Verification of Feigenbaum's Universality in One Dimensional Nonlinear Mathematical Model

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Abstract:

Here we consider a rational nonlinear mathematical model $f(x) = \frac{\mu x}{1+(a x)^b}$ with μ as control parameter and a,b are taken as constants and examine period-doubling route to chaos . We observe that the route followed by this map is universal in the sense of Feigenbaum's universality constant. In order to verify an universal route from order to chaos through period doubling bifurcations appropriate numerical methods such as Newton-Raphson method, Bisection method etc. are used to obtain periodic points and bifurcation points of different period 2^0 ,2^1 ,2^2 ,2^3,2^4……and find Feigenbaum Universal Constant(δ)=4.66920161029… with the help of bifurcation points calculated numerically. Also with the help of experimental bifurcation points and Feigenbaum delta, the accumulation point is calculated numerically .The period doubling scenario explains us how the behaviour of the map changes from regularity to chaotic one. Then periodic behaviours of the map are established by plotting the Time-series graphs. Lastly chaotic region has also been confirmed by obtaining positive Lyapunov Exponents at some parametric values.

Key words: Period –Doubling Bifurcation, Periodic points, Feigenbaum Universal Constant, Time Series, Lyapunov Exponent

1. **Introduction:**

In this paper we pay our attention to study period doubling route to chaos of an inverse non linear algebraic one dimensional model. We consider our model as $f(x) = \frac{\mu x}{1+(a x)^b}$ where a, b are constants and μ is the control parameter. We take this model from [10]. Here we first provide the Feigenbaum tree of bifurcation points along with one of the periodic points, which leads to chaos. Secondly, we determine the accumulation point and draw the bifurcation graph of the model and verify that chaos occurs beyond accumulation point. Thirdly the graphs of the time series analysis are confirmed in order to support our periodic orbits of period $2^0, 2^1, 2^2, 2^3$.....and lastly the graph of Lyapunov exponent confirms about the existence of the chaotic region. [4, 5]

2. **Some useful definitions**: [2, 7, 8, 13, 14]

2.1 Orbit: Let $f : X \to X$ and assume $x \in X$ .The Orbit of x under f is the sequence $\{x, f(x), f^2(x), \ldots, f^n(x) \ldots\}$ and is denoted by O(x).

2.2 Fixed point: A point x^* is said to be a fixed point of f if $f(x^*) = x^*$ .Note that fixed points never move under iteration. Since $f(x^*) = x^*$

It follows that $f(f(x^*)) = f(x^*) = x^*$ and in general $f^n(x^*) = x^*$ .So the orbit of a fixed point x^* is a constant sequence at the fixed point, i.e. $\{x^*, x^*, x^* \ldots\ldots x^* \ldots.\}$

2.3 Periodic point and Periodic orbit: An orbit is periodic or cycle if it eventually returns to where it began. That is ,orbit of x^* is periodic if there is an integer n such that $f^n(x^*) = x^*$, the point x^* is called periodic point of period n .The least such positive integer n is called the prime period of the orbit.

Suppose x^* lies on an orbit that is a cycle of period 4, we may write

$$x_1 = f(x^*)$$

$$x_2 = f(x_1) = f^2(x^*)$$

$$x_3 = f(x_2) = f^3(x^*)$$

$$x^* = f(x_3) = f^4(x^*)$$

Since x^* have period 4. The orbit of x^* therefore repeats cyclically.

$x^*, x_1, x_2, x_3, x^*, x_1, x_2, x_3, x^*, x_1, x_2, x_3, x^*, x_1, \ldots..$

2.4 Stability of the fixed point: [7]

Let the fixed point be defined by the equation $f(x) = x$ .and start the orbit a small distance away from x. Define the small quantities $\in_n$ by

$$x_n = x + \in_n$$

Inserting this and expanding to the first order in the small quantities gives

$$x + \in_{n+1} = f(x + \in_n)$$

$$\approx f(x) + f'(x) \in_n$$

$$\in_{n+1} \approx f'(x) \in_n$$

From this we can say that the fixed point is (i) stable (attractor) if $|f'(x)| < 1$

(ii) Unstable (repellor) if $|f'(x)| > 1$

2.5 Bifurcation:

The qualitative structure of the flow can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called bifurcations, and the parameter values at which they occur are called bifurcation points [13]

Bifurcation means a splitting into two parts. The term bifurcation is commonly used in the study of nonlinear dynamics to describe any sudden change in the behaviour of the system as some parameter is varied. The bifurcation then refers to the splitting of the behaviour of the system into two regions: one above, the other below the particular parameter value at which the change occurs [8]

2.6 Period-doubling bifurcation:

When a sudden change in the behaviour of the system refers to the splitting the system into two region i.e. Period 1 orbit bifurcates into period 2 which again bifurcates into a period 4 and so on, as the parameters are varied; such phenomenon is occurred when $f^{/}(x) = -1$, which is known as period doubling bifurcation.

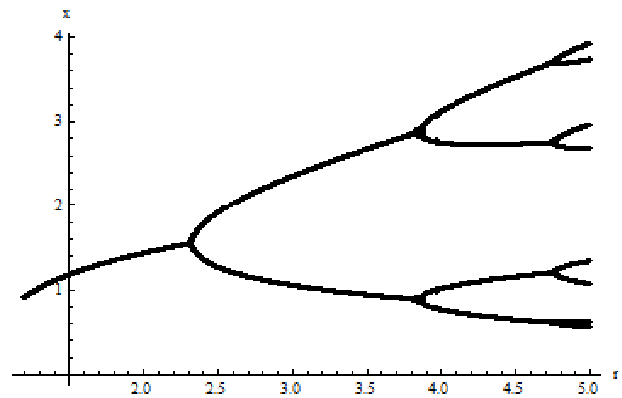


Fig2.1 : Showing period doubling bifurcation at $\mu = 2.3333$ of our model. (Period1 -> period 2), at $\mu = 2.3333$ of our model. (Period2 ->period 4), at $\mu = 2.3333$ of our model. (Period4 ->period 8)

2.7 Feigenbaum Universal Constant

Mitchell J. Feigenbaum, a renowned American particle theorist is known as the founder of the period-doubling bifurcation that may be described as a universal route to chaos- an exciting discovery in non linear dynamical systems. Many new universal properties have been discovered by Feigenbaum for families of maps which depend on a parameter μ .One of his fascinating discoveries is that if a family f represents period-doubling bifurcation then there is an infinite sequence $\{\mu_n\}$ of bifurcation values such that $\lim_{n\to\infty}\frac{\mu_n-\mu_{n-1}}{\mu_{n+1}-\mu_n} = \delta$, where δ is a universal number known as the Feigenbaum universal constant, which does not depend at all on the form of the specific family of maps.

2.8 Accumulation point:

We see in period doubling bifurcation, period one orbit bifurcates into period two orbit which again bifurcates into a period four orbit and this sequence of successive bifurcation continues. In fact, there is an infinity of period doubling bifurcations occurring at shorter and shorter intervals. After n bifurcations the length of the period is 2^n, so after infinity of bifurcations the period is infinitely long, i.e. there is no periodicity anymore and the attractor of the system has become aperiodic or chaotic. The point where this happens, λ_∞ is an accumulation point of the period doubling sequence.[7]

2.9 Lyapunov Exponent :

For a dynamical system, sensitivity to initial conditions is quantified by the Lyapunov exponents. For example, consider two trajectories with nearby initial conditions on an attracting manifold. When the attractor is chaotic, the trajectories diverge, on average, at an exponential rate characterized by the Lyapunov exponent.

The Lyapunov exponents measure quantities which constitute the exponential divergence or convergence of nearby initial points in the phase space of a dynamical system. A positive

Lyapunov exponent measures the average exponential divergence of two nearby trajectories

whereas a negative Lyapunov exponent measures exponential convergence of two nearby

trajectories. If a discrete nonlinear system is dissipative, a positive Lyapunov exponents quantifies a measure of chaos.

3. Our Vital Study:

Considering above definitions as our tool, we first try to find out the fixed points and it's nature.

Here we our model is $f(x) = \frac{\mu x}{1+(a x)^b}$.

So $f'(x) = \frac{\mu(1+(b-1)(a x)^b}{(1+(a x)^b)^2}$

Solving $f'(x) = o$, we get $x = \frac{1}{a(b-1)^{\frac{1}{b}}}$ .At this point we have $f''(x) < 0$,so maximum value for f(x) is $\frac{\mu(b-1)^{1-\frac{1}{b}}}{a b}$ for μ>0 .We may take the range as[0, $\frac{\mu(b-1)^{1-\frac{1}{b}}}{a b}$][6, 14].

The solution of $f(x) = x$ gives the fixed points of f(x). We get the fixed points as

$x = 0$, $x = \frac{(-1+\mu)^{\frac{1}{b}}}{a}$.

Now at x=0 ,$|f'(x)| = \mu > 1$, so x=0 is an unstable point.

Also at $x = \frac{(-1+\mu)^{\frac{1}{b}}}{a}$, we get $f'(x) = \frac{\mu(1-b)+b}{\mu}$

Now,

$f'(x) = -1$ implies $\frac{\mu(1-b)+b}{\mu} = -1$

Or, $\mu = \frac{b}{b-2}$

Also $\frac{df^2}{dx} = -1$

$$=> \frac{\mu^2(b\,(ax)^b - (ax)^b - 1)\left(b\left(\frac{ax\mu}{1+(ax)^b}\right)^b - \left(\frac{ax\mu}{1+(ax)^b}\right)^b - 1\right)}{(1+(ax)^b)^2(1+\left(\frac{ax\mu}{1+(ax)^b}\right)^b)^2} = -1 \quad ---(**)$$

Now $f^2(x) = \frac{x\,\mu^2}{(1+(ax)^b)(1+(\frac{ax\mu}{1+(ax)^b})^b)}$

Solving $f^2(x) = x$,we get

$$\frac{\mu^2}{(1+(ax)^b)(1+(\frac{ax\mu}{1+(ax)^b})^b)} = 1$$

----(*)

Let $\left(\frac{ax\mu}{1+(ax)^b}\right)^b = p$ and let $(ax)^b = t$

$(*) => \frac{\mu^2}{(1+t)(1+p)} = 1 \quad => \mu^2 = (1+t)(1+p)$

$(**) => \frac{\mu^2(\,bt-t-1)(\,bp-p-1)}{(1+t)^2(1+p)^2} = -1$

$=> (\,bt-t-1)(\,bp-p-1) + \mu^2 = 0$

$$=> (\,bt-t-1)\left(b(\frac{\mu^2}{(1+t)} - 1) - \frac{\mu^2}{(1+t)}\right) + \mu^2 = 0$$

$$=> (\,bt-t-1)(b\mu^2 - b - bt - \mu^2) + \mu^2(1+t) = 0$$

Which is quadratic in t , solving for t we have

$$t = \frac{2\mu^2 + b(b-2)(\mu^2-1) + \sqrt{b^4 - 2(b-2)b^3\mu^2 + (2+(b-2)b)^2\mu^4}}{2b(b-1)}$$

And

$$t = \frac{2\mu^2 + b(b-2)(\mu^2-1) - \sqrt{b^4 - 2(b-2)b^3\mu^2 + (2+(b-2)b)^2\mu^4}}{2b(b-1)}$$

Thus for the particular value of μ ,where the second bifurcation occurs the periodic points are of the form

$$x = \frac{1}{a^b}\left\{\frac{2\mu^2 + b(b-2)(\mu^2-1) + \sqrt{b^4 - 2(b-2)b^3\mu^2 + (2+(b-2)b)^2\mu^4}}{2b(b-1)}\right\}^{\frac{1}{b}}$$

Also for high value b ie when

$b \to \infty \Rightarrow t \to \mu^2 - 1$

Again from (*) we have

$$\frac{\mu^2}{(1+(ax)^b)(1+(\frac{ax\mu}{1+(ax)^b})^b)} = 1$$

$$\Rightarrow \frac{\mu^2}{(1+t)(1+(\frac{t}{(1+t)^b}\ \mu^b)} = 1$$

$$\Rightarrow \frac{\mu^2}{\mu^2(1+\frac{(\mu^2-1)\mu^b}{\mu^{2b}}} = 1$$

$\Rightarrow 1 + (\mu^2 - 1)\mu^{-b} = 1$ => 1 = 1, this is an identity

i.e. true for all μ for high values of b

Analytical study of the map after this stage is cumbersome.

So we consider a particular value of the parameters to study dynamical behaviour of the map

So let us consider a= 0.7 ,b =3.5, then at $x = \frac{(-1+\mu)^{\frac{1}{b}}}{a}$, $f'(x) = \frac{-2.5\mu+3.5}{\mu}$.

Now for 1< μ <7/3 , the absolute value of $f'(x)$ remain less than 1,and the point is stable.As soon as μ>7/3 the point becomes unstable. Hence μ=7/3 is the 1st bifurcation point of our model.

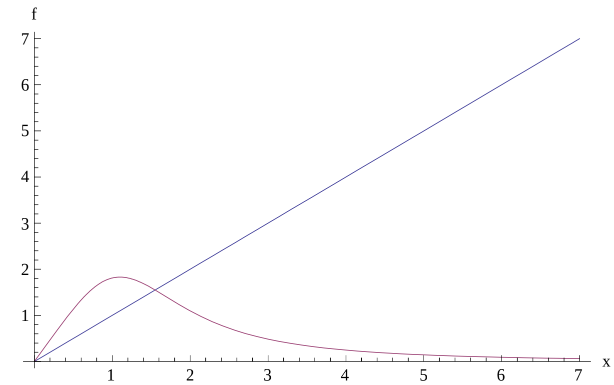


Fig 3.1 Graphs of the model and $f(x) = x$. Their intersection evinces two fixed points of f.

Next we consider the periodic points of period-two and higher.The period-2 points are found by solving the equation $f^2(x) = x$,where $f^2(x) = \frac{x\mu^2}{\{1+(ax)^b\}\{1+\left(\frac{ax\mu}{1+(ax)^b}\right)^b\}}$ .Hence solving this equation analytically is cumbersome one. The following figures shows the graphs of intersection of $f^2(x)$ and $f(x) = x$ at the parameter $\mu = 3.85556$. Their intersection evince four fixed points of f^2 and the graphs of intersection of $f^4(x)$ and $f(x) = x$ at the parameter $\mu = 476588$. Their intersection evince eight fixed points of f^4.

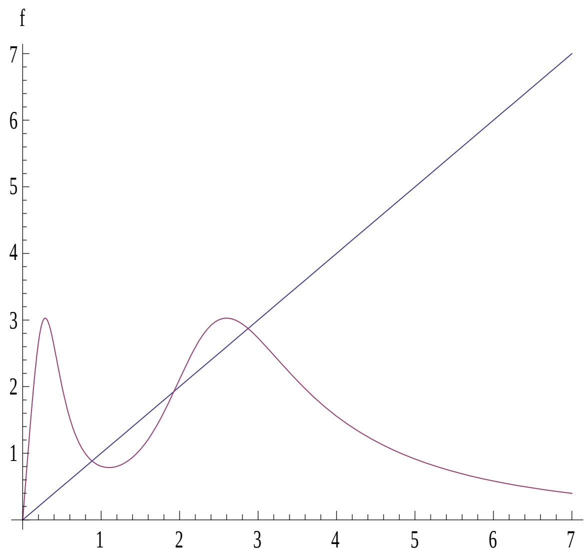


Figure 3.2: Graphs of $f^2(x)$ and $f(x) = x$ at the parameter. $\mu = 3.85556$. Their intersection evinces four fixed points of f^2

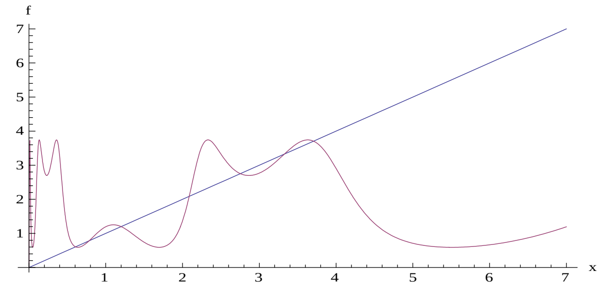


Figure 3.3: Graphs of $f^4(x)$ and $f(x) = x$ at the parameter $\mu = 476588$. Their intersection evinces eight fixed points of f^4.

So to find one of the periodic points we take the help of C-programming and use Newton-Raphson method and bisection method. We built up suitable numerical method and obtain following bifurcation points of different period ,one of the periodic point and Feigenbaum delta(experimental value).

Table 3.4(Calculation of bifucation point)

Bifurcation Point	One of the periodic points	Feigenbaum delta(experimental value) $\delta_n = \frac{\mu_{n+1}-\mu_n}{\mu_{n+2}-\mu_{n+1}}$
μ_1 = 2.3333333333333333	1.550953	
μ_2 = 3.8555640457993676	2.872694	
μ_3 =4.765583846101038470	1.199644	δ_1=1.6727446061738606158048607516547
μ_4 =5.030223122052351670	1.059819	δ_2=3.4387178434885493670955876240024
μ_5 =5.091772523673780530	1.115666	δ_3=4.2996238627797993440090970604845
μ_6 =5.105184125903642570	1.093101	δ_4=4.5892653663992534805402173793947
μ_7 =5.108067585470283940	1.102096	δ_5=4.6512191067356508932967487939334

μ_8 =5.108685635426731150	1.135925	δ_6=4.6654150470564061278015924906657
μ_9 =5.108818026081300090	1.136579	δ_7= 4.6683805474001327266059769334231
μ_{10} =5.108846381166173960	1.132848	δ_8 = 4.6690269190815109636860893857213
μ_{11} =5.108852454005875290	1.133114	δ_9 =4.66916405971657901666282248613255
μ_{12} =5.108853754624501420	1.133119	δ_{10}=4.6691932433720235514769853359827
μ_{13} =5.108854033177172130	1.132883	δ_{11}= 4.6692017808153364159859287819786
μ_{14} =5.108854092834596020	1.132868	δ_{12}=4.66920380644682912740501842678553

From the above table we can establish the Feigenbaum δ up to 4.6692011…..Now the following bifurcation diagram indicates the universal route to chaos for our model. [**5**]

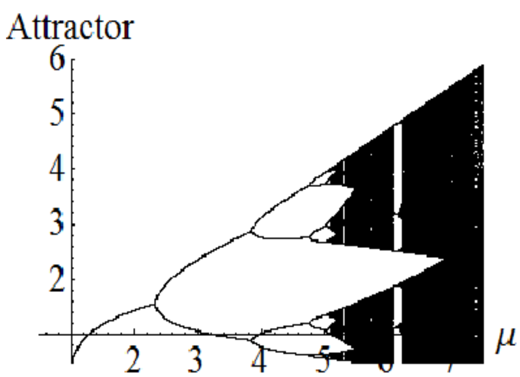


Fig3.5:Bifurcation graph of the model.The abcissa represents the control parameter and ordinate represents the iterated points.

4. Accumulation point

Using the experimental bifurcation points the sequence of accumulation points$\{\mu_{\infty,n}\}$ is calculated with the help of the following formula. [6]

$$\mu_{\infty,n} = \frac{\mu_{n+1} - \mu_n}{\delta - 1} + \mu_{n+1}$$

Table4.1: Accumulation points for different values of n,

$\mu_{\infty,1}$ =4.2704310590181297448747979669338082552

$\mu_{\infty,2}$ =5.0135995991858473731674730137623889376

$\mu_{\infty,3}$ = 5.1023476069252364651526872194419236659

$\mu_{\infty,4}$ =5.10854712702856207215821397216497234016

$\mu_{\infty,5}$=5.10883930859967893551724541914589262381

$\mu_{\infty,6}$= 5.10885534402348430041323135374590805551

$\mu_{\infty,7}$= 5.10885407805329979598053243053621059202

$\mu_{\infty,8}$= 5.10885410767851744909989377950542094182

$\mu_{\infty,9}$= 5.10885410902849788445848490577112387456

$\mu_{\infty, 10}$= 5.1088541090906453020317121452715545726

$\mu_{\infty, 11}$= 5.1088541090936106952360635641857564169

$\mu_{\infty, 12}$= 5.1088541090935976594485553928111681182

$\mu_{\infty, 13}$= 5.1088541090935619329099870012051600306

The above sequence converges to the value 5.1088541090935...which is the required accumulation point. [10]

5. Time series analysis: [4, 6, 11, 12,14]

A time series is a theoretical device for quantifying chaotic behaviour of the data for the system. Using this device one can predict future dynamical behaviour of the system by observing data over a period of time and understanding the changes that have taken place in the past.

We have plotted the following time series graphs. On the horizontal axis, the number of iterations (time) is marked, while on the vertical axis the amplitudes are given for each iterations. Here we take initial value x=0.5, a=0.7, b=3.5 and

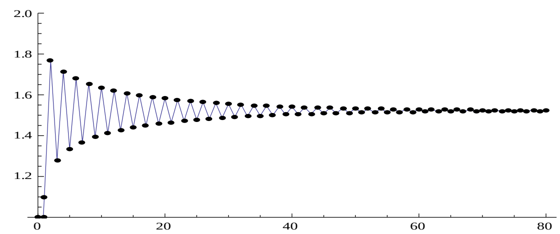
(i) Parameter value $\mu = 2.25$(which is slightly less than 1st bifurcation value) showing period-1 behaviour in figure 5.1

(ii) Parameter value $\mu = 2.50$(which is slightly more than 1st bifurcation value) showing period-2 behaviour in figure 5.2

(iii) Parameter value $\mu = 3.95$(which is slightly more than 2nd bifurcation value) showing period-4 behaviour in figure 5.3

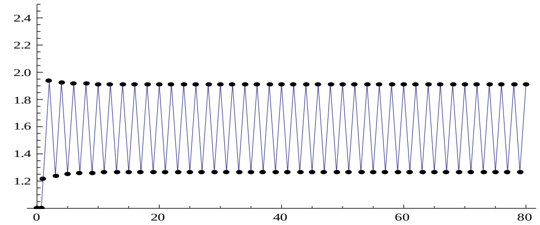
(iv)Parameter value $\mu = 4.95$(which is slightly more than 3rd bifurcation value) showing period-8 behaviour in figure5.4

(v) Showing chaotic behaviour beyond accumulation point in figure5.5



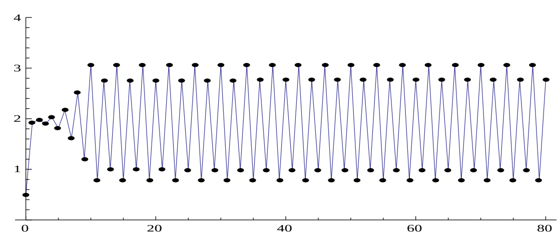
No of iterations $\rightarrow$

Fig 5.1: Time series graph for period 1 at the parameter $\mu = 2.25$



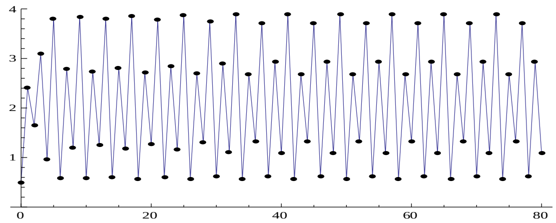
No of iterations $\rightarrow$

Fig 5.2: Time series graph for period 2 at the parameter $\mu = 2.5$



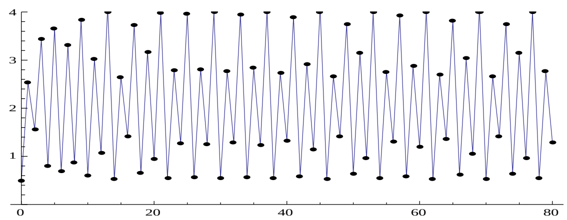
No of iterations $\rightarrow$

Fig 5.3: Time series graph for period 4 at the parameter $\mu = 3.95$



No of iterations $\rightarrow$

Fig 5.4: Time series graph for period 8 at the parameter $\mu = 4.95$



No of iterations $\rightarrow$

Fig 5.5: Time series graph showing chaotic behavior beyond accumulation point

6 .Lyapunov Exponent: [2, 5, 6]

In order to verify how much accurate is the accumulation point, the Lyapunov exponent is calculated. Lyapunov exponent at the parameter greater than the accumulation point is found to be positive whereas Lyapunov exponent less than the accumulation point is negative and at the accumulation point it should be equal to zero. We begin by considering an attractor point xo and calculate the Lyapunov exponent, which is the average of the sum of logarithm of the derivative of the function at the iteration points.

The formula may be summarized as follows:

Lyapunov exponent (μ)

$$=\lim_{n\to\infty}\frac{1}{n}\left(\log|f'(x_0)|+\log|f'(x_1)|+\log|f'(x_2)|+\log|f'(x_3)|+\ldots\ldots+\log|f'(x_n)|\right)$$

From graph of Lyapunov exponent ,we see that some portion lie in the negative side of the parameter axis indicating regular behavior (periodic orbits) and the portion lie on the positive side of the parameter axis confirm us about the existence of chaos for our model .[2]

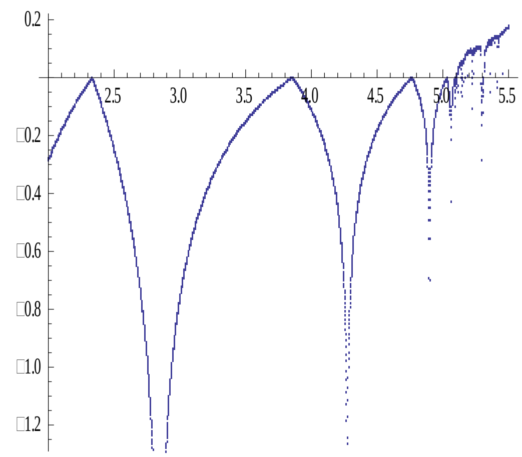


Fig:6.1: Lyapunov exponent of the map.Negative values indicate periodic.Almost zero values indicate bifurcation points and positive values indicate chaos.

7. Acknowledgement:

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