A Few Inherent Attributes of Two Dimensional Nonlinear Map

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Abstract: Here, we consider a two dimensional nonlinear discrete map to find out a few inherent attributes i.e. fixed points, periodic points, bifurcation values of periods2^n,n =0,1,2,3,4…We use suitable numerical methods and have shown how the period doubling bifurcation points ultimately converge to the Feigenbaum constant. We have calculated Feigenbaum value also. We have further verified our findings with the help of bifurcation diagram & the Lyapunov exponent of the map. Computer software package 'Mathematica' and 'C-program' are used prudentially to implement numerical algorithms for our purpose.

Keywords: Fixed points, Periodic points, Bifurcation points, Feigenbaum constant, Accumulation Point.

1. INTRODUCTION[5]:

SCENARIOS of period doubling bifurcations have been observed in many families of dynamical systems, both dissipative and conservative, since the initial discovery by Feigenbaum, and the theory of the phenomenon has been well-studied, see for example, ([8],[10]). One of his fascinating discoveries is that if such a family represents period doubling bifurcations, then there is an infinite sequence $\{\mu_n\}$ of bifurcation values such that

$$\lim_{n\to\infty}\frac{\mu_{n+1}-\mu_n}{\mu_{n+2}-\mu_n}=\delta$$

where δ is a universal number known as the Feigenbaum constant which does not depend at all on the form of the specific family of maps. The value of δ is 4.6692016091029... in the dissipative case and 8.721097200... in the conservative case .

In this paper, consider the two dimensional discrete map$T_{pb}=(k,l):R^2\to R^2$ defined by $k(x,y)=-p+y+x^2, l(x,y)=bx\dots\dots\dots(A)$where p, b are adjustable parameters.

Here our aim is in threefold:

(I) We first disclose some useful numerical algorithm for the determination of Feigenbaum's fascinating sequence of period doubling bifurcation values of periods $2^0, 2^1, 2^2, 2^3,\dots$..in the above map.

(II) Secondly, we determine the accumulation point of the map.

(III)Thirdly, we calculate Lyapunov exponent for confirmation of the existence of the chaotic region.

2.MAIN STUDY[7,15]:

Our discussing map is $T_{pb}=(k,l):R^2\to R^2$ defined by $k(x,y)=-p+y+x^2, l(x,y)=bx\dots\dots(A)$ where p, b are adjustable parameters. TheJacobian of the function(A) is $J_1=\begin{bmatrix}2x & 1\\ b & 0\end{bmatrix}$. Therefore, the determinant$|J_1|=-b$is the amount that the map changes area.$T_{pb}$is dissipativeif$|-b|<1$,area preserving if$b=\pm1$, area expanding if$|b|>1$.In the extremely dissipative limit b=0, we recover the logistic map which follows period doubling route to chaos. If b$\neq 0$,the map invertible and its inverse is$T_{pb}^{-1}(x,y)=(yb^{-1},-y^2b^{-2}+x+p)$.Our taking map have two fixed points, say(x_1,y_1) and (x_2,y_2) whose coordinates are given by the solution is $T_{pb}=(-p+y+x^2,bx)$=(x,y) and they are found to be x=$\frac{-b+1\pm\sqrt{(b-1)^2+4p}}{2}$and y=$\frac{b(1-b)\pm\sqrt{(b-1)^2+4p}}{2}$.Thus the fixed points of the taking map are found to be $x_1 = \frac{-b+1+\sqrt{(b-1)^2+4p}}{2}$, $y_1 = \frac{b(1-b)+\sqrt{(b-1)^2+4p}}{2}$ and $x_2 = \frac{-b+1-\sqrt{(b-1)^2+4p}}{2}$,$y_2 = \frac{b(1-b)-\sqrt{(b-1)^2+4p}}{2}$. From this one finds that T_{pb} has two distinct fixed points if $p > -\frac{(b-1)^2}{4}$ and no fixed point $p < -\frac{(b-1)^2}{4}$ .These fixed points are born simultaneously in a saddle node bifurcation at $p = -\frac{(b-1)^2}{4}$ with larger onex_1 initially stable and smaller one x_2 initially unstable. Thus for fixed$b$,the quantity p can be used as a bifurcation parameter to take the system from a stable fixed point into an unstable one. In this context we also wish to point out that the stability theory is intimately connected with the jacobian matrix of our map. Ifλ_1, λ_2 are the eigen values ofJ_1then $\lambda_1 = \frac{1}{2}\left[trace(J_1) + \sqrt{trace(J_1)^2 - 4det(J_1)}\right]$ and$\lambda_2 = \frac{1}{2}\left[trace(J_1) - \sqrt{trace(J_1)^2 - 4det(J_1)}\right]$. We define trace$(J_1) = k_x + l_y$, det$(J_1) = k_x l_y - k_y l_xand\lambda_1 + \lambda_2 = trace(J_1)$, $\lambda_1\lambda_2 = det(J_1)$.From above relations we get $\lambda_1 + \lambda_2 = 2x \ldots\ldots(i)$, $\lambda_1\lambda_2 = -b \ldots\ldots(ii)$.Since$|J_1| = -b$,contraction occurs within the interval$[-1,1]$.Then for a particular value of b in the interval $[-1,1]$,the map T_{pb}depends on the real parameter p and so a fixed point $\bar{x}_0 = (x_0, y_0)$ (or periodic point $\bar{x}_0$) of this map depend on the parameter p i.e$\bar{x}_0 = \bar{x}_0(p)$.

Now for period doubling bifurcation, one of the eigen values of J_1 must be(-1). So, if we take $\lambda_2 = -1$, then the above equations (ii), (i)reduce to$\lambda_1 = b \ldots.(iii)$, $b - 1 = 2x \ldots\ldots(iv)$. Now putting be$x = \frac{-b+1+\sqrt{(b-1)^2+4p}}{2}$ in (iv) and solving for p we get $p = \frac{3}{4}(b-1)^2$. This ensures that the first period doubling bifurcation for our discussing map T_{pb} takes place at$p_1 = \frac{3}{4}(b-1)^2$ and hence becomes the first bifurcation point. Implication of this is that the fixed point $\bar{x}_0$given by (x_1, y_1)remains stable for all values of lying in the interval $I_1 = \left(-\frac{1}{4}(b-1)^2, \frac{3}{4}(b-1)^2\right)$ and a stable period trajectory of period one appears around it. This means that the two eigenvalues of the Jacobian matrixJ_1at $\bar{x}_0$ remains less than one in absolute value and as a result all the neighboring points i.e. points in the domain of attractor are attracted towards $\bar{x}_0(p)$,for values of p lying in the interval I_1 .Interestingly, for the same set of values of p and b the other fixed point (x_2, y_2) remains unstable for I_1 . Of course some negative values of b for which p lies in the region sandwiched between the boundary curves $p = -b \pm (1-b)\sqrt{-b}$ yield complex eigenvalues for the JacobianJ_1.The successive iterations of the map spiral into the fixed point is the significance of complex eigen values and the consecutive iterations approaches to the stable fixed point along the direction of the eigen vector corresponding to the higher eigen values in modulus is the real eigen values. .If we increase the value of p then stable fixed point becomes unstable and there arises around it two points, say $\bar{x}_{21}(p), \bar{x}_{22}(p)$ forming a stable periodic trajectory of period two. This transition happen at the value$p = p_1 = \frac{3}{4}(b-1)^2$which is the first bifurcation value of p.

Now shift our attention from the first iteration to the second iteration of our map which is given by${T^2}_{pb}$. The fixed points of${T^2}_{pb}$are the periodic points of period two for the map T_{pb} . Solving the equation ${T_{pb}}^2(x,y) = (-p + bx + (-p + y + x^2)^2, b(-p + y + x^2)) = (x.y)$---------$(B)$we get the points as follows

$$\left\{y = \frac{1}{2}b(-1 + b - \sqrt{-3(-1+b)^2 + 4p} \quad x = \frac{1}{2}\left(-1 + b + \sqrt{-3(-1+b)^2 + 4p}\right)\right\},$$

$$\left\{\begin{matrix} y = \frac{1}{2}b\left(-1 + b + \sqrt{-3(-1+b)^2 + 4p}\right), \\ x = \frac{1}{2}\left(-1 + b - \sqrt{-3(-1+b)^2 + 4p}\right) \end{matrix}\right\}$$

$$\left\{y = \frac{1}{2}b\left(1 - b - \sqrt{(-1+b)^2 + 4p}\right), x = \frac{1}{2}\left(1 - b - \sqrt{(-1+b)^2 + 4p}\right)\right\},$$

$$\{y = \frac{1}{2}b(1 - b + \sqrt{(-1+b)^2 + 4p}), x = \frac{1}{2}(1 - b + \sqrt{(-1+b)^2 + 4p})\}$$

Out of the four solutions the first two are interested to us and the last two are imposters that are not genuine period two; they are actually fixed points for which $T_{pb}(x^*, y^*) = (x^*, y^*)$.The Jacobian of $(B)$is given by as follows-----

$$J_2 = \begin{bmatrix} b + 4x(-p + x^2 + y) & 2(-p + x^2 + y) \\ 2bx & b \end{bmatrix}$$

If λ_1, λ_2 are eigen values of then $\lambda_1 + \lambda_2 = 2b + 4x(-p + x^2 + y)$

and $\lambda_1 \lambda_2 = b^2$. Taking $\lambda_2 = -1$,the above equation reduces to $\boldsymbol{4x^3 - 4xp + 4xy + b^2 + 2b + 1 = 0}$------(c) . Putting

$$\left\{\begin{matrix} y = \frac{1}{2}b\left(-1 + b + \sqrt{-3(-1+b)^2 + 4p}\right), \\ x = \frac{1}{2}\left(-1 + b - \sqrt{-3(-1+b)^2 + 4p}\right) \end{matrix}\right\}$$

in (c) solving for p we get $p = \frac{5-6b+5b^2}{4} = \frac{1}{4}(1+b)^2 + (1-b)^2$.This is the second bifurcation value. For our forth iteration we have seen that the enormous computational difficulty with the analytical discussion in that of the periodic points of period two. In this stage analytical discussion is practically impossible. Therefore we use following numerical techniques and for our convenient we consider $b = 0.2$.

3.NUMERICAL METHOD FOR OBTAINING PERIODIC POINTS[4,6,7]:

Although there are so many numerical algorithms to find a periodic fixed point, we have found that Newton Recurrence formula is one of the best numerical methods with negligible error for our purpose. Moreover it gives fast convergence of a periodic fixed point.

The Newton recurrence formula is $\overline{x_{n+1}} = \overline{x_n} - Df(\overline{x_n})^{-1} f(\overline{x_n})$,where $n = 0,1,2,3,4, \ldots \ldots$ and $Df(\bar{x})$ is the Jacobian of the map f is equal to $f^k - 1$in our case, where k is the appropriate period. The Newton formula actually gives the zero(s) of a map, and to apply this numerical tool in our map one needs a number of recurrence formula vector which are given below.

Let the initial point be (x_0, y_0).

Then $f(x_0, y_0) = (-p + y_0 + {x_0}^2, bx_0) = (x_1, y_1)$

$$f^2(x_0, y_0) = f\{f(x_0, y_0)\} = f(x_1, y_1) = (-p + y_1 + {x_1}^2, bx_1) = (x_2, y_2)$$

Preceeding in this way the following recurrence formula for our map can be established. $x_n = -p + y_{n-1} + {x_{n-1}}^2, y_n = bx_n \; where \; n = 1,2,3 \ldots \ldots \ldots ..$

since the Jacobian of f^k (k times of the map) is the product of the Jacobian the of each iteration of the map, we proceed as follows to describe our recurrence mechanism from the Jacobian matrix.

The Jacobian matrix J_1, for the transformation $f(x_0, y_0) = (-p + y_0 + {x_0}^2, bx_0)$ is $\begin{pmatrix} 2x_0 & 1 \\ b & 0 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ where $A_1 = 2x_0, B_1 = 1, C_1 = b, D_1 = 0$.

Next the Jacobian J_2 for the transformation $f^2(x_0, y_0) = (x_2, y_2)$ where x_2, y_2 are as mentioned above; is the product of the jacobians for the transformations $f(x_0, y_0) = (-p + y_0 + {x_0}^2, bx_0)$, $f(x_1, y_1) = (-p + y_1 + {x_1}^2, bx_1)$.Now we obtain $J_2 = \begin{pmatrix} 2x_1 & 1 \\ b & 0 \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} 2x_1 A_1 + C_1 & 1 \\ bA_1 & bB_1 \end{pmatrix} =$

$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ $where A_2 = 2x_1A_1 + C_1, B_1 = 2x_1B_1 + D_1, C_2 = bA_1, D_2 = bB_1$.

Continuing this process in this way we have the Jacobian for f^m as $J_m = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}$with a set of recursive formula as $where A_n = 2x_{n-1}A_{n-1} + C_{n-1}, B_n = 2x_{n-1}B_{n-1} + D_{n-1}, C_{n-1} = bA_{n-1}, D_2 = bB_{n-1}$ $and\ n = 2,3,4,5,6 \ldots \ldots \ldots \ldots \ldots \ldots ..$

Since the fixed point of this map f is zero of the map $F(x, y) = f(x, y) - (x, y)$ the Jacobian of F^k is given by$J_k - I = \begin{pmatrix} A_k - I & B_k \\ C_k & D_k - I \end{pmatrix}$

Its inverse is $(J_k - I)^{-1} = \frac{1}{\Delta}\begin{pmatrix} D_k - I & -B_k \\ -C_k & A_k - I \end{pmatrix}$ $where\ \Delta =$ $(A_k - I)(D_k - I) - B_kC_k$the Jacobian determinant. Therefore, Newtons method given the following recurrence formula in order to yield a periodic point of F^k

$$x_{n+1} = x_n - \frac{(D_k - I)(\overline{x_n} - x_n) - B_k(\overline{y_n} - y_n)}{\Delta}, y_{n+1} = y_n - \frac{(-C_k)(\overline{x_n} - x_n) + (A_k - I)(\overline{y_n} - y_n)}{\Delta} \quad where F^k(\overline{x_n}) = (x_n, y_n).$$

4. NUMERICAL METHODS FOR FINDING BIFURCATION VALUES[4,6]:-

As described above for some value of $r = r_1$say we calculate the fixed point of f^k and hence calculate the eigen values ofJ_k at the fixed point. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots \ldots \ldots \ldots \ldots .. (x_k, y_k)$ be the periodic points of f at r_1.Let λ_1, λ_2 be the two eigen values of $J_k$at r_1. Let $I(k, r_1) = min\{\lambda_1, \lambda_2\}, where\ n = 2^k$ is the period number. Then we search two values of "r" say r_{11}, r_{22} such that $\{I(k, r_{11}) + 1\}\{I(k, r_{22}) + 1\} < 0$ .Then the existence of k^{th}bifurcation point is confirmed in betweenr_{11}, r_{22}. Then we may apply some of the numirical techniques viz. Bisection method or RegulaFalsi method, on $r_{11}\ and\ r_{22}$ for sufficient number of iterations to get r such that $I(k, r) = -1$.Our numerical results are as follows :-

Table of Bifurcation points and one of the periodic points :

Bifurcation Point	One of the Period Points	Eigen Values
0.4800000000000000	(-0.4000000000000,-0.0800000000)	0.20000000000,-1.00000000000
1.0000000000000000	(0.321115509280,-0.2242220510586)	{-0.04000000,-1.000000000}
1.1161208089879030	(0.00542493168672,-0.199827466773)	{-0.001600,-1.000000000000028}
1.1419676460941652	(0.1263092319526,-0.21475634429294)	{-0.000003,-1.00000000000023}
1.1475284145464234	(0.55829638412655,-0.26326319018433)	{-0.000000,-1.00000000003597}
1.1487218502161435	(0.67369626732979,-0.26727845341802)	{-0.000000,-1.0000000032523}

1.14897683663 56123	(0.695929651805 2,- 0.2699168832476 3)	{-0.0000000,- 1.000000000 08036}

The Feigenbaum universal constant is calculated using the experimentally calculated bifurcation point using the following formula$\delta_n = \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$ where μ_n represents nth bifurcation point. The values of δ_n are as follows :-

$$\delta_1 = 4.474731497104102383962$$

$$\delta_2 = 4.646974732306157737039$$

$$\delta_3 = 4.653744874477371956324$$

$$\delta_4 = 4.659462251227029843682$$

$$\delta_5 = 4.669210164328754322145$$

It may be observed that the map obeys Feigenbaum universal behaviour as the sequence $\{\delta_n\}$ converges to δ as n becomes very large.

5. ACCUMULATION POINT [4,6] :

Since our model follows a period doubling bifurcation, therefore we can consider that $\{\mu_n\}$ be the sequence of bifurcation points. With the help of Feigenbaum delta(δ), if we know first(μ_1) and second (μ_2)bifurcation points, then we get $\mu_3 \approx \frac{\mu_2 - \mu_1}{\delta} + \mu_2 \ldots \ldots . . (\text{i})$. Similarly we get $\mu_4 \approx \frac{\mu_3 - \mu_2}{\delta} + \mu_3 \ldots \ldots . . (\text{ii})$. From (i), (ii) we get$\mu_4 \approx (\mu_2 - \mu_1)\left(\frac{1}{\delta} + \frac{1}{\delta^2}\right) + \mu_2$. If we go on this procedure to calculate μ_5 , μ_6 and so on, we just obtain more terms in the sum involving powers of $\left(\frac{1}{\delta}\right)$ .We acknowledge this sum as a geometric series and after simplification we obtain the result. [10]

$\mu_\infty \approx \frac{\mu_{n+1} - \mu_n}{\delta - 1} + \mu_{n+1} \ldots \ldots . . (\text{iii})$ .The expression (iii) is exact when the bifurcation ratio $\delta_n = \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}$ is equal $\forall n$ and then$\lim_{n\to\infty} \delta_n = \delta$. Hence $\{\mu_{\infty,n}\}$ is the sequence and $\lim_{n\to\infty} \mu_{\infty,n} = \mu_\infty$ .Using the experimental bifurcation points the sequence of accumulation points $\{\mu_{\infty,n}\}$ are calculated for some values of n and the points are mentioned under in this regard - ---------

$$\mu_{\infty,1} = 1.149065185092240321654 6 5231$$

$$\mu_{\infty,2} = 1.149066111979130964559 77163$$

$$\mu_{\infty,3} = 1.149041713384609224977 12442$$

$$\mu_{\infty,4} = 1.149047973451638156732 31453$$

$$\mu_{\infty,5} = 1.149048012564934200632134756321$$

The above sequence converges to the value 1.1490…… , which is the required accumulation point.

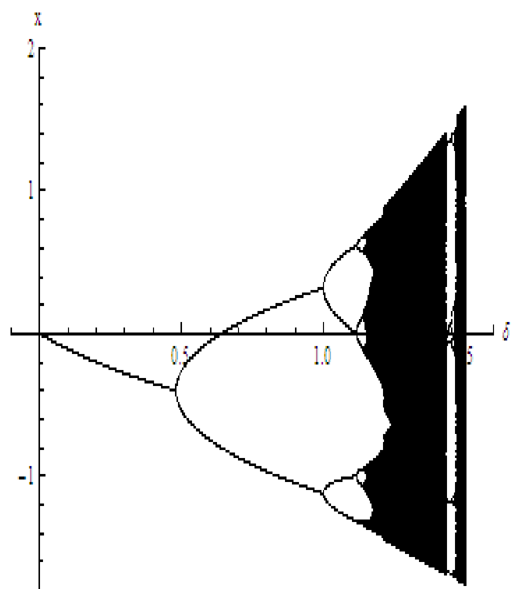

Fig: 1 Bifurcation diagram of the model

6. LYAPUNOV EXPONENT[11]:

The Lyapunov exponent is an experimental device. It has ability to separate unstable, chaotic behavior from that which is stable and predictable. Lyapunov exponent quantifies the exponentially divergence of two trajectories starting very close to each other. This exponent has two types…….

(i) The first one is positive Lyapunov exponent. It indicates the exponential divergence of the trajectory which confirms chaos.
(ii) The second is negativeLyapunov exponent. This is associated with regular behavior (periodic orbit).

Here consider the map of the form $x_{i+1}^i = f^i(x_i^1, x_i^2, x_i^3, \ldots \ldots \ldots . x_i^n)$

$i = 1,2, \dots \dots \dots n$ .... (iv)

and that the Jacobian matrix is given by $j_{ik}(x_i) = \frac{\partial f^i}{\partial x^k}$ .... (v)

To compute the rthLyapunov exponent , consider the component of an $\binom{n}{r}$ dimwnsional vector v_{t+1}^i given by the recursive relation

$$v_{t+1}^i = \sum_{k=1}^{\binom{n}{r}} d_{ik}^{(r)}(j) v_t^k \dots\dots \dots \dots\dots (vi)$$

Here $d^{(r)}(j)$ is a $\binom{n}{r} x \binom{n}{r}$ matrix constructed from the Jacobian matrix by crossing out $(n-r)$ columns in all possible ways. For each combination , the determinant of the remaining elements of j is a matrix element of $d^{(r)}$. The technical name of this matrix is the r- compound matrix of j, or the rth exterior power of j. For our purpose the matrices $d^{(k)}$ may be transformed by letting any row change sign as long as the sign is also changed for the column with same index . Also, any two rows may be interchanged provided also the two columns with the same indices are interchanged.

In particular we notice that $d^{(1)} = j$, and $d^{(n)} = Det\, j$. Let $v_0^k = 1$ for all k . In all but a few special cases the answer will be the same if only one of the components v_0 is different from zero. With these initial conditions, the vector v_t is computed along with the trajectory up to some problem dependent, long time T. The sum of the r biggest Lyapunov exponents is then given by

$$\sum_{i=1}^{r} \lambda_i = \lim_{T \to \propto} \ln\left(max\left|v_T^k\right|\right) / T \quad \dots\dots (vii)$$

where max $\left|v_T^k\right|$ is the component of v_T with the biggest absolute value. Thus, using $r = 1$, i.e,

$d^{(r)}$=j, one finds the biggest Lyapunov exponent. For two dimensional maps $d^{(2)} = Det$ j, and using equation (vi) *and* (vii) one can find $\lambda_1 + \lambda_2$. Here

Lyapunovexponent is calculated, to verify how much accurate are the accumulation points.

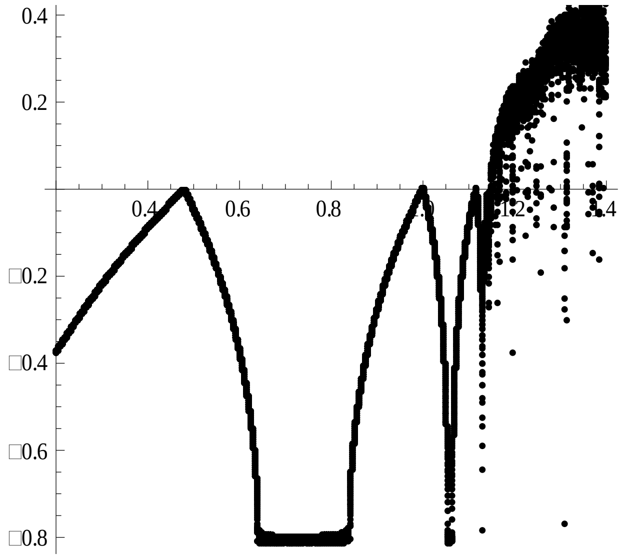

Fig: 2 Graph of Lyapunov exponent obtained by iterating 1,00,000 points at every control parameter

From the graph of Lyapunov experiment, we see that some portions lie in the negative side of the parameter axis indicating regular behavior (periodic orbit) and the portions lie on the positive side of parameter axis confirm us about the assistance of chaos for the model.

Lyapunov exponent near the accumulation point:

Control parameter	Lyapunov exponent value	No of iterations
1.14	- 0.00741221	1,00,000
1.14904	- 0.00353751	1,00,000
1.149048	- 0.00240377	1,00,000
1.14904801	- 0.00372758	1,00,000
1.149048012	- 0.00318032	1,00,000
1.1490480128	- 0.00293009	1,00,000

The above table says that chaos starts very near to the parameter value $b = 1.1490 \dots \dots \dots$.

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