Asymptotic Stability of Testable Measure Differential Equations[1]

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Abstract- In this paper, we consider sufficiency conditions for the stability of trivial solutions of measure differential equations. The bound for $\prod_{k=1}^{\infty} B_k^{-1}$ in the Pandit's problem is estimated in a systematic way and used to establish criteria for the stability of trivial solutions. Results on asymptotic stability for testable perturbed measure differential equations are obtained using some growth properties, Pandit inequality and Brascamp-Lieb inequality. An example is used to illustrate the application of results obtained.

Keywords: stability, testable system, measure differential equations and impulses.
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[1]Dedicated to evergreen memory of one of my PhD supervisors and Co-Researcher ,late Professor Maligie Sasay

I. INTRODUCTION

Physical and biological systems governed by ordinary differential equations (odes) are often assumed to be continuous([1],[2]&[4]).In practice, it has been discovered that some models in applied sciences such as control theory, economics and engineering are not always continuous as often assumed. The state of a system may sometimes be discontinuous and the solution characterized by short time perturbations (impulses) whose actions are negligible when compared to the total time taken by the whole process ([4], [6], [7], [12].& [13]).

The study of impulsive systems through functions with bounded variation or measure differential equations as it is often called is essentially aimed at providing a broad framework for studying impulsive systems using distribution theory. This theory is more general than that of ordinary differential equations (o.d.e.). Hence, measure differential equations (m.d.e.) circumvent some shortcomings in the (o.d.e.) (see [10], [12], [13] & [17]).

Measure differential equations, primarily was motivated as a field of study from pulse frequency modulation models for biological neutral nets and automatic control systems ([8], [11], [12], [14] and [17]). In spite of appreciable advances made in the field of impulsive differential equations, there are still much needed to be studied in the field. For instance, several analogues of ode models are now available but few are present in the literature on measure differential equations.

In the present paper, the crux of our study is to investigate the asymptotic stability of some testability measure differential equations using an analogue of Gronwall-Bellman inequality due to Pandit ([16] & [17]) and Brascamp-Lieb inequality (see [3]). We will obtain sufficiency conditions for asymptotic stability of the m.d.e. and it applied to an example.

II. PRELIMINARY DEFINITIONS AND NOTATIONS

Let R^n denote a n dimensional Euclidean space with norm

$$\|x\| = \max_{1\le i\le n}(|x_i|)$$

Let $u(t) = t + \sum_{k=1}^{\infty} a_k H_k(t), H_k(t) = \{\begin{matrix} 1 \text{ if } t < t_k \\ 0 \text{ if } t \ge t_k \end{matrix}$

Let $M_n(J)$ be the set of $n \times n$ matrices defined on $J = [0, +\infty)$ with the norm

$$\|x\|_{M_n(J)} = \sum_{i=1}^{n}\sum_{j=1}^{n}|x_{ij}|, \ \forall\ x \in M_n(J)$$

Furthermore, the following notation will be adopted: $V(J, R^n)$ will represent the set of vector-valued functions defined on J and whose components are of bounded variation.
The set

$$V_f = \sup\left\{ \begin{matrix} \sum_{k=1}^{n}|f(x_k) - f(x_{k-1})| : x_k \in P(x_0, x_k), \\ P(x_0, x_k) \text{being the partition of } J = [0, +\infty) \text{for } k = 1, 2, \ldots, 3 \end{matrix} \right\}$$

is finite for every $f \in V(J, R^n)$.
The space of bounded variation (BV) will be represented as $BV(J) = BV(J, R^n)$. This is a Banach space with the norm

$$\|x\|^* = V(x, J) + \|x(0+)\|, \ x \in BV(J)$$

Where

$$V(x, J) = \sup_{x\ge 0} V_f$$

is the total variation of f on J as defined above. $GL_n(J)$ will represent the class of $n \times n$ invertible matrices on J . $SL_n(J)$ denotes a subclass of $GL_n(J)$ defined as

$$SL_n(J) = \{A \in GL_n(J) : \det A = 1\}$$

$\lambda_i(A)$ is the set of eigenvalues of the matrix $A \in M_n(J)$ .Furthermore, $CL(\cdot)$ will represent the closure of the set $\{\cdot\}$ and finally.

$$R_H^n = \{x \in R^n : \|x\| < H = \}.$$

II.A Distribution and their Derivatives

The formulation of measure differential equations relies on distribution derivatives. Therefore, there is the need for us to give a preliminary treatment on the distribution theory relevant to our research work.

Let $C_c^\infty(J)$ be a linear space of a class of infinitely partially differentiable complex valued functions defined on the set J which has a compact support i.e., the set

$$Supp\ V = CL\{X \in R^n : \psi(x) \neq 0\} \quad (1)$$

is compact. The support of $\psi(x)$ is the smallest closed set on R^n outside of which the function $\psi(x)$ vanishes. It's canonical norm being

$$\|\psi(x)\|_{C_c^\infty(\Omega)} = \sup_{x\in\Omega}|\psi(x)| < +\infty, \quad \forall\ \psi \in C_c^\infty(\Omega).$$

The space $C_c^\infty(\Omega)$ is called the space of Test functions and it is a normed linear space. Any continuous linear functional on $C_c^\infty(\Omega)$ is what we shall henceforth call distributions, that is, the dual space of continuous functions on $C_c^\infty(\Omega)$.

The distribution on J can be identified with the Lebesque-Stieltjes' measure $d\psi(t)$, on the closed interval $J_0 = [0, T] \subset J$ and

$$F(\phi) = \int_{J_0} \phi(s) d\psi(s), \forall \phi \in C_0^\infty(J_0), \psi \in BV(J_0, R^n) \quad (2)$$

Definition 1

The derivative DF of a distribution F on J is the distribution defined as

$$DF(\phi) = -F^{/}(\phi) \quad (3)$$

Where

$$/ \equiv \frac{d}{dx}$$

The notation D will be adopted for distribution derivative.

Remark 1

The space of distribution on J is a matrisable complete locally convex topological space (LCTS) with the topology

$$P_{M,K}(\phi) = \max_{x \in K} \sup_{|r| \le m} |\partial^r \phi(x)|, \forall \phi \in C_0^\infty(J_0) \quad (4)$$

Where K is a compact subset of Ω, $r = (r_1, r_2, \ldots, r_n) \in R^n$, $|r| = \sum_{i=1}^{m} |r_i|$, m is a positive integer and the multi-index derivative, ∂^r is defined as $\partial^r = \frac{\partial^r}{\partial x^r} = \frac{\partial^r}{\partial x_1^r, \partial x_2^r, \ldots, \partial x_m^m}$.
The dual space of $C_c^\infty(\Omega)$ with the topology defined in eq. (4) is a Frechet space. For definiteness, we restrict our study to some subset of this space.

With this background information on distribution theory, we may now proceed to formulate the measure differential equations. For further insight into distribution theories refer to [Shilov and Gurevich [18], Taylor [19], Treves [20] and Yosida [21]).

III. STATEMENT OF THE PROBLEM

Consider the measure differential equations

$$Dx(t) = Ax(t) + f(t, x(t)) + g(t, x(t))Du \quad (5)$$

Where $A \in M_n(J)$, $f, g \in BV(J, R^n)$ and $x \in R^n$.
$u : J \to R^n$ is a right continuous functions with bounded variation on every compact subinterval $J_0 = [0, T] \subset J$; $f : J \times R_H^n \to R^n$ is a Lebesque integrable function and $g : J \times R_H^n \to R^n$ is integrable with respect to Lebesque-Stieltjes' measure du.
The function $u(t)$ is assumed to be impulsive for an increasing sequence of times $\{t_i\}$, $i = 1, 2, \ldots$ .such that $0 < t_1 < t_2 < \ldots < t_k, \lim_{k \to \infty} t_k = \infty$.

It will also be assumed that $u(t)$ is right continuous and of bounded variation. Moreover, its discontinuities at the impulsive points $\{t_i\}$ are isolated. It was shown in ([16] & [17]) that a necessary and sufficient condition for $x(t) = x(t, t_0, x_0)$ to be a solution of eq. (5) passing through (t_0, x_0) is that the integral equation

$$x(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x(s))ds + \int_{t_0}^{t} [Ax(s) + g(s, x(s))]du(s) \quad (6)$$

Must be satisfied.

Remark 2
If $u \equiv 0$ in eq. (5) then the whole study reduces to that of the classical perturbation in ordinary differential equations. In the monograph ([17], pp. 12) Pandit and Deo presented results that showed that the solution of the (m.d.e.) exists and is unique. Similar results were established by Pandit [16].

Definition 2

The set of impulsive points $\{t_i\}_1^\infty$ in the subinterval $[t, t+w]$, $w > 0$ of J for $i = 1, 2, \ldots$ is called a counting number. We will denote this by $\#(t, t+w)$ and

$$\lim_{w\to\infty} \frac{\#[t, t+w]}{w} = K^* \tag{7}$$

K^* Will be called a counting ratio if the limit in eq. (7) exists and is finite. If this condition is satisfied we shall call the system in the eq. (5) a testable measure differential equation. If otherwise, we refer to the system as attestable measure differential equation.

Definition 3

The zero solution $x \equiv 0$ of the system in the eq. (5) is called:

(a) Stable if $(\forall t \geq 0)(\forall t_0 \in J)(\exists \delta = \delta(t_0, \varepsilon) > 0)$

$$\|x(t, t_0, x_0)\| < \epsilon;$$

(b) Uniformly stable if the number in (a) does not depend on $t_0 \in J$

(c) Attractive if

$$(\forall\ t_0 \in J)\ (\exists \lambda = \lambda(t_0) > 0)\ \ (\forall\ \epsilon > 0)$$

$$(\forall\ t \geq t_0 + \sigma : \|x(t, t_0, x_0)\| \leq \epsilon;$$

(d) Equi-attractive if the number σ from (c) does not depend on $x_0 \in R_H^n$;

(e) Asymptotically stable if it is stable and attractive.

In this paper, asymptotic stability will be studied relation to the zero solution since stability is invariant with respect to non singular change of the equilibrium point.

Consider the prototype (m.d.e.)

$$Dx = AxDu \tag{8}$$

Where A and u satisfy the foregoing conditions stipulated in the preliminaries. Furthermore, let

$$B_k = I - \alpha_k A \qquad k = 1, 2, \ldots,$$

$$(9)$$

$$B_k \in GL_n(J) \qquad I \in SL_n(J).$$

That is, the multipliers α_k are not eigenvalues of A . The unique solution of the eq. (8) is

$$x(t) = \prod_{k=1}^{n-1} B_k^{-1} e^{(t-t_0)} A x_0; x(t_0) = x_0$$

(See [14])

Remark 3

If $\lambda_i(A) = \alpha_k^{-1}$, $\alpha_k \neq 0$ for $k = 1,2,\ldots$. Then the set $(\lambda_i I - A)^{-1}$ is the resolvent of $A \in M_n(J)$ and it is analytic if $\lambda \in \lambda_i(A)$ for $i = 1,2,\ldots$.

Definition 4

The spectrum of $A \in M_n(J)$ is defined as

$$Spec(A) = \max_i |\lambda_i(A)|, \quad i = 1,2,\ldots,n.$$

III.B Auxiliary Results

We state without proofs the following auxiliary lemmas:

Lemma 1 (see Pandit [14])

Let $g(t,x(t))$ be a non-negative function and integrable on J is the sense described above. Suppose that f is non-negative and locally integrable on J such that $\lambda_k f(t_k) < 1$ for such $x \geq 1$. Suppose further that the series $\lambda_k f(t_k)$ converges absolutely. If

$$x(t) \leq C + \int_{t_0}^{t} g(s,x(s))ds + \int_{t_0}^{t} f(s,x(s))du(s) \qquad t \in J$$

Then

$$x(t) \leq P^{-1} C \exp\left(\int_{t_0}^{t} f(s)g(s)u'(s)\right) \qquad t \in J$$

Where

$$P = \prod_{k=1}^{\infty}(1 - \lambda_k f(t_k))$$

We will consider the Brascamp-Lieb inequality which is a powerful extension of Holders' inequality for $1 \leq i \leq m$.

Lemma 2 (Brascamp-Lieb inequality) (See [3])

Let $1 \leq i \leq m$ and $B_i : R^n \to R^d$ be subjective map and $c_i \epsilon [0,1]$. the best constant $k \epsilon [0,+\infty)$ such that the inequality

$$\int_{R^n} \Pi_{i=1}^{m} f_i(B_i x)^{c_i} dx \leq k \Pi_{i=1}^{m} \left(\int_{R^n} f_i dx\right)^{c_i}$$

Holds for all integrable functions $f_i : R^n \to J$ can be computed by taking any centered Gaussian function $f_i(x) = \exp -< A_i x, x >$

Where A_i 's are symmetric positive definite matrices of size D and $k = D^{-1/2}$.

$$D = \inf_{A_i > 0} \frac{\det\left(\sum_{k=1}^{n} C_i B_i^* A_i B_i\right)}{\Pi_{i=1}^{n} \det(A_i)^{C_i}}$$

Where B_i^* is adjoint matrix to B_i.

IV. Main Results

We will estimate the bounds for $\prod_{k=1}^{n-1} B_k^{-1}$ and consequently relative it to the estimation of the solution of eq. (8) and this will be followed by the asymptotic stability theorems.

Lemma 3

$$\left\| \prod_{k=1}^{n-1} B_k^{-1} \right\| \leq c e^{\beta n_s}, \ c \geq 1$$

Where

$$\beta = \max_{1 \leq i \leq n} Re \lambda_i(k)$$

And

$$Q_s := \sum_{k=1}^{n-1} B_k^{-1} \tag{10}$$

Proof

$$B_k = I - \alpha_k A$$

Thus

$$B_k^{-1} = (I - \alpha_k A)^{-1} \qquad B_k \in GL_n(J)$$

Therefore,

$$\left\|\prod_{k=1}^{n-1} B_k^{-1}\right\| = \left\|\prod_{k=1}^{n-1} (I-\alpha_k A)^{-1}\right\|$$
$$\le \prod_{k=1}^{n-1} \| (I-\alpha_k A)^{-1} \|$$
$$\le \left\| e^{\sum_{k=1}^{n-1} \alpha_k A} \right\|$$
$$\le ce^{Q_n} \text{ (by Lemma 5.1) in [12]}$$

Remark 4

In [12] it was demonstrated that the sum of the multipliers, Q_s is related to the counting ratio by the relation $a_s = \beta + K^* In\alpha$, $\alpha > 0$ and even showed that $\|\prod_{k=1}^{n-1} B_k^{-1}\|$ gives the bound for $\alpha^{\#(t_0,t)}$, we will generalize Lemma 1 using a stronger condition that the matrix B_k in eq. (9) is not a sparse matrix.

Theorem 1

Suppose that B_k for $k = 1, 2, \ldots,$ is a matrix which is not sparse such that there exists a constant $N > 0$ such that $0 < |P_k(A)| < 1 \le \|A\| \le N$, $N < +\infty$ for $k = 0, 1, 2, \ldots$.

$P_k(A)$ is the characteristic polynomial for matrix A

Then

$$\left\|\prod_{k=1}^{\infty} B_k^{-1}\right\| \le Ce^{S_k^{(n)} \|B_k\|^n}$$
$$\le Ce^{S_k^{(n)} N^n}, \ C \ge 1.$$

Where

$$S_k^{(n)} = \sum_{k=1}^{\infty} \alpha_k^n; \qquad n = 1, 2, \ldots$$

And

$$S_{mk}^{(n)} = \sum_{k=1}^{m} \alpha_k^n, \quad m = 1, 2, \ldots, n; \quad S_{mn}^{(n)} \to S_k^{(n)} \qquad n \to \infty.$$

Proof

$$B_k^{-1} = (I - \alpha_k A)^{-1}$$

Let

$$Q_s = \lim_{m\to\infty} \sum_{k=0}^{m} (I - \alpha_k A)^{-1}.$$

Then

$$\begin{aligned} Q_s &= \lim_{m\to\infty} \sum_{k=0}^{m} \log_e (I - \alpha_k A)^{-1} \\ &= \lim_{m\to\infty} \sum_{0}^{m} \left\{ \alpha_k A + \alpha_k^2 \frac{A^2}{2} + \frac{\alpha_k^3}{3} A^3 + \cdots \right\} \quad (\, Spec(A) < 1) \\ &= S_k^{(1)} A + S_k^{(2)} \frac{A^2}{2} + O\left(S_k^{(3)} \frac{A^3}{3} \right) \end{aligned}$$

Let $\tilde{J}$ be $n \times n$ matrix in Jordan canonical form with the property that

$$A^k = \tilde{J} P^k \tilde{J}^{-1}, \; k = 0, 1, 2, \ldots$$

And

$$J = diag(\tilde{J}_0, \tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_n),$$

That is, quasi-diagonal is $\tilde{J}_1$ for $i = 1, 2, \ldots, n$, such that $\tilde{J}_i = \lambda_i I_i + N_i$ .N_i Being $n \times n$ nilpotent matrix and I_i is $n \times n$ identity matrix.
Hence

$$\log_e Q = \tilde{J} S_k^{(0)} \tilde{J}^{-1} + \tilde{J} S_k^{(1)} P \tilde{J}^{-1} + \tilde{J} S_k^{(2)} \frac{P^2}{2} \tilde{J}^{-1} + O(\tilde{J} S_k^{(3)} \frac{P^3}{3!} \tilde{J}^{-1}).$$

Define

$$S_k(P) = \sum_{r=1}^{\infty} S_k^{(r)} \frac{P^r}{r}$$

Such that the sequence $(S_k^{(r)})$ is convergent in some compact subset J^* of J and $Spec(P) < 1$, then

$$Q = \tilde{J} e S_k^{(r)}(P) \tilde{J}^{-1}$$

Hence,

$$\|Q\| \leq \|\tilde{J}\|\,\|\tilde{J}^{-1}\left\|\prod_{k=1}^{\infty}\tilde{J}e^{S_k^{(r)}(P)}\tilde{J}^{-1}\right\|$$
$$\leq C_1C_2e^{S_k^{(n)}}N^n$$

Where

$$C_1 = \|\tilde{J}\|\,\|\tilde{J}^{-1}\| \geq 1$$

And

$$C_2 = \max_i(|C_i|) < +\infty \quad i = 1,2,\ldots,$$

Therefore,

$$\left\|\prod_{k=1}^{\infty}e^{S_k^{(r)}PJ}\right\|C_ke^{S_k^{(n)}\|P\|^n}$$

For $k = 1,2,\ldots,$
Hence

$$\|Q\| \leq C_1C_k\exp(S_k^{(n)}N^n); \quad C_k \coloneqq C_1C_2\ldots C_k.$$

Theorem 2
The trivial solution, $x(t,t_0+0,x_0) = 0$ if eq. (6) is asymptotically stable if

$$\beta = \max Re\lambda_i(A) < 0 \quad \forall\ t \geq t_0 - a_s$$

Proof
The solution of eq. (9) is

$$x(t) = x(t,t_0,x_0) = \prod_{k=1}^{n-1}B_k^{-1}e^{(t-t_0)A}x_0$$

Hence by Lemma 5.1 in [12], we have the estimate

$$\|x(t,t_0,x_0)\| \leq \left\|\prod_{k=1}^{n-1}B_k^{-1}\right\|\|e^{(t-t_0)A}\|\,\|x_0\|$$
$$\leq C_1e^{\beta}C_2e^{\beta(t-t_0)}\|x_0\|,$$
$$= C_1C_2\|x_0\|e^{\beta(Q_s+t-t_0)},\ C_1,C_2 \geq 1.$$

Thus given $\epsilon > 0$ and set $\sigma = \ln\left(\frac{\varepsilon}{C_1C_2\delta}\right) - Q_s$. Therefore $\exists\ \delta > 0$ such that whenever

$\|x_0\| < \delta$ it implies that $\|x(t,t_0,x_0)\| < \epsilon,\ \forall\ t \geq t_0 + \sigma$ .Hence $x(t,t_0,x_0) = 0$ is attractive and even stable by Definition 3 therefore $x \equiv 0$ is asymptotically stable.

Remark 5

In fact $x(t,t_0,x_0) = 0$ is equi-attractive since $\delta(t,x_0,\epsilon)$ is independent of x_0 . Moreover, $\|x(t,t_0,x_0)\| \to 0$ as $t \to \infty$.

Next we consider a Theorem which gives a sufficient condition for asymptotic stability of perturbed measure differential equations.

Theorem 3

Consider the (m.d.e.)

$$Dx = AxDu + f(t,x(t))(12)$$

Where $A\epsilon M_n(J)$ .Suppose that following conditions are also satisfied:

There exists a constant such that

H1: $\beta = \max_{1\leq i\leq n} Re\lambda_i(A) < 0$

H2: $\|f(t,x(t))\| \leq \tilde{\epsilon}\|x(t)\|$ for some small real number $\tilde{\epsilon} > 0$.

Then the trivial solution $x \equiv 0$ of eq. (12) is asymptotically stable.

Proof

The solution of the eq. (12) is

$$x(t,t_0,x_0) = \prod_{k=1}^{n-1} B_k^{-1} e^{(t-t_0)A} x_0 + \int_{t_0}^{t} \prod_{k=1}^{n-1} B_k^{-1} e^{(s-t_0)A} f(s,x(s))ds \quad (13)$$

$$\psi(t,t_0,x_0) \overset{def}{=} x(t,t_0,x_0) e^{\beta(Q_s - t - t_0)} \quad (14)$$

Then

$$\|\psi(t,t_0,x_0)\| \leq C\|\psi_0\| + \int_{t_0}^{t} \varepsilon C\psi(s)ds \quad (15)$$

Using the Gronwall-Bellman inequality, we get

$$\|\psi(t,t_0,x_0)\| \leq C\|\psi_0\| e^{\varepsilon C(t-t_0)} \quad (16)$$

Hence

$$\|x(t,t_0,x_0)\| \leq Ce^{\beta a_s}\|x_0\|e^{C\bar{\epsilon}(t-t_0)}e^{\beta(a_s+t-t_0)}$$
$$= C\|x_0\|e^{(C\bar{\epsilon}+\beta)(t-t_0)}$$

The proof follows immediately if we choose

$$\sigma = 1+\frac{1}{C\epsilon-\beta}\ln\left(\frac{\epsilon}{c\delta}\right) \qquad C\epsilon-\beta>0.$$

This ends the proof.

Corollary 1

Let all the conditions in the Theorem 3 be satisfied except H2 which is replaced by

$$f(t,x(t)) = \iint_{R^n} p(t)\Pi_{i=1}^{m} f_i(K_i x)^{\alpha_i}dx$$

Where $p(t)$ is a polynomial of degree n in t and there exists a constant f_0 such that $|\Pi_{i=1}^{m} f_i(x)^{\alpha_i}| = f_0$.

Then the trivial solution to the eq. (12) is asymptotically stable.

Proof

Just like the Theorem 3 by setting $p_0 = \max|p(t)|$ and applying the Brascamp-Lieb inequality to get

$$\|\psi(t)\| \leq C\|\psi_0\| + \int kp_0|\Pi_{i=1}^{m} f_i{}^{\alpha_i}|\|\psi(s)\|ds$$

Where $\psi(t)$ is define in the eq. (14).Application of Gronwall's inequality leads to

$$\|x(t,t_0,x_0)\| \leq C\|x_0\|e^{(C\bar{\epsilon}+\beta+p_0 f_0 k)(t-t_0)} .$$

Therefore if we choose $\sigma = 1+\frac{1}{C\bar{\varepsilon}+\beta+p_0 f_0 k}\ln\left(\frac{\varepsilon}{C\delta}\right)$ and hence the proof.

Remark 6

The application of Gronwall-Bellman inequality poses no difficulty in the eq. (15) to get eq. (16). Suppose we replace the eq. (12) by

$$Dx = Ax + f(t,x)Du \qquad (17)$$

In this case, Gronwall-Bellman inequality is inapplicable. In this situation, it becomes necessary to use the Lebesque-Stieltjes' integrable version of the Gronwall-Bellman inequality (GBI).

We will present a measure differential equation analogue of (GBI) due to Pandit ([16], see Lemma 4.2) and subsequently apply it to obtain stability theorems later.

Remark 7

The condition that $\sum_{k=1}^{\infty} \lambda_k f(t_k)$ converges absolutely guarantees the absolute convergence of

$$P = \prod_{k=1}^{\infty}(1 - \lambda_k f(t_k)),$$

We will indeed demonstrate this assertion in Lemma 4. Let us now derive the scalar analogue of the bound for $\prod_{k=1}(1 - \lambda_k \tilde{\epsilon})$.

Lemma 4

Let

$$P^{-1} = \prod_{k=1}^{\infty}(1 - \lambda_k f(t_k)) \qquad \lambda_k f(t_k) < 1 \text{ and } \sum_{k=1}^{\infty} \lambda_k f(t_k)$$

is absolutely convergent.

Define $\tilde{\tau}(\epsilon) = e^{\epsilon a_s^{(1)}} \cdot e^{\tilde{\epsilon}^2/2 \cdot a_s^{(2)}}$ and let

$$a_s^{(n)} = \sum_{k=1}^{\infty} \lambda_k^n$$

Then $P^{-1} \sim e^{-\tilde{\epsilon} a_s^{(1)}} e^{-\tilde{\epsilon} a_s^{(2)}} = T(\tilde{\epsilon})$.

Proof

From $P^{-1} = \prod_{k=1}^{\infty}(1 - \lambda_k \tilde{\epsilon})$ we have

$$\log_e P^{-1} = \lim_{k \to 0} \sum_{k=1}^{m} \log_e (1 - \lambda_k \tilde{\epsilon})$$

Since $\lambda_k \epsilon > 1$ for $k \in \{1, 2, \ldots\}$ then the power series of $\log(1 - \lambda_k \tilde{\epsilon})$ is absolutely convergent and hence

$$\sum_{k=1}^{\infty} \log(1 - \lambda_k \tilde{\epsilon})$$

But

$$\log(1 - \lambda_k \tilde{\epsilon}) = -\lambda_k \tilde{\epsilon} - \frac{\lambda_k^2}{2} \tilde{\epsilon}^2 - \frac{\lambda_k^2}{3} \tilde{\epsilon}^3 - \cdots -.$$

Hence,

$$\sum_{k=1}^{\infty} \log(1-\lambda_k\tilde{\epsilon}) - \tilde{\epsilon}\sum_{k=1}^{\infty}\lambda_k - \frac{\epsilon^{-1}}{2}\sum_{k=1}^{\infty}\lambda_k^2 - \frac{\tilde{\epsilon}^3}{3}\sum_{k=1}^{\infty}\lambda_k^3 - \cdots -$$

Since $\tilde{\epsilon}$ is an arbitrarily small real number by hypothesis, we may neglect the term $\frac{\epsilon^3}{3}a_s^{(3)}$ and higher terms. Thus

$$\log_e P^{-1} \sim -\tilde{\epsilon}a_s^{(1)} - \frac{\tilde{\epsilon}^2}{2}a_s^{(2)}$$

Or

$$P^{-1} \sim e^{-\tilde{\epsilon}a_s^{(1)}} \bullet e^{\frac{\tilde{\epsilon}^2}{2}a_s^{(2)}} = T(\tilde{\epsilon})$$

This ends the proof.

Theorem 4
Consider the measure differential equations

$$Dx(t) = Ax(t)Du + f(t,x(t))dt + g(t,x(t)) \tag{18}$$

Where $A \in M_n(J)$, the vector-valued functions, $f(t,x(t))$ and $G(t,x(t))$ are integrable in Lebesque-Stieltjes and Lebesque sense respectively for $x \in R^n$.

Suppose further that the following conditions are also satisfied:

H1: $Re\max_{1\leq i\leq n} \lambda_i(A) < 0$;
H2: $\|G(t,x(t))\| \leq g(t)\|x(t)\|$
Where $g(t)$ is a non-negative function for $t \in J$ and it also satisfies the property that

$$\lim_{t\to\infty} \sup \int_t^{t+1} g(s)u'(s) = 0$$

H3: $\|f(t,x(t))\| \leq \tilde{\epsilon}\|x(t)\|$ Where $\tilde{\epsilon}$ is an arbitrary small real number.

Then the trivial solution $x \equiv 0$ of eq. (18) is asymptotically stable.

Proof
The solution $x(t)$ of the eq. (18) is

$$x(t) = \prod_{k=1}^{\infty} B_k^{-1} e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(s-t)} G(s,x(s)) du(s)$$

$$+ \int_{t_0}^{t} \prod_{k=0}^{\infty} B_k^{-1} e^{A(s-t)} f(s,x(s)) ds \qquad (19)$$

$$\|x(t)\| \le C_1 e^{a_s} e^{(t-t_0)} \|x_0\| + \int_{t_0}^{t} C_1 e^{\alpha(s-t)} g(s) \|x(s)\| du(s)$$

$$+ \int_{t_0}^{t} \tilde{\epsilon} C_1 e^{a_s} e^{\beta(s-t)} \|x(s)\| ds$$

Let $\psi \overset{def}{=} x(t) e^{-\beta(t-t_0)}$ where $\beta = Re \max_{1 \le i \le n} \lambda_i(A)$ and $C_1 \ge 1$.
Then

$$\|\psi\| \le C_1 e^{Q_s} \|\psi_0\| + C_1 \int_{t_0}^{t} g(s)\psi(s) du(s) + \int_{t_0}^{t} \varepsilon \psi(s) ds \quad (20)$$

Without loss of generality (w.l.g.) we can choose $C_1 = 1$ .Then, by Lemma 2, we have the estimation

$$\|\psi\| \le e^{a_s} \|\psi_0\| P^{-1} \exp\left(\int_{t_0}^{t} \tilde{\epsilon} g(s) u^{/}(s)\right) \quad t \in J$$

Where $P^{-1} = \prod_{k=1}^{\infty}(1 - \lambda_k \tilde{\varepsilon})$ has the property that $\tilde{\epsilon}\lambda < 1$.

Whence

$$\| x(t) \| \le e^{Q_s} e^{\mathrm{Re}\, A(t-t_0)} \|\psi_0\| P^{-1} \exp\left(\int_{t_0}^{t} \varepsilon g(s) u^{/}(s)\right) \quad (21)$$

Now, by condition H2. in the hypothesis

$$\int_{t_0}^{t} g(s) u^{/}(s) = \left(\int_{t_0}^{t^*} + \int_{t^*}^{t^*+1} \right) g(s) u^{/}(s)$$

For which t^* exists in such a way that $t^* < t \in J$. Therefore, the integral $\int_{t_0}^{t} g(s) u^{/}(s)$ can be evaluated as

$$\int_{t_0}^{t} g(s) u^{/}(s) = g(t^*) M(t^*, t_0)$$

Where

$$M(t^*, t_0) = u(t^*) - u(t_0).$$

Now let $g(t^*), M(t_1, t^*) = b < +\infty,$ then by Lebesque-Stieltjes integrability condition imposed on $g(s)$, the second integral

$$\int_{t^*}^{t^*+1} g(s)u^{/}(s) \to 0 \qquad t^* \longrightarrow \infty.$$

Thus P^{-1} is convergent since it is an infinite product of an absolutely convergent sequence real numbers $(1-\lambda_k\tilde{\epsilon})$ $k = 1,2,\ldots.$ where $\lambda_k\tilde{\epsilon} < 1$ .Hence, we can majorize P^{-1} by some real number $T(a_s)$ (See Lemma 3).

Now given $\epsilon > 0$ choose

$$\delta(t,\varepsilon,x_0) = \log_e(\tilde{\varepsilon}\tau^{-1})^{1/\beta} - a_s\beta^{-1} - \tilde{\varepsilon}^b\beta^{-1} \qquad (22)$$

Then we can find a real number $\delta > 0$ such that if $\|x_0\| < \delta$. Then $\|x(t)\| < \epsilon$ for $t \geq t_0 + \delta$ .For the proof of this claim, we proceed as follows: If $\|x_0\| < \delta$ then $\forall\ \epsilon > 0$

$$e^{a_s} e^{\operatorname{Re}\lambda_i(A)(t-t_0)} \|x_0\| P^{-1} \exp\left(\int_{t_0}^{t} \tilde{\epsilon} g(s)u^{/}(s)\right)$$

$$< \delta\tau e^{a_s} e^{\alpha(t-t_0)} e^{\tilde{\epsilon}b} < \epsilon$$

Therefore,

$$e^{\operatorname{Re}\lambda_i(A)(t-t_0)} \leq \epsilon\lambda^{-1}\tau^{-1}e^{-a_s-\tilde{\epsilon}b}$$

But $\beta < 0 \leq 1$ by hypothesis, thus

$$\beta(t-t_0) \geq \log_e[\epsilon\delta^{-1}\tau^{-1}] - a_s - \tilde{\epsilon}b$$

i.e.

$$t \geq t_0 + \log_e(\epsilon\delta^{-1}\tau^{-1})^{1/\beta} - a_s\beta^{-1} - \tilde{\epsilon}b\beta^{-1} = t_0 + \sigma.$$

This ends the proof.

Theorem 5

Consider the intergro-measure differential equations

$$Dx = Ax + f(t, x(t)Du + \int_{t_0}^{t} k(s,t)x(s)ds \qquad (23)$$

Assume that

H1: There exist $k_0 > 0$ and $\alpha > 0$ such that

$$|k(s,t)| \leq \frac{k_0}{s-t} e^{-\alpha(s-t)}$$

H2: For every $\varepsilon > 0$ and $\delta > 0$ such that

$$|f(t,x(t))| \leq \epsilon |x(t)|$$

H3: There exist $P, P^{-1} \epsilon GL_2(J)$ such that

$$e^{A(t-t_0)} = Pe^{J(t-t_0)}P^{-1}$$

$$\text{And } \quad |e^{J(s-t)}| \leq e^{-\lambda(s-t)}$$

Then the trivial solution to the eq. (23) is asymptotically stable for $0 < t < s$.

Proof

It is not difficult to show that

$$|x(t)| \leq Ce^{-a_S(t-t_0)}|x_0| + \int_{t_0}^{t} CPe^{J(s-t_0)}P^{-1}f(s,x(s))ds$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s} \Pi_{k=1}^{\infty} B_k^{-1} e^{-a_S(t-t_0)} \frac{k_0}{s-t} e^{-\alpha(s-t)} ds$$

And clearly if we let $\beta = \max_{1 \leq i \leq n} Re\lambda_i(A) < 0$ and $|f(t,x(t)| \leq \tilde{\epsilon}\, |x(t)|$, $\lim_{s\to\infty} \frac{e^{-\alpha(s-t)}}{s-t} = 0$ then the trivial solution to the eq. (23) is asymptotically stable for $0 < t < s$. Hence the proof.

Example

Consider the measure differential equation

$$Dx = AxDu + f(t,x) \qquad \text{(E1)}$$

$$A = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

Where

$f(t,x) = (f_1, f_2, f_3, f_4, f_5, f_6)^T, T$ is the transpose of the vector $f(t,x)$.

$$f_1 = 0.78e^{-2t}x_1 - 0.80(2t+1)^{-1}x_2 - 0.23x_3$$
$$+0.20(t^2+2)^{-1}x_4 - 0.30x_5 + 0.20x_6 + d_1.$$

$$f_2 = 0.2(t^4+2t^2+1)^{-1}e^{-\cos t}x_1 - 0.89e^{-2t}x_2 - 0.12e^{-t}x_3$$
$$+0.23x_4\cos 2t + 0.12\sin e^{t}x_5 + 3.2x_6 + d_2.$$

$$f_3 = 0.90x_1 + 0.20e^{-2t}(2t+1)^{-1}x_2 + 0.80x_3$$
$$+0.50\cos t^3(\tfrac{1}{1+t^2})x_4 + 0.20x_5 + 0.06x_6 + d_3.$$

$$f_4 = 0.25e^{-x_1}x_1 + 0.20x_2 + 0.50x_3$$
$$+0.80x_4 + 0.80\sin e^{t}x_5 + 0.78e^{-t^2}x_6 + d_4.$$

$$f_5 = 0.50e^{-\cos t}x_1 + 0.34x_2 - e^{-t}x_3$$
$$+0.23x_4\cos 3t + 0.78\sin e^{2t}x_5 + 7.5x_6 + d_5.$$

$$f_6 = 0.30x_1 - 0.70(1+t^6)^{-1}x_2 + 0.58(t+1)^{-1}x_3$$
$$+0.12x_4 + 0.56e^{-2t}x_5 + 0.35x_6 + d_6.$$

Clearly, $\beta = \operatorname{Re}\lambda(A) = -3 < 0$

$\|f_1\| \leq 0.8\|x\| + d_1, \|f_2\| \leq 089\|x\| + d_2, \|f_3\| \leq 0.90\|x\| + d_3,$
$\|f_4\| \leq 0.80\|x\| + d_4, \|f_5\| \leq 0.78\|x\| + d_5$ And $\|f_6\| \leq 0.7\|x\| + d_6$.

Therefore
$\|f(t,x)\| \leq \max[0.80, 0.89, 0.90, 0.80, 0.78, 0.70, |d_i|]\|x\|$
If we choose $\tilde{\epsilon} = \max_{1\leq i\leq 6}[0.09, |d_i|] < 1,$ hence $\|f(t,x)\| \leq \tilde{\epsilon}\, \|x\|.$

Then the trivial solution $x \equiv 0$ of eq. (E1) is asymptotically stable by Theorem 4 for $G(t,x(t)) = 0$ in the eq. (18).

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