Common Unique Fixed Point Theorems for Compatible mappings of Type (A) in Complete Metric Space

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Abstract **-The object of this paper is to obtain common unique fixed-point theorems for three and four compatible mappings of type (A) in complete metric space using rational inequality .Our result generalizes the results of Jungck [4, 5], Aage and Salunke [2], Shukla, Tiwari and Shukla [3].**

Keywords- Complete metric Space, compatible mappings of types (A), common fixed point.

Mathematics Subject Classification- 54H25.

I. INTRODUCTION

In 1976, Jungck [4], introduced the concept of commuting mapping and proved a common fixed point theorem for commuting mappings. In 1986, Sessa [6], introduced the concept of weak commutativity, which is weaker than commutativity and proved a common fixed point theorem for weakly commuting maps. In 1993, Jungck, Murthy and Cho [5] introduced the concept of compatible of type (A) which is weaker than weakly commutativity and proved some common fixed point theorems in complete metric space. Recently, some common fixed point theorems of three and four compatible mappings of type (A) were proved by Aage and Salunke [2], Shukla, Tiwari and Shukla [3].

In this paper we have proved unique common fixed point theorems for three and four self maps satisfying a new rational inequality by using the concept of compatible of type (A) mappings in a complete metric space. Also we generalize the results Jungck [4, 5] Aage and Salunke [2], Shukla, Tiwari and Shukla [3] by using concept of compatible mappings of type (A) satisfy another type of rational inequality. To illustrate our main theorems, an example is also given.

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II. PRELIMINARIES

Definition 2.1 : Two mapping S and T from a metric space (X, d) into itself, are called commuting on X, if $d(STx, TSx) = 0$ i.e. $STx = TSx$ for all x in X.

Definition 2.2 : Two mapping S and T from a metric space (X, d) into itself, are called weakly commuting on X, if $d(STx, TSx) \leq d(Sx, Tx)$ for all x in X. Clearly, commuting mappings are weakly commuting, but converse is not necessarily true, given by following example :

Example2.1[1]

Let $X = [0, 1]$ with the Euclidean metric d. Define S and $T : X \to X$ by

$Sx = \frac{x}{3-x}$ and $Tx = \frac{x}{3}$ for all x in X.

Then for any x in X,

$$d(STx, TSx) = \left|\frac{x}{9-x} - \frac{x}{9-3x}\right|$$

$$= \left|\frac{-2x^2}{(9-x)(9-3x)}\right|$$

$$\leq \frac{x^2}{9-3x}$$

$$= \left|\frac{x}{3-x} - \frac{x}{3}\right|$$

$$= d(Sx, Tx)$$

i.e. $d(STx, TSx) \leq d(Sx, Tx)$ for all x in X.

Thus S and T are weakly commuting mappings on X, but they are not commuting on X, because

$$STx = \frac{x}{9-x} < \frac{x}{9-3x} = TSx \quad \text{for any } x \neq 0 \text{ in } X.$$

i.e. $STx < TSx$ for any $x \neq 0$ in X.

Definition 2.3. If Two mapping S and T from a metric space (X, d) into itself, are called compatible mappings of type (A) on X, if $\lim_{m\to\infty} d(STx_m, TTx_m) = 0$ and $\lim_{m\to\infty} d(TSx_m, SSx_m) = 0$ when $\{x_m\}$ is a sequence in X such that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some x in X.

Clearly Two mapping S and T from a metric space (X, d) into itself, are called compatible mappings of type (A) on X, then $d(STx, TTx) = 0$ and $d(TSx, SSx) = 0$ when $d(Sx, Tx) = 0$ for some x in X. Note that weakly commuting mappings are compatible of type (A), but the converse is not necessarily true.

Example2.2[1]

Let $X = [0, 1]$ with the Euclidean metric d. Define S and $T : X \to X$ by

$Sx = x$ and $Tx = \frac{x}{x+1}$ for all x in X.

Then for any x in X,

$$STx = S(Tx) = S\left(\frac{x}{x+1}\right) = \frac{x}{x+1}$$

$$TSx = T(Sx) = T(x) = \frac{x}{x+1}$$

$$d(Sx, Tx) = \left|x - \frac{x}{x+1}\right| = \left|\frac{x^2}{x+1}\right|$$

Thus we have

$$d\left(STx, TSx\right) = \left|\frac{x}{x+1} - \frac{x}{x+1}\right|$$

$$= 0 \le \frac{x^2}{x+1} \text{ for all } x \text{ in } X.$$

$$= d\left(Sx, Tx\right)$$

i.e. $d\left(STx, TSx\right) \le d\left(Sx, Tx\right)$ for all x in X.

Thus S and T are weakly commuting mappings on X, and then obviously S and T are compatible of type (A) mappings on X .

Example2.3

Let $X = R$ with the Euclidean metric d. Define S and $T : X \to X$ by $Sx = x^3$ and $Tx = 2x^2$ for all x in X.

Then for any x in X ,

$$STx = S\left(Tx\right) = S\left(2x^2\right) = \left(2x^2\right)^3 = 8x^6$$

$$TSx = T\left(Sx\right) = T\left(x^3\right) = 2\left(x^3\right)^2 = 2x^6$$

$$SSx = S\left(Sx\right) = S\left(x^3\right) = \left(x^3\right)^3 = x^9$$

$$TTx = T\left(Tx\right) = T\left(2x^2\right) = 2\left(2x^2\right)^2 = 8x^4$$

Then S and T are compatible of type (A) mappings on X , because

$$d\left(Sx, Tx\right) = \left|x^3 - 2x^2\right| \to 0 \text{ as } x \to 0 .$$

Then

$$d\left(STx, TTx\right) = \left|8x^6 - 8x^4\right| = 8\left|x^6 - x^4\right| \to 0 \ \ as\ x \to 0$$

$$d\left(TSx, SSx\right) = \left|2x^6 - x^9\right| \to 0 \text{ as } x \to 0$$

But $d\left(STx, TSx\right) \le d\left(Sx, Tx\right)$ is not true for all x in X. Thus S and T are not weakly commuting mappings on X.

Hence all weakly commuting mappings are compatible of type (A), but converse is not true.

III. MAIN RESULTS

Theorem 3.1 Let P, S and T be three mappings from a complete metric space (X, d) into itself satisfying the conditions:

$$S(X) \cup T\left(X\right) \subseteq P(X) \tag{3.1}$$

$$d(Sx, Ty) \le \left\{\alpha + \beta \frac{d(Sx, Px)}{1 + d(Px, Py)}\right\} d(Ty, Py) \tag{3.2}$$

for all $x, y \in X$, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$ with $\alpha < 1$ .Suppose that

(i) One of P, S and T is continuous,

(ii) The pairs (S, P) and (T, P) are compatible of type (A) on X.

Then P, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X, by (3.1) we choose a point x_1 in X such that $Px_1 = Sx_0$ and for this point x_1, there exists a point x_2 in X such that $Px_2 = Tx_1$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$y_{2m+1} = Px_{2m+1} = Sx_{2m}$ and $y_{2m} = Px_{2m} = Tx_{2m-1}$ (3.3)

Then we show that the sequence $\{y_m\}$ defined by (3.3) is a Cauchy sequence in X.

By definition (3.3) we have

$$d(y_{2m+1}, y_{2m}) = d(Sx_{2m}, Tx_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(Sx_{2m}, Px_{2m})}{1 + d(Px_{2m}, Px_{2m-1})}\right\} d(Tx_{2m-1}, Px_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(y_{2m+1}, y_{2m})}{1 + d(y_{2m}, y_{2m-1})}\right\} d(y_{2m}, y_{2m-1})$$

$$\leq \alpha d(y_{2m}, y_{2m-1}) + \beta d(y_{2m+1}, y_{2m})$$

i.e $d(y_{2m+1}, y_{2m}) \leq \dfrac{\alpha}{1-\beta} d(y_{2m}, y_{2m-1})$

Hence $d(y_{2m+1}, y_{2m}) \leq h\, d(y_{2m}, y_{2m-1})$

Where $h = \dfrac{\alpha}{1-\beta} < 1$

Similarly we can show that

$$d(y_{2m+1}, y_{2m}) \leq h^{2m} d(y_1, y_0)$$

For k > m , we have

$$d(y_{m+k}, y_m) \leq \sum_{i=1}^{k} d(y_{n+i}, y_{n+i-1})$$

$$\leq \sum_{i=1}^{k} h^{n+i-1} d(y_1, y_0)$$

i.e. $d(y_{m+k}, y_m) \leq \left(\dfrac{h^n}{1-h}\right) d(y_1, y_0) \to 0$ as $n \to \infty$

Hence $\{y_m\}$ is a Cauchy's sequence in X. Hence it converges to some point u in X. Consequently, the subsequences $\{Sx_{2m}\}$, $\{Px_{2m}\}$ and $\{Tx_{2m-1}\}$ of sequence $\{y_m\}$ also converges to u.

Now suppose that P is continuous. Since $Px_{2m} \to u$ as $m \to \infty$, then $P^2x_{2m} \to Pu$ as $m \to \infty$. The pair (S, P) is compatible of type (A) on X, then $SPx_{2m} \to Pu$ as $m \to \infty$.

By (3.2), we obtain

$$d(SPx_{2m}, Tx_{2m-1}) \leq \left\{\alpha + \beta \frac{d(SPx_{2m}, PPx_{2m})}{1 + d(PPx_{2m}, Px_{2m-1})}\right\} d(Tx_{2m-1}, Px_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(SPx_{2m}, P^2x_{2m})}{1 + d(P^2x_{2m}, Px_{2m-1})}\right\} d(Tx_{2m-1}, Px_{2m-1})$$

Letting $m \to \infty$ and using above results we get

$$d(Pu, u) \leq \left\{\alpha + \beta \frac{d(Pu, Pu)}{1 + d(Pu, u)}\right\} d(u, u)$$

Which implies

$d(Pu,u) \le 0$ .So that $u = Pu$.

Now by (3.2)

$$d(Su, Tx_{2m-1}) \le \left\{ \alpha + \beta \frac{d(Su, Pu)}{1 + d(Pu, Px_{2m-1})} \right\} d(Tx_{2m-1}, Px_{2m-1})$$

Letting $m \to \infty$ and using above results we get

$$d(Su,u) \le \left\{ \alpha + \beta \frac{d(Su,u)}{1 + d(u,u)} \right\} d(u,u)$$

Which implies

$d(Su,u) \le 0$

So that $u = Su$. Since $u = Su \in S(X) \Rightarrow u = Su \in S(X) \cup T(X) \Rightarrow u = Su \in P(X)$ hence there exists a point v in X such that $u = Su = Pv$.

$$d(u,Tv) = d(Su,Tv)$$

$$\le \left\{ \alpha + \beta \frac{d(Su, Pu)}{1 + d(Pu, Pv)} \right\} d(Tv, Pv)$$

$$= \left\{ \alpha + \beta \frac{d(u,u)}{1 + d(u, Pv)} \right\} d(Tv,u)$$

Which implies

$(1-\alpha)d(u,Tv) \le 0$,since $\alpha < 1$.So that $u = Tv$.

Hence $u = Tv = Pv$ therefore $d(Tv, Pv) = 0$ also the pair (T, P) is compatible of type (A) on X , then $d(TPv, PPv) = 0$ and $d(PTv, TTv) = 0$ hence $Pu = Tu$. Thus $u = Su = Pu = Tu$.

Therefore, u is a common fixed point of P, S and T. Similarly, we can also complete the proof, when T or S is continuous.

For uniqueness of u , suppose u and z , $u \ne z$, are common fixed points of P, S and T. Then by (3.2), we obtain

$$d(u,z) = d(Su,Tz)$$

$$\le \left\{ \alpha + \beta \frac{d(Su, Pu)}{1 + d(Pu, Pz)} \right\} d(Tz, Pz)$$

$$\le \left\{ \alpha + \beta \frac{d(u,u)}{1 + d(u,z)} \right\} d(z,z)$$

$$\le 0$$

i.e. $d(u,z) \le 0$ which is a contradiction .So that $u = z$. Hence u is a unique common fixed point P, S and T .This completes the proof.

Now we will generalize theorem 3.1 for four self mappings.

Theorem 3.2 Let P, Q, S and T be four mappings from a complete metric space (X, d) into itself satisfying the conditions:

$$S(X) \subseteq Q(X),\ T(X) \subseteq P(X) \tag{3.4}$$

and $d(Sx,Ty) \le \left\{\alpha + \beta \dfrac{d(Sx,Px)}{1+d(Px,Qy)}\right\} d(Ty,Qy)$ (3.5)

for all $x, y \in X$, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$ with $\alpha < 1$.

Suppose that

(i) One of P, Q, S and T is continuous,

(ii) The pairs (S, P) and (T, Q) are compatible of type (A) on X.

Then P, Q, S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X, by (3.4) we choose a point x_1 in X such that $Qx_1 = Sx_0$ and for this point x_1, there exists a point x_2 in X such that $Px_2 = Tx_1$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$y_{2m+1} = Qx_{2m+1} = Sx_{2m}$ and $y_{2m} = Px_{2m} = Tx_{2m-1}$ (3.6)

Then we show that the sequence $\{y_m\}$ defined by (3.6) is a Cauchy sequence in X.

By definition (3.6) we have

$$d(y_{2m+1}, y_{2m}) = d(Sx_{2m}, Tx_{2m-1})$$

$$\le \left\{\alpha + \beta \frac{d(Sx_{2m}, Px_{2m})}{1+d(Px_{2m}, Qx_{2m-1})}\right\} d(Tx_{2m-1}, Qx_{2m-1})$$

$$\le \left\{\alpha + \beta \frac{d(y_{2m+1}, y_{2m})}{1+d(y_{2m}, y_{2m-1})}\right\} d(y_{2m}, y_{2m-1})$$

$$\le \alpha d(y_{2m}, y_{2m-1}) + \beta d(y_{2m+1}, y_{2m})$$

i.e $d(y_{2m+1}, y_{2m}) \le \dfrac{\alpha}{1-\beta} d(y_{2m}, y_{2m-1})$

Hence $d(y_{2m+1}, y_{2m}) \le h\, d(y_{2m}, y_{2m-1})$

Where $h = \dfrac{\alpha}{1-\beta} < 1$

Similarly we can show that

$$d(y_{2m+1}, y_{2m}) \le h^{2m}\, d(y_1, y_0)$$

For k > m , we have

$$d(y_{m+k}, y_m) \le \sum_{i=1}^{k} d(y_{n+i}, y_{n+i-1})$$

$$\le \sum_{i=1}^{k} h^{n+i-1} d(y_1, y_0)$$

i.e. $d(y_{m+k}, y_m) \le \left(\dfrac{h^n}{1-h}\right) d(y_1, y_0) \to 0$ as $n \to \infty$

Hence $\{y_m\}$ is a Cauchy's sequence in X. Hence it converges to some point u in X. Consequently, the subsequences $\{Sx_{2m}\}$, $\{ Px_{2m} \}$, $\{ Tx_{2m-1} \}$ and $\{ Qx_{2m-1} \}$ of $\{y_m\}$ also converges to u.

Now suppose that P is continuous. Since $Px_{2m} \to u$ as $m \to \infty$, then $P^2x_{2m} \to Pu$ as $m \to \infty$. The pair (S, P) is compatible of type (A) on X, then $SPx_{2m} \to Pu$ as $m \to \infty$.

By (3.5), we obtain

$$d(SPx_{2m},Tx_{2m-1}) \le \left\{ \alpha + \beta \frac{d(SPx_{2m},PPx_{2m})}{1+d(PPx_{2m},Qx_{2m-1})} \right\} d(Tx_{2m-1},Qx_{2m-1})$$

$$\le \left\{ \alpha + \beta \frac{d(SPx_{2m},P^2x_{2m})}{1+d(P^2x_{2m},Qx_{2m-1})} \right\} d(Tx_{2m-1},Qx_{2m-1})$$

Letting $m \to \infty$ and using above results we get

$$d(Pu,u) \le \left\{ \alpha + \beta \frac{d(Pu,Pu)}{1+d(Pu,u)} \right\} d(u,u)$$

Which implies

$d(Pu,u) \le 0$ .So that $u = Pu$.

By (3.5) , we have

$$d(Su,Tx_{2m-1}) \le \left\{ \alpha + \beta \frac{d(Su,Pu)}{1+d(Pu,Qx_{2m-1})} \right\} d(Tx_{2m-1},Qx_{2m-1})$$

Letting $m \to \infty$ and using above results we get

$$d(Su,u) \le \left\{ \alpha + \beta \frac{d(Su,Pu)}{1+d(Pu,u)} \right\} d(u,u)$$

Which implies

$d(Su,u) \le 0$ .So that $u = Su$. Thus $u = Su = Pu$. Since $S(X) \subseteq Q(X)$ and there exists a point v in X, such that $u = Su = Qv$.

Consider

$$d(u,Tv) = d(Su,Tv)$$

$$\le \left\{ \alpha + \beta \frac{d(Su,Pu)}{1+d(Pu,Qv)} \right\} d(Tv,Qv)$$

$$\le \left\{ \alpha + \beta \frac{d(u,u)}{1+d(u,u)} \right\} d(Tv,u)$$

i.e. $d(Tv,u) \le \alpha d(Tv,u)$.

$(1-\alpha)d(Tv,u) \le 0$,since $\alpha < 1$.So that $u = Tv$. Thus $u = Tv = Qv$ and so $d(Tv,Qv) = 0$. Since the pair (T, Q) is compatible of type (A) on X , then $d(TQv,QQv) = 0$ and $d(QTv,TTv) = 0$ hence $Qu = Tu$.

Moreover by (3.5), we obtain

$$d(u,Qu) = d(Su,Tu)$$

$$\le \left\{ \alpha + \beta \frac{d(Su,Pu)}{1+d(Pu,Qu)} \right\} d(Tu,Qu)$$

i.e . $d(u,Qu) \le 0$

so that $u = Qu$. Hence $u = Qu = Tu$. Therefore , u is a common fixed point of P, Q, S and T when the mapping P is continuous. Similarly we can prove that u is a common fixed point of P, Q, S and T, when Q is continuous.

Next suppose that S is continuous. Since $Sx_{2m} \to u$ as $m \to \infty$ then $S^2x_{2m} \to Su$ as $m \to \infty$ and the pair (S, P) is compatible of type (A) on X, then $PSx_{2m} \to Su$ as $m \to \infty$.

By (3.5), we obtain

$$d(SSx_{2m}, Tx_{2m-1}) \leq \left\{\alpha + \beta \frac{d(SSx_{2m}, PSx_{2m})}{1 + d(PSx_{2m}, Qx_{2m-1})}\right\} d(Tx_{2m-1}, Qx_{2m-1})$$

$$d(S^2x_{2m}, Tx_{2m-1}) \leq \left\{\alpha + \beta \frac{d(S^2x_{2m}, PSx_{2m})}{1 + d(PSx_{2m}, Qx_{2m-1})}\right\} d(Tx_{2m-1}, Qx_{2m-1})$$

Letting $m \to \infty$ and using above results we get

$$d(Su, u) \leq \left\{\alpha + \beta \frac{d(Su, Su)}{1 + d(Su, u)}\right\} d(u, u)$$

Which implies

$d(Su, u) \leq 0$. So that $u = Su$. Since $S(X) \subseteq Q(X)$ and there exists a point w in X, such that $u = Su = Qw$.

Consider

$$d(S^2x_{2m}, Tw) \leq \left\{\alpha + \beta \frac{d(S^2x_{2m}, PSx_{2m})}{1 + d(PSx_{2m}, Qw)}\right\} d(Tw, Qw)$$

Letting m $\to \infty$ and using above results we get

$$d(Su, Tw) \leq \left\{\alpha + \beta \frac{d(Su, Su)}{1 + d(Su, Qw)}\right\} d(Tw, Qw)$$

Which implies

$$d(u, Tw) \leq \alpha\, d(u, Tw)$$

$(1-\alpha) d(u, Tw) \leq 0$ So that $u = Tw$ since $\alpha < 1$. Thus $u = Tw = Qw$. Since $d(Tw, Qw) = 0$ and the pair (T, Q) is compatible of type (A) on X , then $d(QTw, TTw) = 0$ and $d(TQw, QQw) = 0$ .Hence $Qu = Tu$.

Moreover by (3.5), we have

$$d(Sx_{2m}, Tu) \leq \left\{\alpha + \beta \frac{d(Sx_{2m}, Px_{2m})}{1 + d(Px_{2m}, Qu)}\right\} d(Tu, Qu)$$

Letting $m \to \infty$ and using above results we get

$$d(u, Tu) \leq \left\{\alpha + \beta \frac{d(Su, u)}{1 + d(u, u)}\right\} d(Tu, Tu)$$

i.e. $\quad d(u, Tu) \leq 0$.

so that $u = Tu$. Since $T(X) \subset P(X)$ and there exists a point z in X, such that $u = Tz = Pz$.

Moreover by (3.5), we obtain

$$d(Sz, u) = d(Sz, Tu)$$

$$\le \left\{\alpha + \beta \frac{d(Sz,Pz)}{1+d(Pz,Qu)}\right\} d(Tu,Qu)$$

$$\le \left\{\alpha + \beta \frac{d(Sz,u)}{1+d(u,u)}\right\} d(u,u)$$

i.e. $d(Sz,u) \le 0$

so that $u = Sz$. since $Sz = Pz = u$ that is $d(Sz, Pz) = 0$ and the pair (S, P) is compatible of type (A) on X and, then $d(PSz, SSz) = 0$ and $d(SPz, PPz) = 0$ and so $Pu = Su$. Hence $u = Tu = Pu = Qu = Su$. Therefore, u is a common fixed point of P, Q, S and T, when the mapping S is continuous .Similarly we can prove that u is a common fixed point of P, Q, S and T, when T is continuous.

For uniqueness of u , suppose u and z , $u \ne z$, are common fixed points of P, Q, S and T. Then by (3.5), we obtain

$$d(u,z) = d(Su,Tz)$$

$$\le \left\{\alpha + \beta \frac{d(Su,Pu)}{1+d(Pu,Qz)}\right\} d(Tz,Qz)$$

$$\le \left\{\alpha + \beta \frac{d(u,u)}{1+d(u,z)}\right\} d(z,z)$$

i.e. $d(u,z) \le 0$

which is a contradiction .Hence $u = z$. Therefore, u is unique common fixed point of P, Q, S and T. This completes the proof.

To illustrate our main theorems 3.2, following example is also given.

Example 3.1 Let $X = [0, 1]$, with $d(x,y) = |x-y|$ .Let P, Q, S and T be mappings from a complete metric space (X, d) into itself defined by

$$Px = Qx = \{x \quad \text{if} \quad x \in [0,1]$$

$$Tx = Sx = \{1-x \quad \text{if} \quad x \in [0,1]$$

Let $x_n = \frac{1}{2} - \frac{1}{n}$ for $n \ge 2$ be a sequence in X converges to $\frac{1}{2}$ as $n \to \infty$.Hence from definition of sequence (3.5) , the subsequences $\{Sx_n\}$,$\{Px_n\}$, $\{Tx_n\}$ and $\{Qx_n\}$ of $\{x_n\}$ also converges to $\frac{1}{2}$ as $n \to \infty$.

Since $SPx_n = S\left(\frac{1}{2}-\frac{1}{n}\right) = 1-\left(\frac{1}{2}-\frac{1}{n}\right) = \frac{1}{2}+\frac{1}{n} \to \frac{1}{2}$ as $n \to \infty$ and $PSx_n = P\left(1-\frac{1}{2}+\frac{1}{n}\right) = P\left(\frac{1}{2}+\frac{1}{2}\right) \to \frac{1}{2}$ as $n \to \infty$.
Also $SSx_n = S\left\{1-\left(\frac{1}{2}-\frac{1}{n}\right)\right\} = S\left(\frac{1}{2}+\frac{1}{n}\right) = \frac{1}{2}-\frac{1}{n} \to \frac{1}{2}$ as $n \to \infty$ and $PPx_n = P\left(\frac{1}{2}-\frac{1}{n}\right) = \frac{1}{2}-\frac{1}{n} \to \frac{1}{2}$ as $n \to \infty$.

Therefore $\lim_{n\to\infty} d\left(PSx_n, SSx_n\right) = d\left(\frac{1}{2},\frac{1}{2}\right) = 0$ and $\lim_{n\to\infty} d\left(SPx_n, PPx_n\right) = d\left(\frac{1}{2},\frac{1}{2}\right) = 0$. Showing that the pair (*S*, *P*) and (*T*, *Q*) are compatible of type (A). Clearly condition $S(X) \subseteq Q(X), T\left(X\right) \subseteq P(X)$ hold and the condition $d(Sx,Ty) \leq \left\{\alpha + \beta \frac{d(Sx,Px)}{1+d(Px,Qy)}\right\} d(Ty,Qy)$ becomes $|y-x| \leq \left\{\alpha + \beta \frac{|1-2x|}{1+|x-y|}\right\} |1-2y|$ is holds on [0,1] and all other conditions of theorem 3.2 are satisfied by $P,Q.S$ and T. Also it is clear that $\frac{1}{2}$ is common fixed point of $P,Q.S$ and T.

IV. CONCLUSION

In this paper we proved common fixed point theorems for three and four compatible of type (A) mappings in complete metric space by using new rational inequality which is different and new from some earlier condition given by Jungck [2] Aage and Salunke [1], Shukla, Tiwari and Shukla [6].

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