T_3 and T_4-Spaces in Smooth Topological Spaces

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***Abstract* — In [25], we introduce the notion of *r*-fuzzy neighborhood filters in smooth topological spaces in sense of Gähler [10], and used it to define and study separation axioms $T_i, i = 0, 1, 2$. Here we continue our study of the axioms of separation in smooth topological spaces. Therefore, we introduce the notion of *r*-fuzzy neighborhood filter at a set. Then by using this notion we define and study separation axioms $T_i, i = 3, 4$. These axioms are related only to usual points and ordinary subsets and reduce to axioms defined in [5], if $\tau: I^X \to \{0, 1\}$. So the current separation axioms are generalization of the old one. In addition, we show T_i-space not necessarily be a T_{i-1}-space for *i*=3,4. We give a condition for which, T_i-space is a T_{i-1}-space for *i*=3,4. Finally, these axioms are good extension from the point of view of Aygün et al. [2].**

***Keywords*– Smooth topological space, fuzzy filter, *r*-fuzzy neighborhood filter, regular space, normal space.**
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I. INTRODUCTION

The concept of fuzzy topology was first defined by Chang [6], as a family of fuzzy sets which contains $\overline{0}, \overline{1}$ and closed under arbitrary union and finite intersection, and later re-defined in a somewhat different way by Lowen [19] and by Hutton [16]. In all these definitions, a fuzzy topology is a crisp subfamily of some family of fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Therefore, a new definition of fuzzy topology introduced by Badard [3] under the name 'smooth topology'. After that, several authors [7, 8, 21] have re-introduced the same definition and studied smooth topological spaces. Separation axioms are one of the important notions which are defined and studied in smooth topological spaces by several authors and many ways (c.f [1, 9, 17, 23, 27]). A notion related to usual points, called *r*-fuzzy neighborhood filter at a point, is used to define these axioms.

The notion of fuzzy filter has been introduced by Eklund and Gähler [10]. By means of an extension of this notion of fuzzy filter, a point-based approach to fuzzy topology related to usual points has been developed by Gähler [12, 13]. In this approach several notions are related to usual points, between these notions the notion of fuzzy neighbourhood filter which is defined by means of the notion of interior of a fuzzy set. For each fuzzy topological space, the mapping which assigns to each point x the fuzzy neighborhood filter at x can be considered itself as the fuzzy topology. Gähler [14] and [15] defined separation axioms for the convergence space using convergence of an *L*-filter M to a crisp point *x*, where *L* be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Bayoumi and Ibedou [4, 5] introduced and studied separation axioms in *L* -topological space. These axioms are equivalent to Gähler axioms [14, 15] in the special case of a fuzzy topology. This specialization obtained in replacing the convergence $\mathrm{M} \to x$ by $\mathrm{M} \leq \mathrm{N}(x)$ where $\mathrm{N}(x)$ is the fuzzy neighborhood filter at x. In [24], we generalized separation axioms given in [4, 5] to *L* -bitopological spaces.

Recently [25], we introduce the notion of *r*-fuzzy neighborhood filters in smooth topological spaces, which is a generalization of fuzzy neighborhood filters of Gähler in fuzzy topological spaces, and used it to define and study separation axioms $T_i, i = 0,1,2$ in smooth topological spaces. We show these axioms reduce to axioms defined in [4], if $\tau: I^X \to \{0,1\}$. So these separation axioms are generalization of the old one. In this paper, we define *r*-fuzzy neighborhood filter at a set, and using it to define separation axioms T_3 and T_4 as a complement to the our study of the separation axioms that introduced in [25]. These axioms are related only to usual points and ordinary subsets. If $\tau: I^X \to \{0,1\}$, these axioms lead to the axioms defined in [5]. Moreover, when $\tau: 2^X \to \{0,1\}$, these axioms are the ordinary once. These separation axioms are good extension in sense of Aygün et al. [2], this means the smooth topological space $(X, w(T))$ is a T_i if and only if the underlying topological space (X, T) is a $T_i, i = 3,4$. Furthermore, we show that the initial smooth topological space of a family of T_i-spaces, $i = 3,4$, is also a T_i-space. Finally, we found the implication between these axioms not goes well. That is mean, in general not each T_i-space is a T_{i-1}-space for $i = 3,4$. We give a condition for which, T_i-space is a T_{i-1}-space for $i = 3,4$.

II. PRELIMINARIES

Throughout this paper, let $I = [0,1], I_0 = (0,1]$ and $I_1 = [0,1)$. A fuzzy set μ on a set X is a mapping $\mu: X \to I$ of X into *I*. Denote by I^X and $P(X)$ for the set of all fuzzy sets and all ordinary subsets of X, respectively. For each fuzzy set $\lambda \in I^X$, $\sup \lambda = \vee_{x\in X} \lambda(x)$ and $\inf \lambda = \wedge_{x \in X} \lambda(x)$. For each $x \in X$ and $t \in I_0$, the fuzzy set x_t of X whose value t at x and 0 otherwise is called the fuzzy point in *X*. For each $\alpha \in I$, the constant fuzzy set of *X* with value α will denoted by $\bar{\alpha}$. For each fuzzy set $\mu \in I^X$, the strong α-cut and weak α-cut of μ are the subsets $S_\alpha \mu = \{x \in X \mid \mu(x) > \alpha\}$and $\omega_\alpha \mu = \{x \in X \mid \mu(x) \geq \alpha\}$ of *X*, respectively. For any subset *U* of *X*, the characteristic function of *U*, denoted by 1_U is a mapping $1_U: X \to \{0,1\}$ defined by $1_U(x) = 1$ if $x \in U$ and 0 if $x \notin U$.

Definition 2.1 *[3, 9, 21] A smooth topology on X is a mapping $\tau: I^X \to I$ which satisfies the following properties:*
1. $\tau(\bar{0}) = \tau(\bar{1}) = 1$,
2. $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2), \forall\, \mu_1, \mu_2 \in I^X$,
3. $\tau(\vee_{i\in J} \mu_i) \geq \wedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. For $r \in I_0$, μ is an r-open fuzzy set of X if $\tau(\mu) \geq r$, and μ is an *r*-closed fuzzy set of X if $\tau(\bar{1} - \mu) \geq r$.

Subsequently, the fuzzy closure for any fuzzy set in smooth topological space is given as follows:

Definition 2.2 *[8] Let (X, τ) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, a fuzzy closure of λ is a mapping $C_\tau : I^X \times I_0 \to I^X$ defined as*

$$C_\tau(\lambda, r) = \wedge \{\mu \in I^X \mid \mu \geq \lambda, \tau(\bar{1} - \mu) \geq r\}. \tag{1}$$

And, a fuzzy interior of λ is a mapping $I_\tau : I^X \times I_0 \to I^X$ define as

$$I_\tau(\lambda, r) = \vee \{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r\}. \tag{2}$$

Definition 2.3 *[21] A mapping $f : (X, \tau) \to (Y, \tau^*)$ from a smooth topological space (X, τ) to another one (Y, τ^*) is said to be:*
1. fuzzy continuous (F-continuous, for short) if and only if $\tau(f^{-1}(\mu)) \geq \tau^*(\mu)$ for each $\mu \in I^Y$.
2. fuzzy closed if and only if $\tau^*(f(\bar{1} - \mu)) \geq \tau(\bar{1} - \mu)$ for each $\mu \in I^X$.

Definition 2.4 *[21] Let (X, τ) be a smooth topological space. Then for each $\alpha \in I$, the family $\tau_\alpha = \{\mu \in I^X \mid \tau(\mu) \geq \alpha\}$ is a Change fuzzy topology on X with respect to τ .*

Definition 2.5 *[22] Let (X, T) be a topological space and define $\omega(T) : I^X \to I$ by $\omega(T)(\lambda) = \vee \{\alpha \in I_1 : \lambda^{-1}(\alpha, 1] \in T\}$. Clearly, $\lambda^{-1}(\alpha, 1] \in T$ if $\omega(T)(\lambda) > \alpha \geq 0$.*

Theorem 2.1 *[22] Let T and $\omega(T)$ be as above. Then $\omega(T)$ is a smooth topology and $\omega(T)(\lambda) = 1$ iff $S_\alpha \lambda \in T$.*

This provides a ‘goodness of extension’ criterion for smooth topological properties. Recall that a fuzzy extension of a topological property of (X, T) is said to be good when it is possessed by $\omega(T)$ if and only if the original property is possessed by T .

Next, definition of initial smooth topological space. This definition depended on the notions of fuzzy basis and the smooth topology generated from that fuzzy basis, for more details see [18, 20].

Definition 2.6 *[18] Let $\{(X_i,\tau_i)\}_{i\in\Gamma}$ be a family of smooth topological spaces, X a set and $f_i : X \to X_i$ a function, for each $i\in\Gamma$. The initial smooth topology τ on X with respect to $(X, f_i,(X_i,\tau_i),\Gamma)$ is the coarsest smooth topology on X for which all f_i, $i\in\Gamma$, are F-continuous.*

Recall now the definition of fuzzy filter and some basic definitions related with fuzzy filters, which need it in the sequel of the paper.

Definition 2.7 *[10, 12] Let X be a non-empty set. A fuzzy filter on X is a mapping $\mathsf{M}: L^X \to L$ such that the following conditions are fulfilled:*

1. $\mathsf{M}(\overline{\alpha}) \le \alpha$ holds for all $\alpha\in L$ and $\mathsf{M}(\overline{1}) = 1$,
2. $\mathsf{M}(\lambda\wedge\mu) = \mathsf{M}(\lambda)\wedge\mathsf{M}(\mu)$ for all $\lambda,\mu\in L^X$.

A fuzzy filter M is called homogeneous if $\mathsf{M}(\overline{\alpha}) = \alpha$ for all $\alpha\in L$. If M and N are fuzzy filters on X, M is called to be finer than N, denoted by $\mathsf{M}\le\mathsf{N}$, provided $\mathsf{M}(\lambda)\ge\mathsf{N}(\lambda)$ holds for all $\lambda\in L^X$. By $\mathsf{M}\not\le\mathsf{N}$ we mean that M is not finer than N. Since L is a complete chain, then

$$\mathsf{M}\not\le\mathsf{N} \Leftrightarrow \text{there is } \lambda\in L^X \text{ such that } \mathsf{M}(\lambda) < \mathsf{N}(\lambda). \qquad (3)$$

Proposition 2.1 *[12] Let A be a set of fuzzy filters on X. Then the following are equivalent:*

1. The infimum $\bigwedge_{\mathsf{M}\in A}\mathsf{M}$ of A with respect to the finer relation of fuzzy filter exists.
2. For each non-empty finite subset $\{\mathsf{M}_1,\mathsf{M}_2,...,\mathsf{M}_n\}$ of A we have $\mathsf{M}_1(\lambda_1)\wedge\mathsf{M}_2(\lambda_2)\wedge...\wedge\mathsf{M}_n(\lambda_n)\le$ sup $(\lambda_1\wedge\lambda_2\wedge...\wedge\lambda_n)$ for all $\lambda_1,\lambda_2,...,\lambda_n\in L^X$.

Definition 2.8 *[13] For each fuzzy topological space (X,τ) and each $x\in X$ the mapping $\mathsf{N}(x): L^X\to L$ defined by $\mathsf{N}(x)(\lambda) = \text{int}_\tau\lambda(x)$ for all $\lambda\in L^X$, is a fuzzy filter on X, called the fuzzy neighborhood filter of the space (X,τ) at x, where $\text{int}_\tau\lambda$ is the interior of a fuzzy set λ.*

For each $x\in X$, the mapping $\dot{x}: L^X\to L$ defined by $\dot{x}(\lambda) = \lambda(x)$ for all $\lambda\in L^X$ is a homogeneous fuzzy filter on X.

Proposition 2.2 *[11, 12] $\bigvee_{x\in X}\dot{x}$ is the coarsest homogeneous fuzzy filter on X. For each $\lambda\in L^X$ we have $(\bigvee_{x\in X}\dot{x})(\lambda) = inf\ \lambda$.*

Definition 2.9 *[5] A fuzzy topological space (X,τ) is called:*

1. regular space if $\mathsf{N}(x)\wedge\mathsf{N}(F)$ does not exist for all $x\in X$, $F\subseteq X$ with $x\notin F$ and $cl_\tau F = F$. A fuzzy topological space (X,τ) is called a T_3-space if it is a regular space and T_1-space.
2. normal space if for all $F_1,F_2\subseteq X$ such that $cl_\tau F_1 = F_1$, $cl_\tau F_2 = F_2$ and $F_1\cap F_2 = \varnothing$, we have $\mathsf{N}(F_1)\wedge\mathsf{N}(F_2)$ does not exist. A fuzzy topological space (X,τ) is called a T_4-space if it is normal and T_1-space.

Definition 2.10 *[25] Let (X,τ) be a smooth topological space, for all $x\in X$ and $r\in I_0$, the mapping $\mathsf{N}(x): I^X\times I_0\to I$ defined by*

$$\mathsf{N}(x)(\lambda,r) = I_\tau(\lambda,r)(x) = \bigvee_{\rho\le\lambda,\, \tau(\rho)\ge r} \rho(x) \qquad (4)$$

for all $\lambda \in I^X$ is a fuzzy filter on X, called the r-fuzzy neighborhood filter of the space (X,τ) at x.

Remark 2.1 *For simply we write* $\mathsf{N}_\tau^r(x)(\lambda)$ *for* $\mathsf{N}(x)(\lambda,r)$, *for each* $\lambda \in I^X$ *and* $r \in I_0$.

The r-fuzzy neighborhood filters fulfill the following conditions:

1. $\dot{x} \le \mathsf{N}_\tau^r(x)$ hold for all $x \in X$ and $r \in I_0$.
2. $\mathsf{N}_\tau^r(x)(I_\tau(\lambda,r)) = \mathsf{N}_\tau^r(x)(\lambda)$ for all $x \in X$, $r \in I_0$ and $\lambda \in I^X$.

It is worth noting that in Definition 2.8 there is a one-to-one correspondence between the fuzzy topologies τ and the mapping $x \to \mathsf{N}(x)$ of X into the set of all fuzzy filters on X, this correspondence is given by $\mathsf{N}(x)(\lambda) = int_\tau \lambda(x)$. But in our definition of r-fuzzy neighborhood filter we have, for each $x \in X$ there are infinite neighborhood filters correspondence it since $r \in I_0$, and when $r = 1$ we return to fuzzy neighborhood filter in Definition 2.8. So, r-fuzzy neighborhood filter is a generalization of fuzzy neighborhood filter.

Definition 2.11 *[25] Let* (X,τ) *be a smooth topological space and let* $\mathsf{N}_\tau^r(x)$ *be the* r*-fuzzy neighborhood filter of* (X,τ) *at* $x \in X$. *Then for each* $\lambda \in I^X$, *the fuzzy closure* $C_\tau(\lambda,r)(x)$ *of* λ *with respect to* τ *is defined by*

$$C_\tau(\lambda,r)(x) = \bigvee_{\mathsf{M}\le \mathsf{N}_\tau^r(x)(\lambda)} \mathsf{M}(\lambda) \text{ forall } x \in X. \qquad (5)$$

Next, the fuzzy closure for any fuzzy filters in smooth topological spaces is given as follows:

Definition 2.12 *[25] For each smooth topological space* (X,τ), *the fuzzy closure operator of* τ *is the mapping* C_τ *which assigns to each fuzzy filter* $C_\tau(\mathsf{M},r)$, *where*

$$C_\tau(\mathsf{M},r)(\lambda) = \bigvee_{C_\tau(\rho,r)\le\lambda} \mathsf{M}(\rho) \qquad (6)$$

$C_\tau(\mathsf{M},r)$ is called the fuzzy closure of M at the degree r, $r \in I_0$. C_τ is isotone and is hull operator, that is for all fuzzy filters M and N on X, we have

$$\mathsf{M} \le \mathsf{N} \text{ implies } C_\tau(\mathsf{M},r) \le C_\tau(\mathsf{N},r) \qquad (7)$$

and moreover C_τ fulfills that $\mathsf{M} \le C_\tau(\mathsf{M},r)$ for all $r \in I_0$.

Definition 2.13 *[25] A smooth topological space* (X,τ) *is called:*

1. a T_0-space if for all $x, y \in X$ with $x \ne y$, there exists $r \in I_0$ such that $\dot{x} \nleq \mathsf{N}_\tau^r(y)$ or $\dot{y} \nleq \mathsf{N}_\tau^r(x)$.
2. a T_1-space if for all $x, y \in X$ with $x \ne y$, there exist $r, s \in I_0$ such that $\dot{x} \nleq \mathsf{N}_\tau^r(y)$ and $\dot{y} \nleq \mathsf{N}_\tau^r(x)$.
3. a T_2-space or Hausdorff space if for all $x, y \in X$ with $x \ne y$, there exist $r, s \in I_0$ such that $\mathsf{N}_\tau^r(x) \wedge \mathsf{N}_\tau^s(y)$ does not exist.

Proposition 2.3 *[25] A topological space* (X,T) *is a* T_1*-space if and only if the smooth topological space* $(X, w(T))$ *is a* T_1*-space.*

Proposition 2.4 *[25] Let* (X_i,τ_i) *be a* T_1*-spaces for all* $i \in \Gamma$ *and Let* $f_i : X \to X_i$ *be an injective mapping for some* $i \in \Gamma$. *Then the initial smooth topological space* (X,τ) *is also a* T_1*-space.*

III. T_3-SPACES

In this section we introduce the notion of r-fuzzy neighborhood filter at a set, and using this r-fuzzy neighborhood filter to introduce the notion of regular spaces and T_3-spaces.

Definition 3.1 *Let (X,τ) be a smooth topological space, the r-fuzzy neighborhood filter at a set $F \subseteq X$, is a mapping $\mathrm{N}(F): I^X \times I_0 \to I$ defined by*

$$\mathrm{N}(F)(\lambda,r) = \bigvee_{x\in F} \mathrm{N}(x)(\lambda,r) \tag{8}$$

for all $\lambda \in I^X$ and $r \in I_0$. For simply we write $\mathrm{N}_\tau^r(F)(\lambda)$ for $\mathrm{N}(F)(\lambda,r)$, for each $\lambda \in I^X$ and $r \in I_0$.

Definition 3.2 *A smooth topological space (X,τ) is called a regular space if for all $x \in X$, $F \subseteq X$ with $x \notin F$ and $\tau(\overline{1} - 1_F) \geq t$, for some $t \in I_0$ there exist an $r,s \in I_0$ such that $\mathrm{N}_\tau^r(x) \wedge \mathrm{N}_\tau^s(F)$ does not exist.*

Definition 3.3 *A smooth topological space (X,τ) is called a T_3-space if it is a regular space and T_1-space.*

Remark 3.1 *When $\tau: I^X \to \{0,1\}$ and r=s=1 we return to axioms that introduced in [5] in fuzzy topological spaces. And when $I=\{0,1\}$ and r=s=1 these axioms are the usual ones, such that $\mathrm{N}_\tau^r(x)$ and $\mathrm{N}_\tau^s(F)$ is the set of all neighborhood of x and F, respectively. In this case $\mathrm{N}_\tau^r(x) \wedge \mathrm{N}_\tau^s(F)$ does not exist mean that there is a neighborhood of x and a neighborhood of F such that the intersection of them is the empty set.*

Example 3.1 *Let $X = \{x,y\}$, λ_1 and λ_2 are fuzzy sets of X defined as* $\lambda_1 = x_1$ and $\lambda_2 = y_1$. Define a smooth topology $\tau: I^X \to I$ by

$$\tau(\lambda) = \begin{cases} 1 & \text{if}\lambda = \overline{0}, \overline{1}, \\ \frac{1}{2} & \text{if}\lambda = \lambda_1, \\ \frac{1}{3} & \text{if}\lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then (X,τ) is a regular space. Since, for $x \notin F_1 = \{y\} \subseteq X$ with $\tau(\overline{1} - 1_{F_1}) \geq \frac{1}{2}$, there exist $r = \frac{1}{2}, s = \frac{1}{3} \in I_0$ an $\lambda_1, \lambda_2 \in I^X$ such that $\mathrm{N}_\tau^{\frac{1}{2}}(x)(\lambda_1) \wedge \mathrm{N}_\tau^{\frac{1}{3}}(F_1)(\lambda_2) = 1 > 0 = \mathit{sup}\,(\lambda_1 \wedge \lambda_2)$. Thus, $\mathrm{N}_\tau^{\frac{1}{2}}(x) \wedge \mathrm{N}_\tau^{\frac{1}{3}}(F_1)$ does not exist. Similarly, for $y \notin F_2 = \{x\} \subseteq X$ with $\tau(\overline{1} - 1_{F_2}) \geq \frac{1}{3}$, we get $\mathrm{N}_\tau^{\frac{1}{3}}(y) \wedge \mathrm{N}_\tau^{\frac{1}{2}}(F_2)$ does not exist.

Theorem 3.1 *Let (X,τ) be a smooth topological space. Then the following statements are equivalent:*

1. (X,τ) is a regular space.
2. For all $x \in X$, $F \subseteq X$ with $x \notin F$ and $\tau(\overline{1} - 1_F) \geq t$ for some $t \in I_0$, there exist $r,s \in I_0$ such that $C_\tau(N_\tau{}^r(x), r) \nleq N_\tau{}^s(y)$ and $C_\tau(N_\tau{}^s(y), s) \nleq N_\tau{}^r(x)$ for each $y \in F$.

Proof. (1)$\Rightarrow$(2) Let $x \in X$, $F \subseteq X$ such that $x \notin F$ and $\tau(\overline{1} - 1_F) \geq t$ for some $t \in I_0$. Since (X,τ) is a regular space, there exist $r,s \in I_0$ such that $\mathrm{N}_\tau^r(x) \wedge \mathrm{N}_\tau^s(F)$ does not exist. That is mean,

$$\mathrm{N}_\tau^r(x) \wedge (\bigvee_{y\in F} \mathrm{N}_\tau^s(y)) = \bigvee_{y\in F} (\mathrm{N}_\tau^r(x) \wedge \mathrm{N}_\tau^s(y)) \text{ doesnotexi st.}$$

Hence,

$$\mathrm{N}_\tau^r(x) \wedge \mathrm{N}_\tau^s(y) \text{ doesnotexi stforall } y \in F, x \neq y.$$

Suppose $C_\tau(\mathrm{N}^r_\tau(x),r)\le \mathrm{N}^s_\tau(y)$ or $C_\tau(\mathrm{N}^s_\tau(y),s)\le \mathrm{N}^r_\tau(x)$. Then, for all $\lambda,\mu\in I^X$ we have

$$C_\tau(\mathrm{N}^r_\tau(x),r)(\lambda)\ge \mathrm{N}^s_\tau(y)(\lambda)\ or\ C_\tau(\mathrm{N}^s_\tau(y),s)(\mu)\ge \mathrm{N}^r_\tau(x)(\mu).$$

Take $\alpha_1,\alpha_2\in I_0$ such that $\mathrm{N}^s_\tau(y)(\lambda)\le\alpha_1$ and $\mathrm{N}^r_\tau(x)(\mu)\le\alpha_2$. Then

$$\mathrm{N}^s_\tau(y)(\lambda)\wedge \mathrm{N}^r_\tau(x)(\mu)\le\alpha< C_\tau(\mathrm{N}^r_\tau(x),r)(\lambda)\wedge C_\tau(\mathrm{N}^s_\tau(y),s)(\mu)\le \mathrm{N}^r_\tau(x)(\lambda)\wedge \mathrm{N}^s_\tau(y)(\mu)$$

where $\alpha=min\{\alpha_1,\alpha_2\}$. Thus $\mathrm{N}^r_\tau(x)(\lambda)\wedge \mathrm{N}^s_\tau(y)(\mu)\le\lambda(x)\wedge\mu(y)< sup(\lambda\wedge\mu)$, implies $\mathrm{N}^r_\tau(x)\wedge \mathrm{N}^s_\tau(y)$ exist for all $y\in F$ which is a contradiction. Hence, $C_\tau(N_\tau{}^r(x),r)\nleq N_\tau{}^s(y)$ and $C_\tau(N_\tau{}^s(y),s)\nleq N_\tau{}^r(x)$ for each $y\in F$.

(2) $\Rightarrow$ (1) Let (2) hold, this implies there exist $\lambda,\mu\in I^X$ such that $C_\tau(\mathrm{N}^r_\tau(x),r)(\lambda)<\mathrm{N}^s_\tau(y)(\lambda)$ and $C_\tau(\mathrm{N}^s_\tau(y),s)(\mu)<\mathrm{N}^r_\tau(x)(\mu)$ for each $y\in F$. If we take $\mu=1_{F^c}$ and $\lambda=1_F$, then we have $\mathrm{N}^r_\tau(x)(1_{F^c})=1>0=C_\tau(\mathrm{N}^s_\tau(y),s)(1_{F^c})$ and $\mathrm{N}^s_\tau(y)(1_F)>0=C_\tau(\mathrm{N}^r_\tau(x),r)(1_F)$. Thus, $\mathrm{N}^r_\tau(x)(1_{F^c})\wedge \mathrm{N}^s_\tau(y)(1_F)>0=sup(\lambda\wedge\mu)$ which is mean $\mathrm{N}^r_\tau(x)\wedge \mathrm{N}^s_\tau(y)$ does not exist for all $y\in F$. Hence, (X,τ) is a regular space.

Remark 3.2 *T_3-spaces need not to be a T_2-space as seen from the following example.*

Example 3.2 *Let $X=\{x,y,z\}$, define a smooth topology $\tau:I^X\to I$ by*

$$\tau(\lambda)=\begin{cases}1 & \text{if}\lambda=\bar{0},\bar{1},\\ \frac{1}{2} & \text{if}\lambda=x_1,\,y_1\vee z_1,\,x_1\vee y_{\frac{1}{2}},\\ \frac{1}{3} & \text{if}\lambda=x_1\vee z_{\frac{1}{2}},\,y_{\frac{1}{2}}\vee z_{\frac{1}{2}},\,y_{\frac{1}{2}},\,z_{\frac{1}{2}},\,x_1\vee y_{\frac{1}{2}}\vee z_{\frac{1}{2}},\\ 0 & otherwise.\end{cases}$$

Then (X,τ) is a T_3-space but not T_2-space. Since there exist $y\neq z$ in X such that for any $r,s\in I_0$ and $\lambda,\mu\in I^X$ we have $\mathrm{N}^r_\tau(y)(\lambda)\wedge \mathrm{N}^s_\tau(z)(\mu)\le sup\,(\lambda\wedge\mu)$.

The following proposition gives the condition makes every T_3-spaces is a T_2-space. But before that we need to introduce the following lemma.

Lemma 3.1 *Let (X,τ) be a smooth topological space, for each $x\in X$ and $r\in I_0$. If $C_\tau(\dot{x},r)=\dot{x}$, then $C_\tau(1_{\{x\}},r)=1_{\{x\}}$.*

Proof. Let $C_\tau(\dot{x},r)=\dot{x}$ for any $x\in X$ and $r\in I_0$. Then, for any $\lambda\in I^X$ we have

$$\bigvee_{C_\tau(\rho,r)\le\lambda}\rho(x)=\lambda(x). \tag{9}$$

Now, $C_\tau(1_{\{x\}},r)(y)=\bigvee_{\mathrm{M}\le \mathrm{N}^r_\tau(y)}\mathrm{M}(x_1)=\mathrm{N}^r_\tau(y)(x_1)=I_\tau(x_1,r)(y)$.

By applying (9), we get $I_\tau(x_1,r)(y)=\bigvee_{C_\tau(\rho,r)\le I_\tau(x_1,r)}\rho(y)\le\bigvee_{C_\tau(\rho,r)\le x_1}\rho(y).$

Again by applying (9), we get $\bigvee_{C_\tau(\rho,r)\le x_1}\rho(y)=x_1(y)$. Hence, $C_\tau(1_{\{x\}},r)=1_{\{x\}}$.

Proposition 3.1 *If $C_\tau(\dot{x},r)=\dot{x}$ for any $x\in X$ and $r\in I_0$, then every regular spaces is a T_2-space.*

Proof. Let $x,y\in X$ with $x\neq y$. Since $C_\tau(\dot{x},r)=\dot{x}$ for any $x\in X$ and $r\in I_0$, then by Lemma 3.1 we have $C_\tau(1_{\{x\}},r)=1_{\{x\}}$. That is mean $1_{\{x\}}$ is an r-closed fuzzy set of X and $y\notin\{x\}$. By regularity of X, there exist $s,t\in I_0$ such that $\mathrm{N}^s_\tau(y)\wedge \mathrm{N}^t_\tau(x)$ does not exist. Hence, (X,τ) is a T_2-space.

Corollary 3.1 *If $C_{\tau}(\dot{x}, r) = \dot{x}$ for any $x \in X$ and $r \in I_0$, then every T_3-spaces is a T_2-space.*

A topological space (X,T) is said to be a regular space. If for each $x \in X$ and T-closed set F in X such that $x \notin F$, there exist T-open set U and T-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.

Theorem 3.2 *A topological space (X,T) is a T_3-space if and only if the smooth topological space $(X, w(T))$ is a T_3-space.*

Proof. Let (X,T) be a T_3-space. From Proposition 2.3, we have (X,T) is a T_1-space equivalent to $(X, w(T))$ is a T_1-space. Now, let (X,T) be a regular space and let $x \in X$, $F \subseteq X$ such that $x \notin F$, $C_{w(T)}(1_F, t) = 1_F$ for some $t \in I_0$. By means of $w(T)$ we have F is a T-closed set in X. Since (X,T) is a regular, then there are T-open set U and T-open set V such that $x \in U, F \subseteq V$ and $U \cap V = \varnothing$. Since $U, V \in T$, then $S_{\alpha} 1_U = U$ and $S_{\alpha} 1_V = V$ for all $\alpha \in I_1$ and from Theorem 2.1, we get $w(T)(1_U) = 1$ and $w(T)(1_V) = 1$ and that is mean 1_U and 1_V are 1-open fuzzy sets in $(X, w(T))$. If we take $\lambda = 1_U, \mu = 1_V$, then we have

$$\mathsf{N}^1_{w(T)}(x)(\lambda) \wedge \mathsf{N}^1_{w(T)}(F)(\mu) = I_{w(T)}(\lambda,1)(x) \wedge I_{w(T)}(\mu,1)(F) = 1 > sup(\lambda \wedge \mu).$$

Hence, there exist $r = s = 1 \in I_0$ such that $\mathsf{N}^r_{w(T)}(x) \wedge \mathsf{N}^s_{w(T)}(F)$ does not exist, and thus $(X, w(T))$ is a regular space.

Conversely, if $(X, w(T))$ is a regular space and $x \in X$, F is a T-closed set such that $x \notin F$ we have $C_{w(T)}(1_F, 1) = 1_F$. Since $(X, w(T))$ is a regular space, then there exist $r, s \in I_0$ and $\lambda, \mu \in I^X$ such that

$$\mathsf{N}^r_{w(T)}(x)(\lambda) \wedge \mathsf{N}^s_{w(T)}(F)(\mu) > sup(\lambda \wedge \mu)$$

This means $I_{w(T)}(\lambda, r)(x) \wedge (\bigvee_{y \in F} \mathsf{N}^s_{w(T)}(y))(\mu) = I_{w(T)}(\lambda, r)(x) \wedge \bigwedge_{y \in F} I_{w(T)}(\mu, s)(y) > sup(\lambda \wedge \mu)$. Taking $\alpha = sup(\lambda \wedge \mu)$, then we get $I_{w(T)}(\lambda, r)(x) > \alpha$ and $I_{w(T)}(\mu, s)(y) > \alpha$ for all $y \in F$. Since $I_{w(T)}(\lambda, r)$ and $I_{w(T)}(\mu, s)$ are r and s-open fuzzy sets in $w(T)$, respectively. Then, $x \in S_{\alpha}(I_{w(T)}(\lambda, r)) = (I_{w(T)}(\lambda, r))^{-1}(\alpha, 1] \in T$ and $y \in S_{\alpha}(I_{w(T)}(\mu, s)) = (I_{w(T)}(\mu, s))^{-1}(\alpha, 1] \in T$ for all $y \in F$. That is mean, $x \in U = (I_{w(T)}(\lambda, r))^{-1}(\alpha, 1]$ and $F \subseteq V = (I_{w(T)}(\mu, s))^{-1}(\alpha, 1]$ and therefore $U \cap V = \varnothing$. Otherwise we have a contradiction. Hence, (X,T) is a regular space.

Example 3.3 *Let X be the real line with T be neighborhoods of any nonzero point being as in the usual topology, while neighborhoods of 0 will have the form $U - A$, Where U is the neighborhood of 0 in the usual topology and $A = \{\frac{1}{n} \mid n = 1,2,...\}$ [26].*

Then (X,T) is a T_2-space. But A is a closed set in X and cannot be separated from 0 by disjoint open sets, so X is not a T_3-space. Thus, from Proposition 3.2 the smooth topological space $(X, w(T))$ is a T_2-space but not T_3-space.

In the following propositions we show the initial smooth topological space (X,τ) of a family $\{(X_i, \tau_i)\}_{i \in \Gamma}$ of a T_3-spaces is also a T_3. Notice that the initial smooth topological space (X,τ) of a family $\{(X_i, \tau_i)\}_{i \in \Gamma}$ of T_0, T_1 and T_2-spaces the mappings f_i for some $i \in \Gamma$ must be injective [25], but in the case of T_3 and T_4 the mapping f_i must be also closed.

At first consider the case of Γ being a singleton.

Proposition 3.2 *Let (X_1,τ_1) be a T_3-space and let $f: X \to X_1$ be an injective and fuzzy closed mapping. Then the initial smooth topological space (X,τ) is also a T_3-space.*

Proof. From Proposition 2.4 it follows that (X,τ) is a T_1-space. Now let $x \in X$ and $F \subseteq X$ such that $x \notin F$ and $\tau(\overline{1}-1_F) \geq t$ for some $t \in I_0$. Since f is an injective and closed, then $f(x) \notin f(F)$, $\tau_1(\overline{1}-1_{f(F)}) \geq t$ and from (X_1,τ_1) is a regular space it follows there exist $r,s \in I_0$ and $\lambda,\mu \in I^{X_1}$ such that $\mathrm{N}^r_{\tau_1}(f(x))(\lambda) \wedge \mathrm{N}^s_{\tau_1}(f(F))(\mu) > sup\,(\lambda \wedge \mu)$, and that is means

$$I_{\tau_1}(\lambda,r)(f(x)) \wedge \bigwedge_{y \in f(F)} I_{\tau_1}(\mu,s)(y) = I_{\tau_1}(\lambda,r)(f(x)) \wedge \bigwedge_{z \in F} I_{\tau_1}(\mu,s)(f(z)) > sup\,(\lambda \wedge \mu). \tag{10}$$

Since $I_{\tau_1}(\lambda,r)$ and $I_{\tau_1}(\mu,s)$ is an r and s-open fuzzy sets in (X_1,τ_1), respectively and f is a fuzzy continuous. Then $\tau(f^{-1}(I_{\tau_1}(\lambda,r))) \geq \tau_1(I_{\tau_1}(\lambda,r))$ and $\tau(f^{-1}(I_{\tau_1}(\mu,s))) \geq \tau_1(I_{\tau_1}(\mu,s))$ implies $f^{-1}(I_{\tau_1}(\lambda,r)) = I_{\tau_1}(\lambda,r)\circ f$ and $f^{-1}(I_{\tau_1}(\mu,s)) = I_{\tau_1}(\mu,s)\circ f$ is an r and s-open fuzzy set in (X,τ), respectively, and that means $I_\tau(I_{\tau_1}(\lambda,r)\circ f, r) = I_{\tau_1}(\lambda,r)\circ f$ and $I_\tau(I_{\tau_1}(\mu,s)\circ f, s) = I_{\tau_1}(\mu,s)\circ f$. Since $I_{\tau_1}(\lambda,r) \leq \lambda$ and $I_{\tau_1}(\mu,s) \leq \mu$, then $I_{\tau_1}(\lambda,r)\circ f \leq \lambda \circ f$ and $I_{\tau_1}(\mu,s)\circ f \leq \mu \circ f$ respectively, implies

$$I_\tau(I_{\tau_1}(\lambda,r)\circ f, r) = I_{\tau_1}(\lambda,r)\circ f \leq I_\tau(\lambda \circ f, r) \text{ forsome } r \in I_0. \tag{11}$$

And

$$I_\tau(I_{\tau_1}(\mu,s)\circ f, s) = I_{\tau_1}(\mu,s)\circ f \leq I_\tau(\mu \circ f, s) \text{ forsome } s \in I_0. \tag{12}$$

Since $sup\,(\lambda \wedge \mu) \geq sup\,((\lambda \circ f) \wedge (\mu \circ f))$, and by applying (12) and (11) in (10), we get

$$I_\tau(\lambda \circ f, r)(x) \wedge \bigwedge_{z \in F}(I_\tau(\mu \circ f, s)(z)) > sup\,((\lambda \circ f) \wedge (\mu \circ f))$$

Thus there exist $\eta_1 = \lambda \circ f, \eta_2 = \mu \circ f \in I^X$ such that

$$I_\tau(\eta_1,r)(x) \wedge \bigwedge_{z \in F}(I_\tau(\eta_2,s)(z)) > sup\,(\eta_1 \wedge \eta_2), \text{ whichmeans } \mathrm{N}^r_\tau(x) \wedge \mathrm{N}^s_\tau(F) \text{ doesnotexist}.$$

Hence (X,τ) is a regular space. This means (X,τ) is a T_1 and regular therefore it is a T_3-space.

For any class Γ the proof is similar.

Proposition 3.3 *Let (X_i,τ_i) be a T_3-spaces for all $i \in \Gamma$ and Let $f_i : X \to X_i$ be an injective and fuzzy closed mapping for some $i \in \Gamma$. Then the initial smooth topological space (X,τ) is also a T_3-space.*

IV. T_4-SPACES

In this section we using the notion of r-fuzzy neighborhood filter at a set to introduce normal and T_4-space in smooth topological spaces and study the properties of this space as in the previous section.

Definition 4.1 *A smooth topological space (X,τ) is called a normal space if for all $F_1,F_2 \subseteq X$, $F_1 \cap F_2 = \varnothing$ with $\tau(\overline{1}-1_{F_1}) \geq t_1$ and $\tau(\overline{1}-1_{F_2}) \geq t_2$ for some $t_1,t_2 \in I_0$, there exist an $r,s \in I_0$ such that $\mathrm{N}^r_\tau(F_1) \wedge \mathrm{N}^s_\tau(F_2)$ does not exist.*

Definition 4.2 *A smooth topological space (X,τ) is called a T_4-space if it is a normal and T_1-space.*

Example 4.1 *Let (X,τ) as in Example 3.1, we have $F_1=\{x\}$ and $F_2=\{y\}$ are the only $\frac{1}{2}$ and $\frac{1}{3}$-closed fuzzy sets in (X,τ), respectively such that $\mathsf{N}_\tau^{\frac{1}{2}}(x)\wedge\mathsf{N}_\tau^{\frac{1}{3}}(y)$ does not exist. Thus, (X,τ) is a normal space.*

Theorem 4.1 *Let (X,τ) be a smooth topological space. Then the following statements are equivalent:*

1. (X,τ) is a normal space.
2. For all $F_1,F_2\subseteq X$, $F_1\cap F_2=\varnothing$ with $\tau(\overline{1}-1_{F_1})\geq t_1$ and $\tau(\overline{1}-1_{F_2})\geq t_2$ for some $t_1,t_2\in I_0$, there exist $r,s\in I_0$ such that $C_\tau(N_\tau{}^s(F_2),s)\nleq N_\tau{}^r(F_1)$ and $C_\tau(N_\tau{}^r(F_1),r)\nleq N_\tau{}^s(F_2)$.

Proof. The proof is similar of Theorem 3.1

A topological space (X,T) is said to be normal space if for all $F_1,F_2\subseteq X$ such that F_1 and F_2 are T-closed sets, and $F_1\cap F_2=\varnothing$, there exist $U\in T, V\in T$ such that $F_1\subseteq V$ and $F_2\subseteq U$ and $U\cap V=\varnothing$. (X,T) is called a T_4-space if it is a normal and T_1-space.

Next, the good extension property holds in T_4-spaces as in T_3-spaces.

Proposition 4.1 *A topological space (X,T) is a T_4-space if and only if the smooth topological space $(X,w(T))$ is a T_4-space.*

Proof. Similar of Theorem 3.2.

Remark 4.1 *If (X,τ) is a T_4-space, then it is not necessary to be a T_3-space as the following example show.*

Example 4.2 *Let $X=\{x,y,p,q\}$, define a smooth topology $\tau: I^X\to I$ by*

$$\tau(\lambda)=\begin{cases}1 & \text{if}\lambda=\overline{0},\overline{1},\\ \frac{1}{4} & \text{if}\lambda=y_1\vee p_1\vee q_1, x_1\vee y_1\vee p_1, x_1\vee y_{\frac{2}{3}}\vee p_1\vee q_{\frac{2}{3}}, x_{\frac{2}{3}}\vee y_1\vee p_{\frac{1}{2}}\vee q_1,\\ \frac{2}{3} & \text{if}\lambda=y_1\vee p_1, x_{\frac{2}{3}}\vee y_{\frac{2}{3}}\vee p_{\frac{1}{2}}\vee q_{\frac{2}{3}}, y_{\frac{2}{3}}\vee p_1\vee q_{\frac{2}{3}}, y_1\vee p_{\frac{1}{2}}\vee q_1, y_{\frac{2}{3}}\vee p_{\frac{1}{2}}\vee q_{\frac{2}{3}},\\ \frac{2}{3} & \text{if}\lambda=x_1\vee y_{\frac{2}{3}}\vee p_1, x_{\frac{2}{3}}\vee y_1\vee p_{\frac{1}{2}}, x_{\frac{2}{3}}\vee y_{\frac{2}{3}}\vee p_{\frac{1}{2}}, y_{\frac{2}{3}}\vee p_1, y_1\vee p_{\frac{1}{2}}, y_{\frac{2}{3}}\vee p_{\frac{1}{2}},\\ \frac{2}{3} & \text{if}\lambda=x_{\frac{2}{3}}\vee y_1\vee p_1\vee q_1, x_1\vee y_1\vee p_1\vee q_{\frac{3}{4}}, x_{\frac{2}{3}}\vee y_1\vee p_1\vee q_{\frac{3}{4}}, y_1\vee p_1\vee q_{\frac{3}{4}},\\ \frac{2}{3} & \text{if}\lambda=x_{\frac{2}{3}}\vee y_1\vee p_1, x_1\vee y_{\frac{2}{3}}\vee p_1\vee q_{\frac{3}{4}}, x_{\frac{2}{3}}\vee y_{\frac{2}{3}}\vee p_1\vee q_{\frac{3}{4}},\\ \frac{2}{3} & \text{if}\lambda=x_{\frac{2}{3}}\vee y_1\vee p_{\frac{1}{2}}\vee q_{\frac{3}{4}}, y_1\vee p_{\frac{1}{2}}\vee q_{\frac{3}{4}}, x_{\frac{2}{3}}\vee y_{\frac{2}{3}}\vee p_1,\\ 0 & \text{otherwise}\end{cases}$$

Then (X,τ) is a T_4-space but not regular space. Since there exist $p\notin F=\{x\}$ such that $\tau(\overline{1}-1_F)\geq\frac{1}{4}$ but for any $r,s\in I_0$ we have $\mathsf{N}_\tau^r(p)\wedge\mathsf{N}_\tau^s(F)$ exist. Hence, (X,τ) is not a T_3-space.

Proposition 4.2 *If $C_\tau(\dot{x},r)=\dot{x}$ for all $x\in X$ and $r\in I_0$, then every normal space is a regular space.*

Proof. Let $x\in X$, $F\subseteq X$ with $x\notin F$ and $\tau(\overline{1}-1_F)\geq t$ for some $t\in I_0$. Since $C_\tau(\dot{x},r)=\dot{x}$ for any $x\in X$ and $r\in I_0$, then by Lemma 3.1 we have $C_\tau(1_{\{x\}},r)=1_{\{x\}}$. Let $F_1=\{x\}$ and $F_2=F$. By normality of X, there exist $s,p\in I_0$ such that $\mathsf{N}_\tau^s(x)\wedge\mathsf{N}_\tau^p(F)$ does not exist. Thus, (X,τ) is a regular space.

Corollary 4.1 *If* $C_\tau(\dot{x},r)=\dot{x}$ *for all* $x\in X$ *and* $r\in I_0$*, then every* T_4*-space is a* T_3*-space.*

The following propositions shows the initial smooth topological space (X,τ) of a family $\{(X_i,\tau_i)\}_{i\in\Gamma}$ of a T_4 -spaces is also a T_4, whenever $f_i:X\to X_i$ for some $i\in\Gamma$ is an injective fuzzy closed mapping.

For the case of Γ being a singleton we get the following result.

Proposition 4.3 *Let* (X_1,τ_1) *be a* T_4*-space and let* $f:X\to X_1$ *be an injective and fuzzy closed mapping. Then the initial smooth topological space* (X,τ) *is also a* T_4*-space.*

Proof. From Proposition 2.4 it follows that (X,τ) is a T_1-space. Now, let $F_1,F_2\subseteq X$, $F_1\cap F_2=\varnothing$ with $\tau(\overline{1}-1_{F_1})\geq t_1$ and $\tau(\overline{1}-1_{F_2})\geq t_2$ for some $t_1,t_2\in I_0$. From that f is an injective and fuzzy closed it follows $f(F_1)$ and $f(F_2)$ are also disjoint t_1 and t_2-closed fuzzy set of X_1, respectively. Since (X_1,τ_1) is a normal space. Then there exist $r,s\in I_0$ such that $\mathsf{N}^r_{\tau_1}(f(F_1))\wedge\mathsf{N}^s_{\tau_1}(f(F_2))$ does not exist, that is there exist $\lambda,\mu\in I^{X_1}$ such that

$$\bigvee_{x\in f(F_1)}\mathsf{N}^r_{\tau_1}(x)(\lambda)\wedge\bigvee_{y\in f(F_2)}\mathsf{N}^s_{\tau_1}(y)(\mu)>sup\,(\lambda\wedge\mu).$$

Implies,

$$\bigwedge_{x\in f(F_1)}I_{\tau_1}(\lambda,r)(x)\wedge\bigwedge_{y\in f(F_2)}I_{\tau_1}(\mu,s)(y)>sup\,(\lambda\wedge\mu).$$

And this means,

$$\bigwedge_{z\in F_1}I_{\tau_1}(\lambda,r)(f(z))\wedge\bigwedge_{p\in F_2}I_{\tau_1}(\mu,s)(f(p))>sup\,(\lambda\wedge\mu).$$

Which means,

$$\bigwedge_{z\in F_1}(I_{\tau_1}(\lambda,r)\circ f)(z)\wedge\bigwedge_{p\in F_2}(I_{\tau_1}(\mu,s)\circ f)(p)>sup\,((\lambda\circ f)\wedge(\mu\circ f)).$$

Because of that $f:(X,\tau)\to(X_1,\tau_1)$ is a fuzzy continuous, it follows $I_{\tau_1}(\lambda,r)\circ f\leq I_\tau(\lambda\circ f,r)$ and $I_{\tau_1}(\mu,s)\circ f\leq I_\tau(\mu\circ f,s)$ and thus we have

$$\bigwedge_{z\in F_1}(I_\tau(\lambda\circ f,r))(z)\wedge\bigwedge_{p\in F_2}(I_\tau(\mu\circ f,s))(p)>sup\,((\lambda\circ f)\wedge(\mu\circ f)).$$

Thus, there exist $\gamma_1=\lambda\circ f$, $\gamma_2=\mu\circ f$ such that

$$\bigwedge_{z\in F_1}(I_\tau(\gamma_1,r))(z)\wedge\bigwedge_{p\in F_2}(I_\tau(\gamma_2,s))(p)>sup\,(\gamma_1\wedge\gamma_2).$$

Hence, (X,τ) is a normal space, and therefore (X,τ) is a T_4-space.

For any class Γ the proof is similar.

Proposition 4.4 *Let* (X_i,τ_i) *be a* T_4*-spaces for all* $i\in\Gamma$ *and Let* $f_i:X\to X_i$ *be an injective and fuzzy closed mapping for some* $i\in\Gamma$*. Then the initial smooth topological space* (X,τ) *is also a* T_4*-space.*

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