')]/2, e(y,BSy).e(y',ATy') / e(y,y')\}$$

for all x, x′ in X and y,y′ in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$.

If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

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