Existence of Fixed Points in a Complete Probabilistic Metric Space

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Abstract: In this paper, we prove a fixed point theorem for a self-map on a Menger space and we generalize the theorem of Sastry and Rao [11] for a sequence of self maps on a complete probabilistic metric space.

Keywords: Probabilistic metric space, fixed point theorem.

INTRODUCTION: The concept of a probabilistic metric space was initiated and studied by Menger [9] which is a generalization of the metric space notion. The theory of a probabilistic metric space is an active field and has application in many other branches of the mathematics. It has also been applied to quantum particle physics in connection with both string and ε^∞ theory.

Menger [9], who played an important role in the development of theory of metric spaces, suggested that instead of associating a single non-negative number d(u,v) shows that the distance between two points u, v and a distribution function should associate with a pair of points. Thus for any points u, v in the spaces and for any x > 0, we have a distribution function $F_{u,v}(x)$ and interpret $F_{u,v}(x)$ as the probability that the distance between u and v is less than x.

Fixed Point theory in a probabilistic metric space is an important branch of probabilistic analysis, which is closely related to the existence and uniqueness of solutions of differential equations and integral equations. Many results on the existence of fixed points or solution of nonlinear equations under the various types of conditions in the Menger [9] spaces have been extensively studied by many scholars.

Definition 1.1 : A probabilistic metric space (in brief PM-space) is an ordered pair (X,F), X is a non-empty set and F is a (mapping) function from X×X into Z, the collection of all distribution function such that if p, q and r are points of X then

1. $F_{u,v}(0) = 0$ ($F_{u,v}$ denotes F(p,q);

2. $F_{u,v}(x) = H(x)$ if and only if u = v;

3. $F_{u,v} = F_{v,u}$;

4. If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{w,u}(x + y) = 1$;

For each u and v in X and each real number x, $F_{u,v}(x)$ is to be thought of as the probability that the distance from u to v is less than x.

Remark 1.1: Every metric space may be regarded as a PM-space indeed, if (X,d) is a metric space, then $F_{u,v}(x) =$ H(x-d(x,d)), for all u,v∈X and H is a distribution function defined by

$$H(X) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases} \ .$$

It is not difficult to prove that we have in fact a PM-space.

Condition 4 is called the triangle inequality for the PM-space.

Definition 1.2: A real-valued function defined on the set of real numbers is a distribution function if it is non-decreasing, left continuous and inf f(x) =0, sup f(x) = 1.

Definition 1.3: A mapping T: $[0,1] \times [0,1] \rightarrow [0,1]$ is called a T-norm or t-norm or Δ - norm or triangular norm if it satisfies the following conditions:

T_1. T(0,0) = 0, T(a,1) = a;

T_2. T(c,d) ≥ T(a,b); for c ≥ a, d ≥ b;

T_3. T(a,b) = T (b,a);

T_4. T(a,1) > 0, when a > 0 and T(1,1) =1;

T_5. T(a,T(b,c)) = T(T(a,b),c),for all a,b,c and d in [0,1];

Examples of t-norms are plenty. t(a,b) = a.b and t(a,b) = Min {a,b} for all a,b ∈ [0,1] are two examples of t-norms.

In this "t = min" means, t is the t-norm defined by t(a,b) = min{a,b}.

Definition 1.4: A Menger PM-space is triplet (X,F,T) where X is a set, F is a function defined on the set X×X with values in the set of distribution function such that

1. $F_{u,v}(0) = 0$ ($F_{u,v}$ denotes F(p,q));

2. $F_{u,v}(x) = H(x)$ if and only if u = v;

3. $F_{u,v} = F_{v,u}$;

4. $F_{u,v}(x + y) \ge T\{Fu,w(x), Fw,v(y)\}$; it satisfied for all u,v,w∈X

and x, y ≥ 0;

where T is a two place function on the unit square.

Note. 1.1. A complete metric space (X,d) can be treated as a complete Menger space.

Note. 1.2. A Menger space can be treated as a generalization of metric space.

Definition 1.5. Let (X, F) be a PM-Space and ε, λ are positive real, then (ε, λ) neighbourhood of a point u∈X is denoted by

$N_u(\varepsilon, \lambda)$ is defined as:

$$N_u(\varepsilon, \lambda) = \left\{ v \in X: F_{u,v}(\varepsilon) > 1 - \lambda \right\}, \varepsilon, \lambda > 0 .$$

Definition 1.6: A sequence $\{u_n\}$ in a PM-space (X, F) is said to be converges to u if for every ε, $\lambda > 0$ there exists an integer $N = N(\varepsilon, \lambda)$ Such that $F_{u_n,u}(\varepsilon) > 1 - \lambda$ for all n≥ N.

Definition 1.7: A sequence of points $\{u_n\}$ in a PM-space is Cauchy sequence, if for any ε, $\lambda > 0$ there exists an integer N=N(ε, λ) such that for all n, m ≥ N (ε, λ)

$$F_{u_n,u_m}(\varepsilon) > 1 - \lambda \text{ for all } n \geq N.$$

A PM-space is complete if every Cauchy sequence in it converges to some point of the space.

Achari [1] studied the fixed points of quasi-contraction type mapping in non-Archimedean PM-space and generalized the result of Istrătescu[8].

Recently, Singh and Pant [13] have established common fixed point theorems for weakly commuting quasi-contraction pair of mappings on non-Archimedean Menger space.

Hadzic [6] extended the Bocsan [2] fixed point theorems in random normed spaces. Some of the remarkable results for the contraction mapping to PM-space have been established by Chang and Kim [3], Ciric [5], Istrătescu[8] and Sacwie [6] and Singh and Pant [12].

Ranganathan [10] introduced the concept of diminishing orbital diameter sum for a pair of commuting mappings and obtained some fixed points theorems for such mappings.

Sastry and Srinivasa Rao [11] proved a fixed point theorem for a self map on Menger space. They extended it to a pair of self maps and obtained a fixed point theorem similar to a theorem of Choudhury and Sarkar [4] for a sequence of self maps on a complete Menger space.

Sastry and Rao [11] proved the following theorems:

Theorem 1.1: Let (M, F, t=min) be a complete Menger space.

Let T be a self map of M such that the following hold: ∃ a ∈ (0, 1) such that for any x, y ∈ M, s ≥ 0 and n = 1, 2, ...

$$\left(F_{Tx,Ty}(S) + F_{x,Tx}(S)\right)\left(F_{Tx,Ty}(S) + F_{y,Ty}(S)\right) \ge 2\left(F_{x,Tx}\left(\frac{S}{a}\right).F_{y,Ty}(S) + F_{x,Tx}(s).F_{y,Ty}\left(\frac{S}{a}\right)\right)$$

$$F_{x_m y_m} \to F_{xy} \text{ as } x_m \to x \text{ and } y_n \to y \text{ in M and } F_{x,y}(s) > 0, \forall s > 0 \text{ and } x, y \in M.$$

Then T has a unique fixed point in M.

Theorem 1.2: Let (M, F, t=min) be a complete Menger space.

Let S and T be two self maps of M such that the following hold: $\exists$ a, $0 < a < 1$ such that for any x, y $\in$ M, s > 0 and n = 1, 2, ...

$$\left(F_{Sx,Ty}(S) + F_{x,Sx}(S)\right)\left(F_{Sx,Ty}(S) + F_{y,Ty}(S)\right) \ge 2\left(F_{x,Sx}\left(\frac{S}{a}\right).F_{y,Ty}(S) + F_{x,Sx}(s).F_{y,Ty}\left(\frac{S}{a}\right)\right)$$

$$F_{x_m y_m} \to F_{xy} \text{ as } x_m \to x \text{ and } y_n \to y \text{ in M and } F_{x,y}(s) > 0, \forall t > 0 \text{ and } x, y \in M.$$

Then S and T have the same fixed point set which is a singleton.

Theorem 1.3: Let (M, F, t = min) be a complete Menger space and $\{T_n\}_{n=0}^{\infty}$ be a sequence of self maps on M such that the following hold, $\exists$ a, $0 < a < 1$ such that for any x, y $\in$ M, s > 0 and

n = 1, 2...

$$\left(F_{T_0x,T_0y}(S) + F_{x,T_0x}(S)\right)\left(F_{T_0x,T_ny}(S) + F_{y,T_ny}(S)\right) \ge 2\left(F_{x,T_0x}\left(\frac{S}{a}\right).F_{y,T_ny}(S) + F_{x,T_0x}(s).F_{y,T_ny}\left(\frac{S}{a}\right)\right)$$

$$F_{x_m y_m} \to F_{xy} \text{ as } x_m \to x \text{ and } y_n \to y \text{ in M and } F_{x,y}(s) > 0, \forall t > 0 \text{ and } x, y \in M.$$

then the sequence $\{T_n\}_{n=0}^{\infty}$ has a unique common fixed point.

Theorem 1.4: Let (M,F,t=Min) be a complete Menger space and $\{T_n\}$ be a self map on M satisfying

$$F_{x_m y_n} \to F_{xy} \text{ as } x_m \to x \text{ and } y_n \to y \text{ in M and } F_{x,y}(S) > 0, \text{ for all } S > 0 \text{ and } x, y \in M, \text{ and there exist}$$

$0 < a < 1$ such that for any x, y $\in$ M, S_1, $S_2 > 0$ and n = 1, 2,...

$$\left(F_{T_0x,T_ny}(S_1) + F_{x,T_0x}(S_2)\right)\left(F_{T_0x,T_ny}(S_1) + F_{y,T_ny}(S_2)\right) \ge 4\left(F_{x,T_0x}\left(\frac{S_1}{a}\right).F_{y,T_ny}(S_2)\right)$$

Then {Tn}, n = 0, 1, ... has a unique common fixed point.

Main Result:

Theorem: Let (X, F) be a complete probabilistic metric space and S be a self map of X such that the following holds and there exist a, b>1 such that for any x, y $\in$ X and t $\geq$ 0

$$F_{S_x,S_y}(t) \geq \frac{F_{x,S_x}\left(\frac{t}{a}\right)\left[1 + F_{y,S_y}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{x,y}\left(\frac{t}{a}\right)\right]^2} + F_{x,y}\left(\frac{t}{b}\right) \tag{1}$$

$$\text{and } F_{x,y} \geq 0, \forall t \geq 0 \text{ and } x, y \in X. \tag{2}$$

Then S has a unique fixed point.

Proof: Let $x_0 \in X$ and define a sequence by $x_n + 1 = Sx_n$ for all n = 0, 1, 2, ...

Now taking t > 0 we get by equation (1) for x = x_2 and y = x_1

$$F_{x_2,x_1}(t) = F_{Sx_1,Sx_0}(t)$$

$$\geq \frac{F_{x_1,Sx_1}\left(\frac{t}{a}\right)\left[1 + F_{x_0,Sx_0}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{x_1,x_0}\left(\frac{t}{a}\right)\right]^2} + F_{x_1,x_0}\left(\frac{t}{b}\right)$$

$$\geq \frac{F_{x_1,x_2}\left(\frac{t}{a}\right)\left[1 + F_{x_0,x_1}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{x_1,x_0}\left(\frac{t}{a}\right)\right]^2} + F_{x_1,x_0}\left(\frac{t}{b}\right)$$

$$\geq F_{x_1,x_2}\left(\frac{t}{a}\right) + F_{x_1,x_0}\left(\frac{t}{b}\right)$$

$$F_{x_2,x_1}(t) - F_{x_1,x_2}\left(\frac{t}{a}\right) \geq F_{x_1,x_0}\left(\frac{t}{b}\right)$$

$$F_{x_2,x_1}\left(\frac{a-1}{a}\right)t \geq F_{x_1,x_0}\left(\frac{t}{b}\right)$$

$$F_{x_2,x_1}(t) \geq F_{x_1,x_0}\left(\frac{a}{(a-1)b}\right)t$$

$$F_{x_2,x_1}(t) \geq F_{x_1,x_0}\left(\frac{t}{k}\right)$$

$$\text{where } k = \left(\frac{a-1}{a}\right)b$$

Now by taking x = x_3 and y = x_2

$$F_{x_3,x_2}(t) = F_{Sx_2,Sx_1}(t)$$

$$\geq \frac{F_{x_2,Sx_2}\left(\frac{t}{a}\right)\left[1 + F_{x_1,Sx_1}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{x_2,x_1}\left(\frac{t}{a}\right)\right]^2} + F_{x_2,x_1}\left(\frac{t}{b}\right)$$

$$\geq \frac{F_{x_2,x_3}\left(\frac{t}{a}\right)\left[1 + F_{x_1,x_2}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{x_2,x_1}\left(\frac{t}{a}\right)\right]^2} + F_{x_2,x_1}\left(\frac{t}{b}\right)$$

$$\geq F_{x_2,x_3}\left(\frac{t}{a}\right) + F_{x_2,x_1}\left(\frac{t}{b}\right)$$

$$F_{x_3,x_2}(t) - F_{x_2,x_3}\left(\frac{t}{a}\right) \geq F_{x_2,x_1}\left(\frac{t}{b}\right)$$

$$F_{x_3,x_2}\left(\frac{a-1}{a}\right)t \geq F_{x_2,x_1}\left(\frac{t}{b}\right)$$

$$F_{x_3,x_2}(t) \geq F_{x_2,x_1}\left(\frac{a}{(a-1)b}\right)t$$

$$F_{x_3,x_2}(t) \geq F_{x_2,x_1}\left(\frac{t}{k}\right)$$

$$\text{where } k = \left(\frac{a-1}{a}\right)b$$

and also $\quad F_{x_3,x_2}(t) \geq F_{x_2,x_1}(t) \cdot \frac{1}{k}$

$$F_{x_3,x_2}(t) \geq F_{x_1,x_0}\left(\frac{t}{k}\right) \cdot \frac{1}{k}$$

$$F_{x_3,x_2}(t) \geq F_{x_1,x_0}\left(\frac{t}{k^2}\right)$$

Now by taking $x = x_{n+1}$ and $y = x_n$ in equation (1)

$$F_{x_{n+1},x_n}(t) = F_{Sx_n,Sx_{n-1}}(t)$$

$$\geq \frac{F_{x_n,Sx_n}\left(\frac{t}{a}\right)\left[1 + F_{x_{n-1},Sx_{n-1}}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{x_n,x_{n-1}}\left(\frac{t}{a}\right)\right]^2} + F_{x_n,x_{n-1}}\left(\frac{t}{b}\right)$$

$$\geq \frac{F_{x_n,x_{n+1}}\left(\frac{t}{a}\right)\left[1 + F_{x_{n-1},x_n}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{x_n,x_{n-1}}\left(\frac{t}{a}\right)\right]^2} + F_{x_n,x_{n-1}}\left(\frac{t}{b}\right)$$

$$\geq F_{x_n,x_{n+1}}\left(\frac{t}{a}\right) + F_{x_n,x_{n-1}}\left(\frac{t}{b}\right)$$

$$F_{x_{n+1},x_n}(t) - F_{x_{n+1},x_n}\left(\frac{t}{a}\right) \geq F_{x_n,x_{n-1}}\left(\frac{t}{b}\right)$$

$$F_{x_{n+1},x_n}\left(\frac{a-1}{a}\right)t \geq F_{x_n,x_{n-1}}\left(\frac{t}{b}\right)$$

$$F_{x_{n+1},x_n}(t) \geq F_{x_n,x_{n-1}}\left(\frac{a}{(a-1)b}\right)t$$

$$F_{x_{n+1},x_n}(t) \geq F_{x_n,x_{n-1}}\left(\frac{t}{k}\right)$$

where $k = \left(\frac{a-1}{a}\right)b$. and also

$$F_{x_{n+1},x_n}(t) \geq F_{x_n,x_{n-1}}(t) \cdot \frac{1}{k}$$

$$\geq F_{x_{n-1},x_{n-2}}\left(\frac{t}{k}\right) \cdot \frac{1}{k}$$

$$\geq F_{x_{n-1},x_{n-2}}\left(\frac{t}{k^2}\right)$$

$$\geq F_{x_1,x_0}\left(\frac{t}{k^n}\right). \tag{6}$$

Now by taking $x = x_n$ and $y = x_m$ for n>m and by using equation (1),

$$F_{x_n,x_m}(t) = F_{Sx_{n-1},Sx_{m-1}}(t)$$

$$= F_{Sx_{n-1},Sx_{n-2}}(t).F_{Sx_{n-2},Sx_{n-3}}(t).F_{Sx_{n-3},Sx_{n-4}}(t)\ldots\ldots..F_{Sx_{m-2},Sx_{m-1}}(t)$$

$$> F_{x_1,x_0}\left(\frac{t}{k^{n-2}}\right).F_{x_1,x_0}\left(\frac{t}{k^{n-3}}\right).F_{x_1,x_0}\left(\frac{t}{k^{n-4}}\right)\ldots\ldots\ldots\ldots.F_{x_1,x_0}\left(\frac{t}{k^{m-1}}\right)$$

$$> (1-\lambda)$$

$$F_{x_n,x_m}(t) > (1-\lambda) \tag{7}$$

Hence, $\{x_n\}$ converges to a point, say z. Now taking $x = z$ and $y = x_n$ in equation (1)

$$F_{Sz,Sx_n}(t) \geq \frac{F_{z,Sz}\left(\frac{t}{a}\right)\left[1 + F_{x_n,Sx_n}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{x_n,x_n}\left(\frac{t}{a}\right)\right]^2} + F_{z,x_n}\left(\frac{t}{b}\right)$$

$$\geq \frac{F_{z,Sz}\left(\frac{t}{a}\right)\left[1 + F_{x_n,x_{n+1}}\left(\frac{t}{a}\right)\right]^2}{\left[1 + F_{z,x_n}\left(\frac{t}{a}\right)\right]^2} + F_{z,x_n}\left(\frac{t}{b}\right)$$

$$\geq \frac{F_{z,Sz}\left(\frac{t}{a}\right)\left[1+F_{z,z}\left(\frac{t}{a}\right)\right]^2}{\left[1+F_{z,z}\left(\frac{t}{a}\right)\right]^2} + F_{z,z}\left(\frac{t}{b}\right) \qquad \text{as } x_n \to z$$

$$\geq F_{z,Sz}\left(\frac{t}{a}\right) + F_{z,z}\left(\frac{t}{b}\right)$$

$$F_{z,Sz}(t) \geq F_{z,Sz}\left(\frac{t}{a}\right) + F_{z,z}\left(\frac{t}{b}\right)$$

$$F_{z,Sz}(t) - F_{Sz,z}\left(\frac{t}{a}\right) \geq F_{z,z}\left(\frac{t}{b}\right)$$

$$F_{Sz,z}\left(\frac{a-1}{a}\right)t \geq F_{z,z}\left(\frac{t}{b}\right)$$

$$F_{Sz,z}(t) \geq F_{z,z}\left(\frac{a}{(a-1)b}\right)t$$

$$F_{Sz,z}(t) \geq F_{z,z}\left(\frac{t}{k}\right)$$

$$F_{Sz,z}(t) \geq 1$$

$$Sz = z$$

Clearly, z is a fixed point of S.

Corollary 1. Let (X, F) be a complete probabilistic metric space. Let S and T be two self-maps which is compatible of X such that the following holds and there exists a, b>1 such that for any x, y ∈ X and t ≥ 0

$$F_{Sx,Ty} \geq \frac{F_{x,Sx}\left(\frac{t}{a}\right)\left[1+F_{y,Ty}\left(\frac{t}{a}\right)\right]^2}{\left[1+F_{x,y}\left(\frac{t}{a}\right)\right]^2} + F_{x,y}\left(\frac{t}{b}\right)$$

and $F_{x,y}(t) \geq 0$, $\forall t \geq 0$ and $x, y \in X$.

then S and T have a Common Fixed Point.

Corollary 2. Let (X, F, Δ) be a complete Menger PM-space. Let S and T be two self-maps which is compatible of X such that the following holds and there exists a, b>1 such that for any x, y ∈ X and t ≥ 0

$$F_{Sx,Ty} \geq \frac{F_{x,Sx}\left(\frac{t}{a}\right)\left[1+F_{y,Ty}\left(\frac{t}{a}\right)\right]^2}{\left[1+F_{x,y}\left(\frac{t}{a}\right)\right]^2} + F_{x,y}\left(\frac{t}{b}\right)$$

and $F_{x,y}(t) \geq 0$, $\forall t \geq 0$ and $x, y \in X$.

then S and T have a Common Fixed Point.

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