On Some Properties of Multiy -Hadamard Matrix

Haider.Baqer ameen

Department Mathematics of Faculty Mathematics and Computer Sciences of Kufa University

Abstract **- In this paper i might like bestowed some necessary definition and theorems on matrix theory , Kroncker product and Hadamard matrix . in final we tend to get some result once extention the kronker product for n sort from they Hadamard matrix**.

***Keywords*— Hadamard matrix , kronecker product , multy – Hardmarad matrix.**

INTRODUCTION

Hadamard matrices appear such straightforward matrix structures: they're sq., have, entries +1 or −1 and have orthogonal row vectors and orthogonal column vectors. however they need been actively studied for over 138 years and still have additional secrets to be discovered. during this paper we tend to consider engineering and applied math applications particularly those in communications systems, digital image process and orthogonal unfolding sequences for direct sequences spread spectrum code division multiple access. the Kronecker product has a number of an equivalent properties as typical matrix operation. each product follow an equivalent properties for multiplication with a scalar. Also, each product ar associative and that they share the distributive property with typical matrix operation ,[1],[2],[5]

1. Kronecker product :-

1.1. Definition [3]:

Let $A \in \mathbb{R}^{m\times n}$, and $B \in \mathbb{R}^{p\times q}$. then the kronecker product of A and B is defined as the matrix :

$$A \otimes B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix} & \cdots & a_{1n}\begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ a_{m1}\begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix} & \cdots & a_{mn}\begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{mp\times nq}$$

Note $A \otimes B \neq B \otimes A$

1.2. Some properties of the Kronecker product

Theorem (1.2.1):[3]

Let $A \in R^{m\times n}$, $B \in R^{r\times s}$, $C \in R^{n\times p}$,and $D \in R^{s\times t}$. Then

$$(A \otimes B)(C \otimes D) = AC \otimes BD(\in \mathbb{R}^{mr\times pt})$$

Theorem (1.2.2):[3]

For all A and B, Then $(A\otimes B)^T = A^T \otimes B^T$.

Theorem (1.2.3):[3]

If A and B are nonsingular , then $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$

Theorem (1.2.4):[3],[4]

If $A \in R^{n\times n}$, $B \in R^{m\times m}$ are normal, then $A \otimes B$ is normal

$(A\otimes B)^T(A\otimes B) = (A\otimes B)(A\otimes B)^T$

Corollary (1.2.5):[3],[4]

If $A \in R^{n\times n}$, $B \in R^{m\times m}$, then :

$tr(A\otimes B) = tr(A) . tr(B)$.

$\det(A\otimes B) = \det(A)^m . \det(B)^n$.

If A and B are square then $(A\otimes B)^n = A^n \otimes B^n$.

2. Hadamarad matrix:

Definition (2.1) : [4]

A Hadamard matrix, **H**n, is a square matrix of order n = 1, 2, or $4k^1$ where k is a positive integer. The elements of **H** are either +1 or –1 and $\mathbf{H_n H_n^T = nI_n}$, where $\mathbf{H_n^T}$ is the transpose of **H**n, and **I**n is the identity matrix of order n. A Hadamard matrix is said to be normalized if all of the elements of the first row and first column are +1.

Example (2.2)

The only normalized Hadamard matrices of orders one and two (i.e., of sizes $1 \times \mathbf{1}$ and 2×2) are:

$H_1=[1]$

$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

$H_4=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ or $H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}$

In general: $H_2^{k} = \begin{bmatrix} H_2^{k-1} & H_2^{k-1} \\ H_2^{k-1} & -H_2^{k-1} \end{bmatrix} = H_2 \otimes H_2^{k-1}$ where k=0,1,2

Lemma (2.3) : [4]

Let H_n be an Hadamard matrix of order n. Then**:**

a) $H_n = (H_n)^T$
b) $H_n.(H_n)^T = nI_n$, where I_n the identity matrix of order n
c) $\det(H_n) = (n)^{\frac{n}{2}}$ for $n \geq 4$.

2.1. Inverse Hadamarad matrix:

In this section we discuss the general method to get (inverse Hadamard matrix)

Theorem (2.1.1) : [4]

Let H_n be an Hadamard matrix of order n. Then the inverse Hadamard matrix be:

$$H_n^{-1} = \frac{1}{n} H_n$$

Proof :from proposition we get

$$H_n.H_n^{-1} = I_n$$

$$H_n.(H_n.H_n^{-1}) = H_n.I_n$$

$$(H_n.H_n).H_n^{-1} = H_n.I_n$$

$$(n\,I_n).H_n^{-1} = H_n$$

$$n.H_n^{-1} = H_n$$

$$H_n^{-1} = \frac{1}{n} H_n$$

3. **Some properties of the Hadamarad Matrix: (New Result)**

In this section we present some new theorems on hadmarad matrix

Theorem (3.1):

For all H_n and H_m are a hadamard matrix , Then $(H_n \otimes H_m)^T = (H_n)^T \otimes (H_m)^T$

Proof:

$(H_n \otimes H_m)^T = (H_{n\times m})^T$ ..let $(n \times m = c)$

$= (H_c)^T$

$= H_c$ (see lemma 3.3)

And $H_n{}^T \otimes H_m{}^T = H_n \otimes H_m$ …………..[since $H_n{}^T = H_n$]

$= H_{n\times m} = H_c$

For the proof, simply verify using the definitions of transpose and Kronecker product.

Theorem (3.2):

For all H_n and H_m are a hadamard matrix are nonsingular ,then

$$(H_n \otimes H_m)^{-1} = (H_n)^{-1} \otimes (H_m)^{-1}.$$

Proof:

$(H_n \otimes H_m)^{-1} = (H_{n\times m})^{-1}$ let $n \times m = c$

$= \frac{1}{c}(H_c)$ (by theorem 3.1.1)

And $(H_n)^{-1} \otimes (H_m)^{-1} = \frac{1}{n\times m}(H_n \otimes H_m) = \frac{1}{c}(H_c)$

For the proof, simply verify using the definitions of inverse and Kronecker product

Theorem (3.3):

For all H_n and H_m are a hadamard matrix are normal, then

$$(H_n \otimes H_m)^T (H_n \otimes H_m) = (H_n \otimes H_m)(H_n \otimes H_m)^T$$

Proof:

$(H_n \otimes H_m)^T (H_n \otimes H_m) = (H_{n\times m})^T (H_{n\times m})$

$= (H_c)^T (H_c)$ but $H^T = H$

$= (H_c)(H_c) = cI_c$

And $(H_n \otimes H_m)(H_n \otimes H_m)^T = (H_{n\times m})(H_{n\times m})^T$

$= (H_c)(H_c)^T$ but $H^T = H$

$=(H_c)(H_c) = cI_c$

Then $(A\otimes B)^T(A\otimes B) = (A\otimes B)(A\otimes B)^T$

4. multiy - Hadamarad matrix

In this section I would lo presented some theorems on extension definition hadamarad matrix to product n time for multiply hadamarad matrix .

Definition 4.1:

Let $H_n, H_m, \ldots, H_k$ are hardamard matrix difference order then $H_r = H_n \otimes H_m \otimes \ldots \otimes H_k$ is hadamarad matrix , where $r = n \times m \times \ldots \times k$.

Example 4.2 : let H_2 , H_4 and H_8 are hadamarad matrix then

$$H_2 \otimes H_4 \otimes H_8 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

$$H_2 \otimes H_4 \otimes H_8 = \begin{bmatrix} H_{32} & H_{32} \\ H_{32} & -H_{32} \end{bmatrix} = \begin{bmatrix} H_{16} & H_{16} & H_{16} & H_{16} \\ H_{16} & -H_{16} & H_{16} & -H_{16} \\ H_{16} & H_{16} & -H_{16} & -H_{16} \\ H_{16} & -H_{16} & -H_{16} & H_{16} \end{bmatrix}$$

And we can write :

$$H_2 \otimes H_4 \otimes H_8 = \prod_{i=1}^{6} \otimes_i H_2$$

Theorem 4.3: For all $H_n, H_m, \ldots, H_k$ are hardmard matrix , Then

$$(H_n \otimes H_m \otimes \ldots \otimes H_k)^T = H_n{}^T \otimes H_m{}^T \otimes \ldots \otimes H_k{}^T$$

Proof:

$(H_n \otimes H_m \otimes \ldots \otimes H_k)^T = (H_r)^T$

$$= H_r = H_n \otimes H_m \otimes \ldots \otimes H_k = H_n{}^T \otimes H_m{}^T \otimes \ldots \otimes H_k{}^T \blacksquare$$

Theorem 4.4: For all $H_n, H_m, \ldots, H_k$ are hardamard matrix are nonsingular ,

then $(H_n \otimes H_m \otimes \ldots \otimes H_k)^{-1} = H_n{}^{-1} \otimes H_m{}^{-1} \otimes \ldots \otimes H_k{}^{-1}$.

Proof:

$(H_n \otimes H_m \otimes \ldots \otimes H_k)^{-1} = (H_r)^{-1}$

$= \frac{1}{r}(H_r)$

$= \frac{1}{n \times m \times \ldots \times k}(H_n \otimes H_m \otimes \ldots \otimes H_k)$

$= (\frac{1}{n} H_n \otimes \frac{1}{m} H_m \otimes \ldots \otimes \frac{1}{k} H_k)$

$= (H_n)^{-1} \otimes (H_m)^{-1} \otimes \cdots \otimes (H_k)^{-1}$▪

Theorem 4.5 : Let H_n be an Hadamard matrix of order n. Then the inverse Hadamard matrix be:

$$H_n^{-1} = (\prod_{i=1}^{k} \otimes_i \frac{1}{m} H_m)$$

where $n = m \times m \times \ldots \times m(k - \text{time})$

proof:

$$(H_n)^{-1} = (H_m \otimes H_m \otimes \ldots \otimes H_m)^{-1} \qquad \text{where } n = m \times m \times \ldots \times m$$

$$= (H_m)^{-1} \otimes (H_m)^{-1} \otimes \cdots \otimes (H_m)^{-1}$$

$= \left(\frac{1}{m}(H_m) \otimes \frac{1}{m}(H_m) \otimes \ldots \otimes \frac{1}{m}(H_m)\right)$

$= (\prod_{i=1}^{k} \otimes \frac{1}{m} H_m)$▪

REFRENCES

[1] C. Cullen, “***An Introduction to Numerical Linear Algebra***” (PWS Publishing Company, Boston, MA, 1994).

[2] R Brualdi, “***Combinatorial Verification of the Elementary Divisors of Tensor Products***”, Linear Algebra Applications. (1985).

[3] RA. Hom and C.R Johnson, “***Matrix Analysis***” (Cambridge University Press, Cambridge, 1985).

[4] J. Seberry and M. Yamada, “***Hadamard matrices, sequences and block designs. surveys in contemporary design theory***,” in Wiley-Interscience Series in Discrete Mathematics. Jhon Wiley, New York, 1992.

[5] Jennifer Seberry & B. J. Wysocki “***On some applications of Hadamard matrices***” Metrika, 62, (2005) 221-239. Copyright Springer-Verlag.