Coefficient Transformation of Polynomial Equations

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Abstract— **This is the new concept to transform the Coefficient of polynomial equation to another set of coefficient to make the roots in between a particular interval by which any person can easily identify how many real roots, after plotting the equation in a graph of x axis having the desired range by compressing the curve.**

Keywords— **Coefficient transformation, roots finding, curve compression, Coefficient, solving polynomial equations, equations, polynomial, transformation, compression**

I. INTRODUCTION

If each coefficient is linear to a variable, then we can substitute another variable to get another equation to that variable. If we design the variable as a function such a way that it cannot exceed certain limits, then we can bring the roots in between an interval which compresses the curve for plotting.

II. DERIVATION

Following derivation is the new concept to transform the Coefficient of $a_0 + a_1 * x + a_2 * x^2 + a_3 * x^3 + \cdots + a_r * x^r$ into another set of $u_0 + u_1 * x + u_2 * x^2 + u_3 * x^3 + \cdots + u_r * x^r$ so that we can either minimize the coefficient of $u_t = 0$ where $t = from\ 1\ to\ r - 1$, to obtain the roots or same can also be used to transform the roots to a particular interval by which any person can easily identify how many real roots, after plotting the equation in a graph of x axis having the desired range.

Let $F(p) = (a_{00} + a_{01} * x + a_{02} * x^2 + a_{03} * x^3 + \cdots + a_{0m} * x^m) * p^n + (a_{10} + a_{11} * x + a_{12} * x^2 + a_{13} * x^3 + \cdots + a_{1m} * x^m) * p^{n-1} + (a_{20} + a_{21} * x + a_{22} * x^2 + a_{23} * x^3 + \cdots + a_{2m} * x^m) * p^{n-2} + \cdots + (a_{n0} + a_{n1} * x + a_{n2} * x^2 + a_{n3} * x^3 + \cdots + a_{nm} * x^m) = 0$

Then $F(p) * (p - x) = E(p) = ((a_{00} + a_{01} * x + a_{02} * x^2 + a_{03} * x^3 + \cdots + a_{0m} * x^m) * p^n + (a_{10} + a_{11} * x + a_{12} * x^2 + a_{13} * x^3 + \cdots + a_{1m} * x^m) * p^{n-1} + (a_{20} + a_{21} * x + a_{22} * x^2 + a_{23} * x^3 + \cdots + a_{2m} * x^m) * p^{n-2} + \cdots + (a_{n0} + a_{n1} * x + a_{n2} * x^2 + a_{n3} * x^3 + \cdots + a_{nm} * x^m)) * (p - x) = (a_{00} + a_{01} * x + a_{02} * x^2 + a_{03} * x^3 + \cdots + a_{0m} * x^m) * p^{n+1} + ((a_{10} + a_{11} * x + a_{12} * x^2 + a_{13} * x^3 + \cdots + a_{1m} * x^m) - (a_{00} + a_{01} * x + a_{02} * x^2 + a_{03} * x^3 + \cdots + a_{0m} * x^m) * x) * p^n + ((a_{20} + a_{21} * x + a_{22} * x^2 + a_{23} * x^3 + \cdots + a_{2m} * x^m) - (a_{10} + a_{11} * x + a_{12} * x^2 + a_{13} * x^3 + \cdots + a_{1m} * x^m) * x) * p^{n-1} + \ldots + (-(a_{n0} + a_{n1} * x + a_{n2} * x^2 + a_{n3} * x^3 + \cdots + a_{nm} * x^m) * x) = 0$

Divide the coefficient of p^{n+1} and get all coefficient of p^r where $r = from\ n\ to\ 0$. Then you will have coefficient of p^{n-r+1} is equal to $\frac{((a_{r0}+a_{r1}*x+a_{r2}*x^2+a_{r3}*x^3+\cdots+a_{rm}*x^m)-(a_{(r-1)0}+a_{(r-1)1}*x+a_{(r-1)2}*x^2+a_{(r-1)3}*x^3+\cdots+a_{(r-1)m}*x^m)*x)}{(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m)}$

Let coefficient of $p^{n-r} = k_r * z + l_r$, i.e.) coefficient of $p^n = k_0 * z + l_0$, coefficient of $p^{n-1} = k_1 * z + l_1$, coefficient of $p^{n-2} = k_2 * z + l_2$, ..., and last coefficient of constant, $p^0 = k_n * z + l_n$ Now obtain the value of z from Coefficient of p^n which is

$$\frac{((a_{10}+a_{11}*x+a_{12}*x^2+a_{13}*x^3+\cdots+a_{1m}*x^m)-(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m)*x)}{(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m)} = k_0 * z + l_0$$

Then you will get

$$z = \frac{((a_{10}+a_{11}*x+a_{12}*x^2+a_{13}*x^3+\cdots+a_{1m}*x^m)-(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m)*x)-(l_0*(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m))}{(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m)*k_0}$$

Now Substitute to all other equations from coefficient of $p^{n-r} = k_r * z + l_r$, where $r = from\ 1\ to\ n$. Then you will get the equation as

$$\frac{((a_{r0}+a_{r1}*x+a_{r2}*x^2+a_{r3}*x^3+\cdots+a_{rm}*x^m)-(a_{(r-1)0}+a_{(r-1)1}*x+a_{(r-1)2}*x^2+a_{(r-1)3}*x^3+\cdots+a_{(r-1)m}*x^m)*x)}{(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m)} =$$

$$k_r * \left(\frac{((a_{10}+a_{11}*x+a_{12}*x^2+a_{13}*x^3+\cdots+a_{1m}*x^m)-(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m)*x)-(l_0*(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m))}{(a_{00}+a_{01}*x+a_{02}*x^2+a_{03}*x^3+\cdots+a_{0m}*x^m)*k_0}\right) + l_r$$

where $r = from\ 1\ to\ n$.

Now expand these to have equations in terms of x^w where $w = from\ 0\ to\ m+1$. After getting each coefficient of x^w, solve variables of a_{nm} by equating every coefficient of x^w where $w = from\ 0\ to\ m+1$ into zero.

Since there are n equations to coefficient of $p^{n-r} = k_r * z + l_r$, where $r = from\ 1\ to\ n$ which will have further sub forms of where $m+2$ equations to each coefficient of x^w, we can resolve $[n*(m+2)]$ equations of a_{nm} which has $[(n+1)*(m+1)]-1$ variables.

We can have solution for a_{nm} only if $[n*(m+2)] \le [(n+1)*(m+1)]-1$, which resolves to the condition of $n \le m$. Hence if $n \le m$, we can find a_{nm} which satisfies the above equations. Otherwise If $n > m$, then consider k_n, l_n, of $2*n$ variables to resolve additional $n-m$ equations.

After substituting those variables, resolve $F(p) = 0$ and Let them are $x_1, x_2, \dots, x_n$ then $F(p) = (p-x_1)*(p-x_2)*\dots*(p-x_n) = 0$ and then $F(p)*(p-x) = E(p) = (p-x)*(p-x_1)*(p-x_2)*\dots*(p-x_n) = 0$ which is also equal to $E(p) = p^{n+1} + p^n*(k_0*z+l_0) + p^{n-1}*(k_1*z+l_1) + \cdots + p^0*(k_n*z+l_n) = 0$

Let $P(x) = a_0 + a_1*x + a_2*x^2 + a_3*x^3 + \cdots + a_r*x^r = 0$ where a_0 and a_r are not zero.

If $P(x)$ has $p_1, p_2, \dots, p_r$ roots. Then $(x-p_1)*(x-p_2)*\dots*(x-p_r) = 0$

Instead of solving $P(x)$ solve $P(x)*P(x_1)*P(x_2)*\dots*P(x_n) = 0$. Since $P(x_t) = (x_t-p_1)*(x_t-p_2)*\dots*(x_t-p_r)$, $P(x)*P(x_1)*P(x_2)*\dots*P(x_n) = 0$ will become $((x-p_1)*(x-p_2)*\dots*(x-p_r))*((x_1-p_1)*(x_1-p_2)*\dots*(x_1-p_r))*((x_2-p_1)*(x_2-p_2)*\dots*(x_2-p_r))*\dots*((x_n-p_1)*(x_n-p_2)*\dots*(x_n-p_r)) = 0$

Now regroup the multiplication to have for every p_t instead of x_t, I.e.) $((x-p_1)*(x_1-p_1)*(x_2-p_1)*\dots*(x_n-p_1))*((x-p_2)*(x_1-p_2)*(x_2-p_2)*\dots*(x_n-p_2))*((x-p_3)*(x_1-p_3)*(x_2-p_3)*\dots*(x_n-p_3))*\dots*((x-p_r)*(x_1-p_r)*(x_2-p_r)*\dots*(x_n-p_r)) = 0$

Since $(x-p_t)*(x_1-p_t)*(x_2-p_t)*\dots*(x_n-p_t) = E(p_t)$, above equation will lead into $E(p_1)*E(p_2)*E(p_3)*\dots*E(p_r) = 0$, Since

$E(p_t) = p_t^{n+1} + p_t^n*(k_0*z+l_0) + p_t^{n-1}*(k_1*z+l_1) + \dots + k_n*z+l_n = 0$, we can get z in terms of p_t, i.e.)

$$z_t = \frac{-(p_t^{n+1}+p_t^n*(l_0)+p_t^{n-1}*(l_1)+\dots+p_t^0*(l_n))}{(p_t^n*(k_0)+p_t^{n-1}*(k_1)+\dots+p_t^0*(k_n))}$$

If $E(p_1)*E(p_2)*E(p_3)*\dots*E(p_r) = 0$, Then $(z-z_1)*(z-z_2)*\dots*(z-z_r) = 0$. Since each $z_t = \frac{-(p_t^{n+1}+p_t^n*(l_0)+p_t^{n-1}*(l_1)+\dots+p_t^0*(l_n))}{(p_t^n*(k_0)+p_t^{n-1}*(k_1)+\dots+p_t^0*(k_n))}$ and each p_t is a solution of $P(x)$, we can substitute

$z_x = \frac{-(x^{n+1}+x^n*(l_0)+x^{n-1}*(l_1)+\dots+x^0*(l_n))}{(x^n*(k_0)+x^{n-1}*(k_1)+\dots+x^0*(k_n))}$ or $\frac{(x^{n+1}+x^n*(l_0)+x^{n-1}*(l_1)+\dots+x^0*(l_n))}{(x^n*(k_0)+x^{n-1}*(k_1)+\dots+x^0*(k_n))}$ In to $E(p_1)*E(p_2)*E(p_3)*\dots*E(p_r) = 0$

Hence $P(x) * P(x_1) * P(x_2) * \dots * P(x_n)$ can substitute z_x and get another polynomial of same degree but with different coefficient. Since there are n variables of k_r and l_r we can find k_r and l_r which will make these coefficient to zero and thereby can be used to resolve polynomial equations.

Let us see whether Same concept also can be extended to have $F(p) * \left((b_0 + b_1 * x + b_2 * x^2 + b_3 * x^3 + \dots + b_s * x^s)p - (c_0 + c_1 * x + c_2 * x^2 + c_3 * x^3 + \dots + c_s * x^s)\right)^q = \left((a_{00} + a_{01} * x + a_{02} * x^2 + a_{03} * x^3 + \dots + a_{0m} * x^m) * p^n + (a_{10} + a_{11} * x + a_{12} * x^2 + a_{13} * x^3 + \dots + a_{1m} * x^m) * p^{n-1} + (a_{20} + a_{21} * x + a_{22} * x^2 + a_{23} * x^3 + \dots + a_{2m} * x^m) * p^{n-2} + \dots + (a_{n0} + a_{n1} * x + a_{n2} * x^2 + a_{n3} * x^3 + \dots + a_{nm} * x^m)\right) * \left((b_0 + b_1 * x + b_2 * x^2 + b_3 * x^3 + \dots + b_s * x^s)p - (c_0 + c_1 * x + c_2 * x^2 + c_3 * x^3 + \dots + c_s * x^s)\right)^q$ and Divide the coefficient of p^{n+q} and get all coefficient of p^r where $r = from\ n + q - 1\ to\ 0$. Let coefficient of $p^{n+q-1} = k_0 * z + l_0$, co − efficient of $p^{n+q-2} = k_1 * z + l_1$, co − efficient of $p^{n+q-3} = k_2 * z + l_2$, ... last coefficient of constant, $p^0 = k_{n+q-1} * z + l_{n+q-1}$, where in there are $n + q - 1$ equations of coefficient p^r which will have further sub forms of where $m + s * q + 1$ equations to each coefficient of x^w, we can resolve $[(n + q - 1) * (m + s * q - 1)]$ equations of a_{nm} which has $[(n + 1) * (m + 1)] - 1$ variables and b_s, c_s having $2 * s + 1$ variables

Hence if $[(n + q - 1) * (m + s * q - 1)] \le [(n + 1) * (m + 1) + 2 * s]$, we can have solution for a_{nm}, b_s, c_s which relates to $n <= \frac{(2-q)(m+s*(q+1)+1)}{s*q}$, otherwise we need to consider k_n, l_n of $2 * (n + q - 1)$ variables to resolve additional equations of $n * s * q + (q - 2)(m + s * (q + 1) + 1)$. If $q = 1$, then above condition leads to if $n <= 2 + \frac{m+1}{s}$, we can have solution for a_{nm}, b_s, c_s otherwise we need to consider k_n, l_n of $2 * (n + q - 1) = 2 * n$ variables to resolve. If $q \ge 2$, then $n * s * q + (q - 2)(m + s * (q + 1) + 1)$ is always greater than 0, hence we need to consider k_n, l_n of $2 * (n + q - 1)$ variables. If $q = 2$, irrespective of whatever m, $[2 * n * s] \le [2 * n + 2]$ which is possible only if $s <= 1 + \frac{1}{n}$. If $q > 2$, then $s <= \frac{(2*(n+q-1)-(q-2)*(m+1))}{(n*q+(q-2)*(q+1))}$ which leads to $s <= \frac{2}{q} + \frac{4-q*(q-2)*(m+1)}{q*(n*q+(q-2)*(q+1))}$ which means $s < 1$ which means q cannot be > 2.

Hence coefficient transformation to the concept of $F(p) * \left((b_0 + b_1 * x + b_2 * x^2 + b_3 * x^3 + \dots + b\ \ * x^s)p - (c_0 + c_1 * x + c_2 * x^2 + c_3 * x^3 + \dots + c_s * x^s)\right)^q = \left((a_{00} + a_{01} * x + a_{02} * x^2 + a_{03} * x^3 + \dots + a_{0m} * x^m) * p^n + (a_{10} + a_{11} * x + a_{12} * x^2 + a_{13} * x^3 + \dots + a_{1m} * x^m) * p^{n-1} + (a_{20} + a_{21} * x + a_{22} * x^2 + a_{23} * x^3 + \dots + a_{2m} * x^m) * p^{n-2} + \dots + (a_{n0} + a_{n1} * x + a_{n2} * x^2 + a_{n3} * x^3 + \dots + a_{nm} * x^m)\right) * \left((b_0 + b_1 * x + b_2 * x^2 + b_3 * x^3 + \dots + b_s * x^s)p - (c_0 + c_1 * x + c_2 * x^2 + c_3 * x^3 + \dots + c_s * x^s)\right)^q$ is possible when $q \le 2$ and by extending the same approach which we did in the beginning, then we will have in general, the following transformation

The equation $\left(P\left(\frac{c_0+c_1*x+c_2*x^2+c_3*x^3+\dots+c_s*x^s}{b_0+b_1*x+b_2*x^2+b_3*x^3+\dots+b_s*x^s}\right)\right)^q * P(x_1) * P(x_2) * \dots * P(x_n) = 0$ can have

$z_x = \frac{\left(x^{n+q} + x^{n+q-1} * (l_0) + x^{n+q-2} * (l_1) + \dots + x^0 * (l_{n+q-1})\right)}{(x^{n+q} * (k_0) + x^{n+q-2} * (k_1) + \dots + x^0 * (k_{n+q-1}))}$ As substitution when $q \le 2$ is possible, and after substituting, it will have same degree of $P(x)$.

Similarly the same concept can also be extended to have the equation to the following form,

$$\left(P\left(\frac{c_{10}+c_{11}*x+c_{12}*x^2+c_{13}*x^3+\dots+c_{1s_1}*x^{s_1}}{b_{10}+b_{11}*x+b_{12}*x^2+b_{13}*x^3+\dots+b_{1s_1}*x^{s_1}}\right)\right)^{q_1} * \left(P\left(\frac{c_{20}+c_{21}*x+c_{22}*x^2+c_{23}*x^3+\dots+c_{2s_2}*x^{s_2}}{b_{20}+b_{21}*x+b_{22}*x^2+b_{23}*x^3+\dots+b_{2s_2}*x^{s_2}}\right)\right)^{q_2} * \dots * \left(P\left(\frac{c_{j0}+c_{j1}*x+c_{j2}*x^2+c_{j3}*x^3+\dots+c_{js_j}*x^{s_j}}{b_{j0}+b_{j1}*x+b_{j2}*x^2+b_{j3}*x^3+\dots+b_{js_j}*x^{s_j}}\right)\right)^{q_j} * P(x_1) * P(x_2) * \dots * P(x_n) = 0$$ Can have

$z_x = \frac{\left(x^{n+q_1+q_2+\cdots+q_j} + x^{n+q_1+q_2+\cdots+q_j-1} * (l_0) + x^{n+q_1+q_2+\cdots+q_j-2} * (l_1) + \ldots + \left(l_{n+q_1+q_2+\cdots+q_j-1}\right)\right)}{\left(x^{n+q_1+q_2+\cdots+q_j-1} * (k_0) + x^{n+q_1+q_2+\cdots+q_j-2} * (k_1) + \ldots + (k_{n+q_1+q_2+\cdots+q_j-1})\right)}$ As substitution and after substituting, it will have same degree of $P(x)$. Since it is the same degree of $P(x)$, we could transform the Coefficient of $a_0 + a_1 * x + a_2 * x^2 + a_3 * x^3 + \cdots + a_r * x^r$ into another set of $u_0 + u_1 * x + u_2 * x^2 + u_3 * x^3 + \cdots + u_r * x^r$ so that we can either minimize the coefficient of $u_t = 0$ where $t = from\ 1\ to\ r - 1$, to obtain the roots or same can also be used to transform the roots to a particular interval by which any person can easily identify how many real roots, after plotting the equation in a graph of x axis having the desired range.

Let us go back to original derivation and explain the derivation with following example.

Let $n = m = 1$.

Then $F(p) = (a_{00} + a_{01} * x) * p^1 + (a_{10} + a_{11} * x) = 0$

Then $F(p) * (p - x) = (a_{00} + a_{01} * x) * p^1 + (a_{10} + a_{11} * x) * (p - x) = (a_{00} + a_{01} * x) * p^2 + \left((a_{10} + a_{11} * x) - (a_{00} + a_{01} * x) * x \right) * p^1 + (-(a_{10} + a_{11} * x) * x) = 0$

Divide the coefficient of p^2 and get all coefficient of p^r where $r = from\ 1\ to\ 0$.

Let coefficient of $p^1 = \frac{((a_{10}+a_{11}*x) - (a_{00}+a_{01}*x)*x)}{(a_{00}+a_{01}*x)} = k_0 * z + l_0$, and coefficient of

$p^0 = \frac{(-(a_{10}+a_{11}*x)*x)}{(a_{00}+a_{01}*x)} = k_1 * z + l_1$

After substituting coefficient of p^1, you will get $z = \frac{\left(((a_{10}+a_{11}*x) - (a_{00}+a_{01}*x)*x) - l_0 * (a_{00}+a_{01}*x)\right)}{(a_{00}+a_{01}*x)*k_0}$

After substituting z and equating coefficient of $p^0 = k_1 * z + l_1$, you will get

$\frac{(-(a_{10}+a_{11}*x)*x)}{(a_{00}+a_{01}*x)} = k_1 * \frac{\left(((a_{10}+a_{11}*x) - (a_{00}+a_{01}*x)*x) - l_0 * (a_{00}+a_{01}*x)\right)}{(a_{00}+a_{01}*x)*k_0} + l_1$

Then comparing coefficient of x^2, x^1 and x^0 from left hand side to right hand side will lead into

$a_{10} * k_1 + a_{00} * (-l_0 * k_1 + l_1 * k_0) = 0$, $a_{11} * k_1 - a_{00} * k_1 + a_{01} * (-k_1 * l_0 + l_1 * k_0) = -a_{10} * k_0$, $-a_{01} * k_1 = -a_{11} * k_0$, Then you will get, $a_{01} = a_{00} * \frac{k_0}{k_1}$, $a_{10} = a_{00} * \frac{k_1 * l_0 - l_1 * k_0}{k_1}$, and $a_{11} = a_{00}$

After putting $F(p) = (a_{00} + a_{01}x) * p^1 + (a_{10} + a_{11}x) = 0$ and solving p,

You will get $p = \frac{(-k_1 * l_0 + l_1 * k_0) - k_1 * x}{k_1 + k_0 * x}$

Hence $P(x) * P\left(\frac{(-k_1 * l_0 + l_1 * k_0) - k_1 * x}{k_1 + k_0 * x}\right)$ will have substitution of $z = \frac{(x^2 + x^1 * (l_0) + x^0 * (l_1))}{(x^1 * (k_0) + x^0 * (k_1))}$

Special cases from this substitution are

1) If $k_0 = 0, k_1 = 1, l_0 = 0, l_1 = 0$, then $P(x) * P(-x)$ will have $z = x^2$ as substitution
2) If $k_0 = 1, k_1 = 0, l_0 = 0, l_1 = 1$, then $P\left(\frac{x}{1}\right) * P\left(\frac{1}{x}\right)$ will have $z = \frac{x}{1} + \frac{1}{x}$ as substitution

If $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_rx^r = 0$ where a_0 and a_r are not zeros, then $P(x)$ is of the polynomial having degree r and it has $p_1, p_2, \ldots, p_r$ roots. Then

$P\left(\frac{x}{1}\right) * P\left(\frac{1}{x}\right) = (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_rx^r) * (a_0x^r + a_1x^{r-1} + a_2x^{r-2} + a_3x^{r-3} + \cdots + a_r) = 0$

Since $P\left(\frac{x}{1}\right) * P\left(\frac{1}{x}\right)$ will have $z = \frac{x}{1} + \frac{1}{x}$ as substitution, then $P\left(\frac{x}{1}\right) * P\left(\frac{1}{x}\right) = G\left(z = \frac{x}{1} + \frac{1}{x}\right)$ will have $z_t = \frac{p_t}{1} + \frac{1}{p_t}$ as root and forming $z^r * G\left(\frac{1}{z}\right)$ will have root as $z_t = \frac{1}{\frac{p_t}{1} + \frac{1}{p_t}} = \frac{p_t}{p_t^2+1}$ since $\frac{p_t}{p_t^2+1}$ is always in between $-\frac{1}{2}$ and $+\frac{1}{2}$, Hence we can plot $z^r * G\left(\frac{1}{z}\right)$ into the graph to know where it crosses x axis in zero, irrespective of larger root of p_t .

After getting each z_t which crosses zero, then we can get $p_t = \frac{1 \pm \sqrt{1-4*z_t^2}}{2*z_t}$ from z_t and one of the root will satisfy $P\left(\frac{x}{1}\right) = 0$ and another root will satisfy $P\left(\frac{1}{x}\right) = 0$

Since from the graph, it will give all real roots in between $-\frac{1}{2}$ and $+\frac{1}{2}$, any person can easily identify how many real roots the equation has.

Hence the transformation, $z^r * P\left(\frac{1+\sqrt{1-4*z^2}}{2*z}\right) * P\left(\frac{1-\sqrt{1-4*z^2}}{2*z}\right) = G(z) = 0$ will bring all the real roots in between $-\frac{1}{2}$ and $+\frac{1}{2}$ and Plotting $G(z) = 0$ can easily identify how many real roots the equation has with the x axis crossing zero in between $-\frac{1}{2}$ and $+\frac{1}{2}$.

Another advantage is that since every root is in between $-\frac{1}{2}$ and $+\frac{1}{2}$, again applying transformation of $G(z) * G(-z) = H(y)$ which will have $y = z^2$ as substitution. This will make every root is in between $\frac{0}{1}$ and $+\frac{1}{4}$. Hence the transformation $y^r * P\left(\frac{1+\sqrt{1-4*y}}{2*\sqrt{y}}\right) * P\left(\frac{1-\sqrt{1-4*y}}{2*\sqrt{y}}\right) * P\left(\frac{1+\sqrt{1-4*y}}{-2*\sqrt{y}}\right) * P\left(\frac{1-\sqrt{1-4*y}}{-2*\sqrt{y}}\right) = H(y) = 0$ will make every root in between $\frac{0}{1}$ and $\frac{1}{4}$. In this case coefficient of y^v can be easily judged whether all are real roots. Since every root cannot be greater than ¼ and less than 0, maximum absolute value of coefficient of y^v will not exceed the binomial coefficient of $\left(y - \frac{1}{4}\right)^r = \sum_{v=0}^{r} \binom{r}{v} y^v \left(-\frac{1}{4}\right)^{r-v}$

Hence Coefficient of y^{v-1} won't be lesser than $\frac{-r}{4}$ and greater than 0 and in general coefficient of y^{r-v} is in between 0 and $\binom{r}{v} * (-4)^{-v}$. If the condition is not satisfied, then the equation has imaginary roots.

If you don't want to have more multiplications and since the transformation, $z^r * P\left(\frac{1+\sqrt{1-4*z^2}}{2*z}\right) * P\left(\frac{1-\sqrt{1-4*z^2}}{2*z}\right) = G(z) = 0$ which will bring all the real roots in between $-\frac{1}{2}$ and $+\frac{1}{2}$, again applying $y = z - \frac{1}{2}$ which will bring transformation of $\left(y + \frac{1}{2}\right)^r * P\left(\frac{1+\sqrt{1-4*\left(y+\frac{1}{2}\right)^2}}{2*\left(y+\frac{1}{2}\right)}\right) * P\left(\frac{1-\sqrt{1-4*\left(y+\frac{1}{2}\right)^2}}{2*\left(y+\frac{1}{2}\right)}\right) = \left(\frac{2*y+1}{2}\right)^r * P\left(\frac{1+2*\sqrt{-y*(y+1)}}{2*y+1}\right) * P\left(\frac{1-2*\sqrt{-y*(y+1)}}{2*y+1}\right) = 0$, will have roots in between -1 and 0. Since every root cannot be lesser than -1 and greater than 0, maximum absolute value of coefficient of y^v will not exceed the binomial coefficient of $(y+1)^r = \sum_{v=0}^{r} \binom{r}{v} y^v$. Hence in general coefficient of y^{r-v} is in between 0 and $\binom{r}{v}$. If the condition is not satisfied, then the equation has imaginary roots.

Same can also be extended for any polynomial function $P(x)$ having degree r to compress the curve to get the roots to desired range between r_1 and r_2, then $(2*y - r_2 - r_1)^r * P\left(\frac{(r_2-r_1)+2*\sqrt{(r_2-y)*(y-r_1)}}{(2*y-r_2-r_1)}\right) * P\left(\frac{(r_2-r_1)-2*\sqrt{(r_2-y)*(y-r_1)}}{(2*y-r_2-r_1)}\right) = G(y) = 0$ will be the transformation having same degree of r. Similarly for any function other than polynomial, $F(x)$ to compress the curve to get the roots to desired range between r_1 and r_2, then $F\left(\frac{(r_2-r_1)+2*\sqrt{(r_2-y)*(y-r_1)}}{(2*y-r_2-r_1)}\right) * F\left(\frac{(r_2-r_1)-2*\sqrt{(r_2-y)*(y-r_1)}}{(2*y-r_2-r_1)}\right) = G(y) = 0$ will be the transformation.