New Matrix Operators to Get an Expansion Series

Sankaralingam Lakshmanaraj

gloStream Technologies Pvt Ltd,

Flat no 307, Ishwar Pratik Apartment, Tapovan road, Bodhale Nagar, Nashik, India,

Abstract— **This discovery of operator is similar to derivative of a function, but this is to derivative of a matrix function. This operator is bringing yet another expansion series to a function. This expansion series can be used to approximate wider range of the values.**

Keywords— **Matrix expansion series, matrix derivative operator, matrix operator, matrix, derivative, operator, series, expansion**

I. Introduction

The matrix multiplication of n x n and n x 1 will give n x 1 matrix. If we expand n x 1 further in to another n x n * n x 1 matrix, we get n x n * n x n * n x 1 and keep on extending, we get an expansion series.

II. Derivation

Let there be a 2 x 1 matrix, $\begin{pmatrix} a \\ b \end{pmatrix}$ and, operator [] to the matrix $\left[\begin{pmatrix} a \\ b \end{pmatrix}\right]$ is $a : b$

Since $k * a : k * b = a : b$ *for where* $k \neq 0$, $\left[\begin{pmatrix} k*a \\ k*b \end{pmatrix}\right]$ is also $\left[\begin{pmatrix} a \\ b \end{pmatrix}\right]$ which is $a : b$

Hence $\left[\begin{pmatrix} \boldsymbol{k*a} \\ \boldsymbol{k*b} \end{pmatrix}\right] = \left[\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}\right]$ **this makes identity no 1**

Since $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right] = \left[\begin{pmatrix} a*\alpha + b*\beta \\ c*\alpha + d*\beta \end{pmatrix}\right]$, then $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right] = (a*\alpha + b*\beta) : (c*\alpha + d*\beta)$

Also $\left[\begin{pmatrix} k*a & k*b \\ k*c & k*d \end{pmatrix}\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right] = (k*a*\alpha + k*b*\beta) : (k*c*\alpha + k*d*\beta) = (a*\alpha + b*\beta) : (c*\alpha + d*\beta) = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right]$

Hence $\left[\begin{pmatrix} \boldsymbol{k*a} & \boldsymbol{k*b} \\ \boldsymbol{k*c} & \boldsymbol{k*d} \end{pmatrix}\begin{pmatrix} \propto \\ \boldsymbol{\beta} \end{pmatrix}\right] = \left[\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}\begin{pmatrix} \propto \\ \boldsymbol{\beta} \end{pmatrix}\right]$ **this makes identity no 2**

If $\left[\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right] = \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} \gamma \\ \delta \end{pmatrix}\right]$, which equates to $\alpha : \beta = (e*\gamma + f*\delta) : (g*\gamma + h*\delta)$ then $\alpha = k * (e*\gamma + f*\delta)$, $\beta = k*(g*\gamma + h*\delta)$ for any $k \neq 0$ and if you substitute value of α and β to the equation, $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right] = (a*\alpha + b*\beta) : (a*\alpha + b*\beta)$ then one can prove that, $(a*\alpha + b*\beta) : (a*\alpha + b*\beta) = \big((k*a*e + k*b*g)*\gamma + (k*a*f + k*b*h)*\delta\big) : \big((k*c*e + d*g)*\gamma + (k*c*f + k*d*h)*\delta\big) = \big((a*e + k*b*g)*\gamma + (a*f + k*b*h)*\delta\big) : \big((c*e + d*g)*\gamma + (c*f + k*d*h)*\delta\big)$ which simply follows to the **identity No 3.**

If $\left[\begin{pmatrix} \propto \\ \boldsymbol{\beta} \end{pmatrix}\right] = \left[\begin{pmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{pmatrix}\begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\delta} \end{pmatrix}\right]$ **then** $\left[\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}\begin{pmatrix} \propto \\ \boldsymbol{\beta} \end{pmatrix}\right] = \left[\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}\begin{pmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{pmatrix}\begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\delta} \end{pmatrix}\right]$

Similarly we can also prove that, If $\left[\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right] = \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} \gamma \\ \delta \end{pmatrix}\right]$, which equates to $\alpha : \beta = (e*\gamma + f*\delta) : (g*\gamma + h*\delta)$ then $\alpha = k * (e*\gamma + f*\delta)$, $\beta = k*(g*\gamma + h*\delta)$ and then $k*\gamma = \frac{(h*\alpha - f*\beta)}{e*h - f*g}$, $k*\delta = \frac{(-g*\alpha + e*\beta)}{e*h - f*g}$ for any $k \neq 0$ which makes the identity, If $\left[\begin{pmatrix}\propto\\ \beta\end{pmatrix}\right] = \left[\begin{pmatrix}e & f\\ g & h\end{pmatrix}\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\right]$, then $\left[\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\right] = \left[\begin{pmatrix}e & f\\ g & h\end{pmatrix}^{-1}\begin{pmatrix}\propto\\ \beta\end{pmatrix}\right] = \left[\begin{pmatrix}\frac{h}{e*h-f*g} & -\frac{f}{e*h-f*g}\\ -\frac{g}{e*h-f*g} & \frac{e}{e*h-f*g}\end{pmatrix}\begin{pmatrix}\propto\\ \beta\end{pmatrix}\right]$, on applying identity no 3,

which gives $\left[\begin{pmatrix}h & -f\\ -g & e\end{pmatrix}\begin{pmatrix}\propto\\ \beta\end{pmatrix}\right]$ if $e*h \neq f*g$

This leads to **Identity no 4** as follows.

If $\left[\begin{pmatrix}\propto\\ \boldsymbol{\beta}\end{pmatrix}\right] = \left[\begin{pmatrix}\boldsymbol{e} & \boldsymbol{f}\\ \boldsymbol{g} & \boldsymbol{h}\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\delta}\end{pmatrix}\right]$, **then** $\left[\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\delta}\end{pmatrix}\right] = \left[\begin{pmatrix}\boldsymbol{e} & \boldsymbol{f}\\ \boldsymbol{g} & \boldsymbol{h}\end{pmatrix}^{-1}\begin{pmatrix}\propto\\ \boldsymbol{\beta}\end{pmatrix}\right] = \left[\begin{pmatrix}\boldsymbol{h} & -\boldsymbol{f}\\ -\boldsymbol{g} & \boldsymbol{e}\end{pmatrix}\begin{pmatrix}\propto\\ \boldsymbol{\beta}\end{pmatrix}\right]$ **if** $\boldsymbol{e*h \neq f*g \rightarrow \frac{e}{g} \neq \frac{f}{h}}$

Let $\left[\begin{pmatrix}n(x)\\ d(x)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_1) & n(x_2)\\ d(x_1) & d(x_2)\end{pmatrix}\begin{pmatrix}n_1(x)*(x-x_2)\\ d_1(x)*(x-x_1)\end{pmatrix}\right]$

Then if $x = x_1$, $\left[\begin{pmatrix}n(x_1)\\ d(x_1)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_1) & n(x_2)\\ d(x_1) & d(x_2)\end{pmatrix}\begin{pmatrix}n_1(x_1)*(x_1-x_2)\\ d_1(x_1)*(x_1-x_1)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_1)*n_1(x_1)*(x_1-x_2)\\ d(x_1)*n_1(x_1)*(x_1-x_2)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_1)\\ d(x_1)\end{pmatrix}\right]$

And if $x = x_2$, $\left[\begin{pmatrix}n(x_2)\\ d(x_2)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_1) & n(x_2)\\ d(x_1) & d(x_2)\end{pmatrix}\begin{pmatrix}n_1(x_2)*(x_2-x_2)\\ d_1(x_2)*(x_2-x_1)\end{pmatrix}\right] = \left[\begin{pmatrix}d_1(x_2)*\beta*(x_1-x_2)\\ d_1(x_2)*\beta*(x_1-x_2)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_2)\\ d(x_2)\end{pmatrix}\right]$

Hence this equation satisfies for any $n_1(x)$ and $d_1(x)$ when $x = x_1\ or\ x_2$. Let us find what is $n_1(x)$ and $d_1(x)$ when x approaches either $x_1\ or\ x_2$

If $\left[\begin{pmatrix}n(x)\\ d(x)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_1) & n(x_2)\\ d(x_1) & d(x_2)\end{pmatrix}\begin{pmatrix}n_1(x)*(x-x_2)\\ d_1(x)*(x-x_1)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_1)*(x-x_2) & n(x_2)*(x-x_1)\\ d(x_1)*(x-x_2) & d(x_2)*(x-x_1)\end{pmatrix}\begin{pmatrix}n_1(x)\\ d_1(x)\end{pmatrix}\right]$

Then $\left[\begin{pmatrix}n_1(x)\\ d_1(x)\end{pmatrix}\right] = \left[\begin{pmatrix}d(x_2)*(x-x_1) & -n(x_2)*(x-x_1)\\ -d(x_1)*(x-x_2) & n(x_1)*(x-x_2)\end{pmatrix}\begin{pmatrix}n(x)\\ d(x)\end{pmatrix}\right]$ using Identity no 4.

Since $n_1(x){:}\,d_1(x) = (d(x_2)*(x-x_1)*n(x)-n(x_2)*(x-x_1)*d(x)) : (-d(x_1)*(x-x_2)*n(x)+n(x_1)*(x-x_2)*d(x))$,

Which leads $n_1(x) = k*\frac{(d(x_2)*n(x)-n(x_2)*d(x))}{(x-x_2)}$ and $d_1(x) = k*\frac{(-d(x_1)*n(x)+n(x_1)*d(x))}{(x-x_1)}$

Hence $\left[\begin{pmatrix}n_1(x)\\ d_1(x)\end{pmatrix}\right] = \left[\begin{pmatrix}k*\frac{(d(x_2)*n(x)-n(x_2)*d(x))}{(x-x_2)}\\ k*\frac{(-d(x_1)*n(x)+n(x_1)*d(x))}{(x-x_1)}\end{pmatrix}\right] = \left[\begin{pmatrix}\frac{(d(x_2)*n(x)-n(x_2)*d(x))}{(x-x_2)}\\ \frac{(-d(x_1)*n(x)+n(x_1)*d(x))}{(x-x_1)}\end{pmatrix}\right]$

Expand $\left[\begin{pmatrix}n_1(x)\\ d_1(x)\end{pmatrix}\right]$ similar way to $\left[\begin{pmatrix}n(x)\\ d(x)\end{pmatrix}\right] = \left[\begin{pmatrix}n(x_1) & n(x_2)\\ d(x_1) & d(x_2)\end{pmatrix}\begin{pmatrix}n_1(x)*(x-x_2)\\ d_1(x)*(x-x_1)\end{pmatrix}\right]$

I.e.) $\left[\begin{pmatrix}n_1(x)\\ d_1(x)\end{pmatrix}\right] = \left[\begin{pmatrix}n_1(x_1) & n_1(x_2)\\ d_1(x_1) & d_1(x_2)\end{pmatrix}\begin{pmatrix}n_2(x)*(x-x_2)\\ d_2(x)*(x-x_1)\end{pmatrix}\right] =$

$$\left[\begin{pmatrix}\frac{(d(x_2)*n(x_1)-n(x_2)*d(x_1))}{(x_1-x_2)}*(x-x_2) & (d(x_2)*n'(x_2)-n(x_2)*d'(x_2))*(x-x_1)\\ (-d(x_1)*n'(x_1)+n(x_1)*d'(x_1))*(x-x_2) & \frac{(d(x_2)*n(x_1)-n(x_2)*d(x_1))}{(x_2-x_1)}*(x-x_1)\end{pmatrix}\begin{pmatrix}n_2(x)\\ d_2(x)\end{pmatrix}\right]$$

Hence,

$$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} n(x_1)*(x-x_2) & n(x_2)*(x-x_1) \\ d(x_1)*(x-x_2) & d(x_2)*(x-x_1) \end{pmatrix}\begin{pmatrix} n_2(x) \\ d_2(x) \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} n(x_1)*(x-x_2) & n(x_2)*(x-x_1) \\ d(x_1)*(x-x_2) & d(x_2)*(x-x_1) \end{pmatrix} * \begin{pmatrix} \frac{(d(x_2)*n(x_1)-n(x_2)*d(x_1))}{(x_1-x_2)}*(x-x_2) & (d(x_2)*n'(x_2)-n(x_2)*d'(x_2))*(x-x_1) \\ (-d(x_1)*n'(x_1)+n(x_1)*d'(x_1))*(x-x_2) & \frac{(d(x_2)*n(x_1)-n(x_2)*d(x_1))}{(x_2-x_1)}*(x-x_1) \end{pmatrix} * \begin{pmatrix} n_2(x) \\ d_2(x) \end{pmatrix}\right]$$

And we can keep expanding these to get a series. For that let us work on the operators needed.

Let the operator, $\underset{x=x_1\#x_2}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \begin{pmatrix} \frac{(d(x_2)*n(x)-n(x_2)*d(x))}{(x-x_2)} \\ \frac{(-d(x_1)*n(x)+n(x_1)*d(x))}{(x-x_1)} \end{pmatrix} = \begin{pmatrix} n_1(x) \\ d_1(x) \end{pmatrix}$ then

$$\underset{x=x_1\#x_2}{{}^{2}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \underset{x=x_1\#x_2}{{}^{1}}\left[\begin{pmatrix} n_1(x) \\ d_1(x) \end{pmatrix}\right] = \begin{pmatrix} \frac{(d_1(x_2)*n_1(x)-n_1(x_2)*d_1(x))}{(x-x_2)} \\ \frac{(-d_1(x_1)*n_1(x)+n_1(x_1)*d_1(x))}{(x-x_1)} \end{pmatrix}$$

And finally $\underset{x=x_1\#x_2}{{}^{r}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \begin{pmatrix} n_r(x) \\ d_r(x) \end{pmatrix} = \underset{x=x_1\#x_2}{{}^{1}}\left[\begin{pmatrix} n_{r-1}(x) \\ d_{r-1}(x) \end{pmatrix}\right] = \begin{pmatrix} \frac{(d_{r-1}(x_2)*n_{r-1}(x)-n_{r-1}(x_2)*d_{r-1}(x))}{(x-x_2)} \\ \frac{(-d_{r-1}(x_1)*n_{r-1}(x)+n_{r-1}(x_1)*d_{r-1}(x))}{(x-x_1)} \end{pmatrix}$

Also $\underset{x=x_1\#x_2}{{}^{1}}\left[\begin{pmatrix} i & j \\ k & l \end{pmatrix}\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \underset{x=x_1\#x_2}{{}^{1}}\left[\begin{pmatrix} i*n(x)+j*d(x) \\ k*n(x)+l*d(x) \end{pmatrix}\right] =$

$$\begin{pmatrix} \frac{\left((k*n(x_2)+l*d(x_2))*(i*n(x)+j*d(x))-(i*n(x_2)+j*d(x_2))*(k*n(x)+l*d(x))\right)}{(x-x_2)} \\ \frac{\left(-(k*n(x_1)+l*d(x_1))*(i*n(x)+j*d(x))-(i*n(x_1)+j*d(x_1))*(k*n(x)+l*d(x))\right)}{(x-x_1)} \end{pmatrix}$$

$= \begin{pmatrix} \frac{(d(x_2)*n_{r-1}(x)-n(x_2)*d_{r-1}(x))}{(x-x_2)} \\ \frac{(-d(x_1)*n_{r-1}(x)+n(x_1)*d_{r-1}(x))}{(x-x_1)} \end{pmatrix} = \underset{x=x_1\#x_2}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]$ If $(i*l-j*k)\neq 0$

Hence $\underset{x=x_1\#x_2}{{}^{1}}\left[\begin{pmatrix} i & j \\ k & l \end{pmatrix}\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \underset{x=x_1\#x_2}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]$ **if** $(i*l-j*k)\neq 0 \rightarrow, \frac{i}{k}\neq\frac{j}{l}$ **which is identity no 5.**

As a special case to identity no 5, if $i=0, j=1, k=1, l=0$, then one can prove that

$$\underset{x=x_1\#x_2}{}^{1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \underset{x=x_1\#x_2}{}^{1}\left[\begin{pmatrix} d(x) \\ n(x) \end{pmatrix}\right]$$

Let another operator, $\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2} = \begin{pmatrix} n(x_1)*(x-x_2) & n(x_2)*(x-x_1) \\ d(x_1)*(x-x_2) & d(x_2)*(x-x_1) \end{pmatrix}$

And then operator, $\underset{x=x_1\#x_2}{}^{r}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2} = \begin{pmatrix} n_r(x_1)*(x-x_2) & n_r(x_2)*(x-x_1) \\ d_r(x_1)*(x-x_2) & d_r(x_2)*(x-x_1) \end{pmatrix}$ and

Let it be operator $^{r}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2}$

Also the inverse operator using identity 4 to $\left(^{r}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1,x_2}\right)^{-1} = \left(\begin{pmatrix} n_r(x_1)*(x-x_2) & n_r(x_2)*(x-x_1) \\ d_r(x_1)*(x-x_2) & d_r(x_2)*(x-x_1) \end{pmatrix}\right)^{-1} =$ $\begin{pmatrix} d_r(x_2)*(x-x_1) & -n_r(x_2)*(x-x_1) \\ -d_r(x_1)*(x-x_2) & n_r(x_1)*(x-x_2) \end{pmatrix}$

Using identity no 2 on dividing $(x-x_2)*(x-x_1)$, we will get $\left(^{r}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1,x_2}\right)^{-1} = \begin{pmatrix} \frac{d_r(x_2)}{(x-x_2)} & \frac{-n_r(x_2)}{(x-x_2)} \\ \frac{-d_r(x_1)}{(x-x_1)} & \frac{n_r(x_1)}{(x-x_1)} \end{pmatrix}$ and Let it be the operator as $^{r}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]^{-1}_{x=x_1\#x_2}$

Using these operators, we can expand the $\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]$ to the following series which is **identity no 6.**

$$\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right] = \left[{}^{0}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1\#x_2} * {}^{1}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1\#x_2} * {}^{2}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1\#x_2} * \dots * {}^{r-1}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1\#x_2} * \underset{x=x_1\#x_2}{}^{r}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]\right]$$

And $\boldsymbol{r}$ can be limited until $\boldsymbol{n_r(x) : d_r(x) = k}$ where $\boldsymbol{k}$ is a constant.

Using identity no 4 to above identity no 6, one can prove that,

$$\underset{x=x_1\#x_2}{}^{r}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\left({}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2} * {}^{1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2} * {}^{2}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2} * \dots * {}^{r-1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2}\right)^{-1} * \begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right],$$

We can arrive at

$$\underset{x=x_1\#x_2}{}^{r}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\left({}^{r-1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2}\right)^{-1} * \left({}^{r-2}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2}\right)^{-1} * \left({}^{r-3}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2}\right)^{-1} * \dots * \left({}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2}\right)^{-1} * \begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[{}^{r-1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]^{-1}_{x=x_1\#x_2} * {}^{r-2}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]^{-1}_{x=x_1\#x_2} * {}^{r-3}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]^{-1}_{x=x_1\#x_2} * \dots * {}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]^{-1}_{x=x_1\#x_2} * \begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]$$

Hence $${}_{x=x_1\#x_2}^{\quad r}\left[\binom{n(x)}{d(x)}\right] = \left[{}^{r-1}\left[\binom{n(x)}{d(x)}\right]^{-1}_{x=x_1\#x_2} * {}^{r-2}\left[\binom{n(x)}{d(x)}\right]^{-1}_{x=x_1\#x_2} * {}^{r-3}\left[\binom{n(x)}{d(x)}\right]^{-1}_{x=x_1\#x_2} * \dots * {}^{0}\left[\binom{n(x)}{d(x)}\right]^{-1}_{x=x_1\#x_2} * \binom{n(x)}{d(x)} \right]$$ **which is identity no 7.**

Using Identity no 6, one can prove for $x_1, x_2, \dots, x_{2*r}$ as $$\left[\binom{n(x)}{d(x)}\right] = \left[{}^{0}\left[\binom{n(x)}{d(x)}\right]_{x=x_1\#x_2} * {}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]_{x=x_3\#x_4} * {}_{x=x_3\#x_4}^{\quad 1}\left[{}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]\right]_{x=x_5\#x_6} * {}_{x=x_5\#x_6}^{\quad 1}\left[{}_{x=x_3\#x_4}^{\quad 1}\left[{}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]\right]\right]_{x=x_7\#x_8} * \dots * \right.$$
$$ {}_{x=x_{2*r-3}\#x_{2*r-2}}^{\quad 1}\left[{}_{x=x_{2*r-5}\#x_{2*r-4}}^{\quad 1}\left[\dots {}_{x=x_5\#x_6}^{\quad 1}\left[{}_{x=x_3\#x_4}^{\quad 1}\left[{}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]\right]\right]\right]\right]_{x=x_{2*r-1}\#x_{2*r}} *$$
$$\left. {}_{x=x_{2*r-1}\#x_{2*r}}^{\quad 1}\left[{}_{x=x_{2*r-3}\#x_{2*r-2}}^{\quad 1}\left[{}_{x=x_{2*r-5}\#x_{2*r-4}}^{\quad 1}\left[\dots {}_{x=x_5\#x_6}^{\quad 1}\left[{}_{x=x_3\#x_4}^{\quad 1}\left[{}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]\right]\right]\right]\right]\right]\right]$$ **Which is identity no 8.**

Let the operator

$${}_{x=x_1\#x_2\#\dots\#x_{2*r}}^{\quad 1}\left[\binom{n(x)}{d(x)}\right] = {}_{x=x_{2*r-1}\#x_{2*r}}^{\quad 1}\left[{}_{x=x_{2*r-3}\#x_{2*r-2}}^{\quad 1}\left[{}_{x=x_{2*r-5}\#x_{2*r-4}}^{\quad 1}\left[\dots {}_{x=x_5\#x_6}^{\quad 1}\left[{}_{x=x_3\#x_4}^{\quad 1}\left[{}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]\right]\right]\right]\right]\right]$$ and the operator

$${}_{x=x_1\#x_2\#\dots\#x_{2*r}}^{\quad s}\left[\binom{n(x)}{d(x)}\right] = \binom{n_s(x)}{d_s(x)} =$$
$${}_{x=x_{2*r-1}\#x_{2*r}}^{\quad 1}\left[{}_{x=x_{2*r-3}\#x_{2*r-2}}^{\quad 1}\left[{}_{x=x_{2*r-5}\#x_{2*r-4}}^{\quad 1}\left[\dots {}_{x=x_5\#x_6}^{\quad 1}\left[{}_{x=x_3\#x_4}^{\quad 1}\left[{}_{x=x_1\#x_2}^{\quad 1}\left[{}_{x=x_1\#x_2\#\dots\#x_{2*r}}^{\quad s-1}\left[\binom{n(x)}{d(x)}\right]\right]\right]\right]\right]\right]\right]$$

And also the operator,

$$\left[\binom{n(x)}{d(x)}\right]_{x=x_1\#x_2\#\dots\#x_{2*r}} = {}^{0}\left[\binom{n(x)}{d(x)}\right]_{x=x_1\#x_2} * {}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]_{x=x_3\#x_4} * {}_{x=x_3\#x_4}^{\quad 1}\left[{}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]\right]_{x=x_5\#x_6} * \dots *$$
$${}_{x=x_{2*r-3}\#x_{2*r-2}}^{\quad 1}\left[{}_{x=x_{2*r-5}\#x_{2*r-4}}^{\quad 1}\left[\dots {}_{x=x_5\#x_6}^{\quad 1}\left[{}_{x=x_3\#x_4}^{\quad 1}\left[{}_{x=x_1\#x_2}^{\quad 1}\left[\binom{n(x)}{d(x)}\right]\right]\right]\right]\right]_{x=x_{2*r-1}\#x_{2*r}}$$ and

then $${}_{x=x_1\#x_2\#\dots\#x_{2*r}}^{\quad s}\left[\binom{n(x)}{d(x)}\right]_{x=x_1\#x_2\#\dots\#x_{2*r}} =$$

$$ {}^{0}\left[{}_{x=x_1\#x_2\#\ldots\#x_{2*r}} {}^{s}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]\right]_{x=x_1\#x_2} * {}_{x=x_1\#x_2}{}^{1}\left[{}_{x=x_1\#x_2\#\ldots\#x_{2*r}}{}^{r}\left[{}_{x=x_1\#x_2\#\ldots\#x_{2*r}}{}^{s}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]\right]\right]_{x=x_3\#x_4} * $$

$$ {}_{x=x_3\#x_4}{}^{1}\left[{}_{x=x_1\#x_2}{}^{1}\left[{}_{x=x_1\#x_2\#\ldots\#x_{2*r}}{}^{s}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]\right]\right]_{x=x_5\#x_6} * \ldots * $$

$$ {}_{x=x_{2*r-3}\#x_{2*r-2}}{}^{1}\left[{}_{x=x_{2*r-5}\#x_{2*r-4}}{}^{1}\left[\ldots {}_{x=x_5\#x_6}{}^{1}\left[{}_{x=x_3\#x_4}{}^{1}\left[{}_{x=x_1\#x_2}{}^{1}\left[{}_{x=x_1\#x_2\#\ldots\#x_{2*r}}{}^{s}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]\right]\right]\right]\right]\right]_{x=x_{2*r-1}\#x_{2*r}} $$

and let it be operator

$$ {}^{s}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{2*r}} $$

If we also expand identity no 8, further as we did in identity no 6, we will get the following series.

$$ \left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right] = $$

$$ \left[{}^{\mathbf{0}}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{2*r}} * {}^{\mathbf{1}}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{2*r}} * \ldots * {}^{\boldsymbol{s-1}}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{2*r}} * {}_{x=x_1\#x_2\#\ldots\#x_{2*r}}{}^{s}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right] \right] $$

And limit to $\boldsymbol{s}$ terms until $\boldsymbol{n_s(x)} : \boldsymbol{d_s(x)} = \boldsymbol{k}$ where $\boldsymbol{k}$ is a constant which is identity no 9.

Instead of even terms, if we have odd terms, then also, we can construct matrix operators as follows:

For the first term x_1 alone we need to have the matrix operator as,

Let $\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} n(x_1) & \alpha*(x-x_1) \\ d(x_1) & \beta*(x-x_1) \end{pmatrix}\begin{pmatrix} n_1(x) \\ d_1(x) \end{pmatrix}\right]$ where α, β can be any constant and $\frac{n(x_1)}{d(x_1)} \neq \frac{\alpha}{\beta}$ and remaining terms will be even and hence there after even terms can be used for the operator as given in identity no 8. I.e.)

$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} n(x_1) & \alpha*(x-x_1) \\ d(x_1) & \beta*(x-x_1) \end{pmatrix}\begin{pmatrix} n_1(x) \\ d_1(x) \end{pmatrix}\right]$, $\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1} = \begin{pmatrix} n(x_1) & \alpha*(x-x_1) \\ d(x_1) & \beta*(x-x_1) \end{pmatrix}$ and

$$ {}_{x=x_1}{}^{1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} \beta*(x-x_1) & -\alpha*(x-x_1) \\ -d(x_1) & n(x_1) \end{pmatrix}\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} \beta & -\alpha \\ -\frac{d(x_1)}{x-x_1} & \frac{n(x_1)}{x-x_1} \end{pmatrix}\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] $$

We can also get what are α, β. Using identity no 2, by dividing $(x-x_1)$, we get $\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] =$ $\left[\begin{pmatrix} n(x_1) & \alpha*(x-x_1) \\ d(x_1) & \beta*(x-x_1) \end{pmatrix}\begin{pmatrix} n_1(x) \\ d_1(x) \end{pmatrix}\right] = \left[\begin{pmatrix} \frac{n(x_1)}{(x-x_1)} & \alpha \\ \frac{d(x_1)}{(x-x_1)} & \beta \end{pmatrix}\begin{pmatrix} n_1(x) \\ d_1(x) \end{pmatrix}\right]$ and on substituting $x = \infty$, $\left[\begin{pmatrix} n(\infty) \\ d(\infty) \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix}\begin{pmatrix} n_1(x) \\ d_1(x) \end{pmatrix}\right] =$ $\left[\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right]$. That simply makes $\left[\begin{pmatrix} n(\infty) \\ d(\infty) \end{pmatrix}\right] = \left[\begin{pmatrix} \propto \\ \beta \end{pmatrix}\right]$.

Hence One can prove that, if $\frac{n(\infty)}{d(\infty)} = 0$, then $\alpha = 0$ *and* $\beta = 1$, If $\frac{n(\infty)}{d(\infty)} = \infty$, then $\alpha = 1$ *and* $\beta = 0$, for other values $\beta = \frac{d(\infty)}{n(\infty)} * \alpha$. And we can make following as **identity no. 10:**

$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1} = \begin{pmatrix} n(x_1) & \alpha*(x-x_1) \\ d(x_1) & \beta*(x-x_1) \end{pmatrix}$ **And** $\underset{x=x_1}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} \beta & -\alpha \\ -\frac{d(x_1)}{x-x_1} & \frac{n(x_1)}{x-x_1} \end{pmatrix}\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]$ **where if** $\frac{n(\infty)}{d(\infty)} = 0$, **then** $\alpha = 0$ *and* $\beta = 1$, **If** $\frac{n(\infty)}{d(\infty)} = \infty$, **then** $\alpha = 1$ *and* $\beta = 0$, **for other values** $\beta = \frac{d(\infty)}{n(\infty)} * \alpha$

Using these operators, Identity no 8 can be expanded for odd numbers as given below.

$$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] =$$

$$\Bigg[{}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1} * \underset{x=x_1}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_2\#x_3} * \underset{x=x_2\#x_3}{{}^{1}}\left[\underset{x=x_1}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]\right]_{x=x_4\#x_5} * \underset{x=x_4\#x_5}{{}^{1}}\left[\underset{x=x_2\#x_3}{{}^{1}}\left[\underset{x=x_1}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]\right]\right]_{x=x_6\#x_7} * \dots *$$

$$\underset{x=x_{2*r-4}\#x_{2*r-3}}{{}^{1}}\left[\underset{x=x_{2*r-6}\#x_{2*r-5}}{{}^{1}}\left[\dots \underset{x=x_4\#x_5}{{}^{1}}\left[\underset{x=x_2\#x_3}{{}^{1}}\left[\underset{x=x_1}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]\right]\right]\right]\right]_{x=x_{2*r-2}\#x_{2*r-1}} *$$

$$\underset{x=x_{2*r-2}\#x_{2*r-1}}{{}^{1}}\left[\underset{x=x_{2*r-4}\#x_{2*r-3}}{{}^{1}}\left[\underset{x=x_{2*r-6}\#x_{2*r-5}}{{}^{1}}\left[\dots \underset{x=x_4\#x_5}{{}^{1}}\left[\underset{x=x_2\#x_3}{{}^{1}}\left[\underset{x=x_1}{{}^{1}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]\right]\right]\right]\right]\right]\Bigg]$$

Which is identity no 11.

And similarly identity no 9 will be expanded for odd numbers as below.

$$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] =$$

$$\left[{}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2\#\dots\#x_{2*r-1}} * {}^{1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2\#\dots\#x_{2*r-1}} * \dots * {}^{s-1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1\#x_2\#\dots\#x_{2*r-1}} * \underset{x=x_1\#x_2\#\dots\#x_{2*r-1}}{{}^{s}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] \right]$$

And limit to s terms until $n_s(x) : d_s(x) = k$ **where k is a constant which is identity no 12.**

Using identity no 12, if we expand only for the term x_1 alone then we gets special expansion as follows **to identity no.13:**

If $\underset{x=x_1}{{}^{s}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \begin{pmatrix} n_s(x) \\ d_s(x) \end{pmatrix}$ **and if** $\alpha = 1$ *and* $\beta = 0$, **then** $n_s(x) = d_{s-1}(x)$ **and** $d_s(x) = \frac{(d_{s-1}(x_1) * n_{s-1}(x) - n_{s-1}(x_1) * d_{s-1}(x))}{(x-x_1)}$ **for** $s = 1, 2, 3 \dots, \infty$ **and after substituting** $n_{s-1}(x)$ *as* $d_{s-2}(x)$, **then** $d_s(x) = \frac{(d_{s-1}(x_1) * d_{s-2}(x) - d_{s-2}(x_1) * d_{s-1}(x))}{(x-x_1)}$ **for** $s = 2, 3, 4 \dots, \infty$

If $\alpha = 0$ *and* $\beta = 1$, **then** $n_s(x) = n_{s-1}(x)$ **and** $d_s(x) = \frac{(-d_{s-1}(x_1) * n_{s-1}(x) + n_{s-1}(x_1) * d_{s-1}(x))}{(x-x_1)}$ **for** $s = 1, 2, 3 \dots, \infty$**. . And Expansion series will be as given below.**

$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[{}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1} * {}^{1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1} * \dots * {}^{s-1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1} * \underset{x=x_1}{{}^{s}}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] \right]$ **And limit to** s **terms until** $n_s(x) : d_s(x) = k$ **where** k **is a constant**

In identity no 6, If $\frac{n(x_1)}{d(x_1)} = \frac{n(x_2)}{d(x_2)}$ then we need to expand the series like following,

If $\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} n(x_1)*(x-x_3) & n(x_3)*(x-x_1)*(x-x_2) \\ d(x_1)*(x-x_3) & d(x_3)*(x-x_1)*(x-x_2) \end{pmatrix} \begin{pmatrix} n_1(x) \\ d_1(x) \end{pmatrix}\right]$ and let operator, ${}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1,x_2\#x_3} = \begin{pmatrix} n(x_1)*(x-x_3) & n(x_3)*(x-x_1)*(x-x_2) \\ d(x_1)*(x-x_3) & d(x_3)*(x-x_1)*(x-x_2) \end{pmatrix}$ then

$$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[{}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1,x_2\#x_3} * {}^{1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1,x_2\#x_3} * {}^{2}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1,x_2\#x_3} * \dots * {}^{r-1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1,x_2\#x_3} * {}_{x=x_1,x_2\#\ 3}^{\quad r}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] \right]$$

Similar way, if $\frac{n(x_1)}{d(x_1)} = \frac{n(x_2)}{d(x_2)} = \cdots = \frac{n(x_{s-1})}{d(x_{s-1})} = \frac{n(x_s)}{d(x_s)}$ and $\frac{n(y_1)}{d(y_1)} = \frac{n(y_2)}{d(y_2)} = \cdots = \frac{n(y_{t-1})}{d(y_{t-1})} = \frac{n(y_t)}{d(y_t)}$ and let operator, ${}^{0}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right]_{x=x_1,x_2,\dots,x_s\ \#y_1,y_2,\dots,y_t} = \begin{pmatrix} n(x_1)*\prod_{r=1}^{t}(x-y_r) & n(y_1)*\prod_{r=1}^{s}(x-x_r) \\ d(x_1)*\prod_{r=1}^{t}(x-y_r) & d(y_1)*\prod_{r=1}^{s}(x-x_r) \end{pmatrix}$ then we need to expand the series like following,

$$\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right] = \left[{}^{\mathbf{0}}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1,x_2,\dots,x_s\ \#y_1,y_2,\dots,y_t} * {}^{\mathbf{1}}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1,x_2,\dots,x_s\ \#y_1,y_2,\dots,y_t} * {}^{\mathbf{2}}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1,x_2,\dots,x_s\ \#y_1,y_2,\dots,y_t} * \dots * {}^{\boldsymbol{r-1}}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right]_{x=x_1,x_2,\dots,x_s\ \#y_1,y_2,\dots,y_t} * {}_{x=x_1,x_2,\dots,x_s\ \#y_1,y_2,\dots,y_t}^{\quad \boldsymbol{r}}\left[\begin{pmatrix} \boldsymbol{n(x)} \\ \boldsymbol{d(x)} \end{pmatrix}\right] \right]$$ **Which is identity no 14.**

Since $a : b = c : d, then \frac{a}{b} = \frac{c}{d}$ and using Identity no 6 or Identity no 13, Let right hand side expands to $\left[\begin{pmatrix} N(x) \\ D(x) \end{pmatrix}\right]$ then $\frac{n(x)}{d(x)} = \frac{N(x)}{D(x)}$ where $n(x)$=numerator and $d(x)$ =denominator

If $n(x)$ is a polynomial of degree r_n and $d(x)$ is a polynomial of degree r_d then above series, expands up to maximum (r_n, r_d) terms and the last term is constant.

For other functions using identity no 13, we can limit r to where $n_r(x) : d_r(x)$ is almost constant or x can be nearer to x_1 or ∞. Similarly using identity no 6, we can limit r to where $n_r(x) : d_r(x)$ is almost constant or x can be nearer to x_1 *or* x_2 Or if $x = \frac{(x_2*(l-l_1)-x_1*(l-l_2))}{l_2-l_1}$, then l can be nearer to l_1 *or* l_2, which leads x nearer to x_1 *or* x_2 and we can expand to l instead of x. This is identity no 15.

Similar concept of 2 x 1 matrixes can be expanded to m x 1 matrixes to all the identities above which is identity no 16.

For example, for identity no 6, m x 1 matrix will be,

$$\left[\begin{pmatrix} y_1(x) \\ \vdots \\ y_m(x) \end{pmatrix}\right] =$$

$$\left[{}^{0}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_m} * {}^{1}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_m} * {}^{2}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_m} * \ldots * {}^{r-1}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_m} * {}_{x=x_1\#x_2\#\ldots\#x_m}^{r}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] \right]$$

And r can be expanded until $y_{1r}(x) : y_{2r}(x): \ldots : y_{mr}(x) = k$ where k is a constant. Where in following are the operators used

$$\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_m} = \begin{pmatrix} \frac{y_1(x_1)*\prod_{\mathrm{u}=1}^{\mathrm{m}}(\mathrm{x}-x_u)}{(\mathrm{x}-x_1)} & \cdots & \frac{y_1(x_m)*\prod_{\mathrm{u}=1}^{\mathrm{m}}(\mathrm{x}-x_u)}{(\mathrm{x}-x_m)} \\ \vdots & \ddots & \vdots \\ \frac{y_m(x_1)*\prod_{\mathrm{u}=1}^{\mathrm{m}}(\mathrm{x}-x_u)}{(\mathrm{x}-x_1)} & \cdots & \frac{y_m(x_m)*\prod_{\mathrm{u}=1}^{\mathrm{m}}(\mathrm{x}-x_u)}{(\mathrm{x}-x_m)} \end{pmatrix},$$

$${}_{x=x_1\#x_2\#\ldots\#x_m}^{1}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] = \begin{pmatrix} \frac{y_1(x_1)*\prod_{\mathrm{u}=1}^{\mathrm{m}}(\mathrm{x}-x_u)}{(\mathrm{x}-x_1)} & \cdots & \frac{y_1(x_m)*\prod_{\mathrm{u}=1}^{\mathrm{m}}(\mathrm{x}-x_u)}{(\mathrm{x}-x_m)} \\ \vdots & \ddots & \vdots \\ \frac{y_m(x_1)*\prod_{\mathrm{u}=1}^{\mathrm{m}}(\mathrm{x}-x_u)}{(\mathrm{x}-x_1)} & \cdots & \frac{y_m(x_m)*\prod_{\mathrm{u}=1}^{\mathrm{m}}(\mathrm{x}-x_u)}{(\mathrm{x}-x_m)} \end{pmatrix}^{-1} * \left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] \text{ And}$$

$${}_{x=x_1\#x_2\#\ldots\#x_m}^{r}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] = {}_{x=x_1\#x_2\#\ldots\#x_m}^{1}\left[{}_{x=x_1\#x_2\#\ldots\#x_m}^{r-1}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]\right]$$

Similarly for identity no 13, m x 1 matrix will be

$$\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] = \left[{}^{0}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{m-1}} * {}^{1}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{m-1}} * {}^{2}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{m-1}} * \ldots * {}^{r-1}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{m-1}} * {}_{x=x_1\#x_2\#\ldots\#x_{m-1}}^{r}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] \right]$$

And r can be expanded until $y_{1r}(x) : y_{2r}(x): \ldots : y_{mr}(x) = k$ where k is a constant. Where in following are the operators used

$$\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]_{x=x_1\#x_2\#\ldots\#x_{m-1}} = \begin{pmatrix} \frac{y_1(x_1)*\prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u)}{(\mathrm{x}-x_1)} & \cdots & \frac{y_1(x_{m-1})*\prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u)}{(\mathrm{x}-x_{m-1})} & y_1(\infty) * \prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{y_m(x_1)*\prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u)}{(\mathrm{x}-x_1)} & \cdots & \frac{y_m(x_{m-1})*\prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u)}{(\mathrm{x}-x_{m-1})} & y_m(\infty) * \prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u) \end{pmatrix},$$

$${}_{x=x_1\#x_2\#\ldots\#x_{m-1}}^{1}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] = \begin{pmatrix} \frac{y_1(x_1)*\prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u)}{(\mathrm{x}-x_1)} & \cdots & \frac{y_1(x_{m-1})*\prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x\ )}{(\mathrm{x}-x_{m-1})} & y_1(\infty) * \prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{y_m(x_1)*\prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u)}{(\mathrm{x}-x_1)} & \cdots & \frac{y_m(x_{m-1})*\prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u)}{(\mathrm{x}-x_{m-1})} & y_m(\infty) * \prod_{\mathrm{u}=1}^{\mathrm{m}-1}(\mathrm{x}-x_u) \end{pmatrix}^{-1} * \left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] \text{ And}$$

$${}_{x=x_1\#x_2\#\ldots\#x_{m-1}}^{r}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right] = {}_{x=x_1\#x_2\#\ldots\#x_{m-1}}^{1}\left[{}_{x=x_1\#x_2\#\ldots\#x_{m-1}}^{r-1}\left[\begin{pmatrix} y_1(\mathrm{x}) \\ \vdots \\ y_m(\mathrm{x}) \end{pmatrix}\right]\right]$$

Another concept also can be looked in to expand the series like the following.

Let If $n_v(x) = a_{v0} + a_{v1} * x + a_{v2} * x^2 + a_{v3} * x^3 + \cdots + a_{vr} * x^r = 0$ where a_{v0} and a_{vr} are not zero, $d_v(x) = b_{v0} + b_{v1} * x + b_{v2} * x^2 + b_{v3} * x^3 + \cdots + b_{vs} * x^s = 0$ where b_{v0} and b_{vs} are not zero and then for $\frac{n_{v0}(x)}{d_{v0}(x)}$, all the variables $a_{v0}, a_{v1}, a_{v2}, \dots, a_{vr}$ and $b_{v0}, b_{v1}, b_{v2}, \dots, b_{vs}$ can be found if we know the values of $\frac{n_{v0}(x_u)}{d_{v0}(x_u)}$ for all $u = 1,2,3, \dots, r + s + 1$. Let us say $t = r + s + 1$, then for any natural number t we can have multiple r and s as whole numbers. For example, if $t = 2$, then $(r = 0 \; and \; s = 1) \; or \; (r = 1 \; and \; s = 0)$. If in case $r = s$, then we can use the identity no 8 to get the variables. If in case $(r = s - 1) \; or \; (r - 1 = s)$, then we can use identity no 11 to get the variables. For all other values of r and s, we can get after solving linear equations via matrixes.

If we can find these variables, then we can expand to the following series which is **identity no 17**,

$$\frac{n(x)}{d(x)} = \frac{n_0(x)}{d_0(x)} + \frac{n_1(x)}{d_1(x)} * \left((x - x_1) * (x - x_2) * (x - x_3) * \dots (x - x_t)\right)^1 + \frac{n_2(x)}{d_2(x)} * \left((x - x_1) * (x - x_2) * (x - x_3) * \dots (x - x_t)\right)^2 + \cdots + \frac{n_v(x)}{d_v(x)} * \left((x - x_1) * (x - x_2) * (x - x_3) * \dots (x - x_t)\right)^v$$

Until the value $\left((x - x_1) * (x - x_2) * (x - x_3) * \dots (x - x_t)\right)^v$ **is very small enough Where** $\frac{n_1(x)}{d_1(x)} = \frac{\left(\frac{n(x)}{d(x)} - \frac{n_0(x)}{d_0(x)}\right)}{((x-x_1) * (x-x_2) * (x-x_3) * \dots (x-x_t))}$ **and**

$$\frac{n_v(x)}{d_v(x)} = \frac{\left(\frac{n_{v-1}(x)}{d_{v-1}(x)} - \frac{n_{v-2}(x)}{d_{v-2}(x)}\right)}{((x-x_1) * (x-x_2) * (x-x_3) * \dots (x-x_t))}$$ **for all** $v = 2, 3, 4, \dots, \infty$

Let us see examples of expansion series below.

For $n(x) = e + f * (x - x_1) + \sqrt{g^2 + h * (x - x_1) + i * (x - x_1)^2}$ and if $d(x) = 1$, then using identity no 13, matrix will be as follows:

Since

$$\frac{n(\infty)}{d(\infty)} = \infty, \; \prod_{x=x_1}^{1}\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & -1 \\ -\frac{d(x_1)}{x-x_1} & \frac{n(x_1)}{x-x_1} \end{pmatrix} \begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] =$$

$$\left[\begin{pmatrix} 0 & -1 \\ -\frac{1}{x-x_1} & \frac{e+g}{x-x_1} \end{pmatrix} \begin{pmatrix} e + f * (x - x_1) + \sqrt{g^2 + h * (x - x_1) + i * (x - x_1)^2} \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ \frac{e + f*(x-x_1) + \sqrt{g^2 + h*(x-x_1) + i*(x-x_1)^2} - e - g}{x - x_1} \end{pmatrix}\right]$$

On multiplying, $-f * (x - x_1) + \sqrt{g^2 + h * (x - x_1) + i * (x - x_1)^2} + g$ both the sides, we get,

$$\left[\begin{pmatrix} -f * (x - x_1) + \sqrt{g^2 + h * (x - x_1) + i * (x - x_1)^2} + g \\ (i - f^2) * (x - x_1) + 2 * f * g + h \end{pmatrix}\right]$$

If $i = f^2 = 0$, then we get $n(x)$ repeats itself with only change in constant variable g. Hence it will be repeating from second term.

I.e.) $$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} e + \sqrt{g^2 + h * (x - x_1)} \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} e + g & x - x_1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 2 * g & x - x_1 \\ h & 0 \end{pmatrix}^\infty * \begin{pmatrix} n_\infty \\ d_\beta \end{pmatrix}\right]$$

If $i = j^2 \neq 0$ then $\frac{n_1(\infty)}{d_1(\infty)} = \frac{1}{j+f}$, hence it will be looking like following:

$$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} e + g & x - x_1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 2 * g & x - x_1 \\ 2 * f * g + h & (f + j) * (x - x_1) \end{pmatrix} * \dots * \begin{pmatrix} n_\infty \\ d_\beta \end{pmatrix}\right]$$

For $n(x) = e + f * x + \sqrt{g * (x - x_1) * (x - x_2) + h^2 + (\frac{(i^2 - h^2) * (x - x_1) + (h^2 - j^2) * (x - x_2)}{(x_2 - x_1)})}$ and if $d(x) = 1$, then using identity no 6, matrix will be repeating after every two terms with all real numbers.

I.e.) $\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\left(\begin{matrix} (e + f * x_1 + j) * (x - x_2) & (e + f * x_2 + i) * (x - x_1) \\ (x - x_2) & (x - x_1) \end{matrix}\right) * \left(\begin{pmatrix} a_1 * (x - x_2) & a_2 * (x - x_1) \\ a_3 * (x - x_2) & a_4 * (x - x_1) \end{pmatrix} * \begin{pmatrix} b_1 * (x - x_2) & b_2 * (x - x_1) \\ b_3 * (x - x_2) & b_4 * (x - x_1) \end{pmatrix}\right)^{\infty} * \begin{pmatrix} n_{\propto} \\ d_{\beta} \end{pmatrix}\right]$ or

$$\left[\begin{pmatrix} n(x) \\ d(x) \end{pmatrix}\right] = \left[\begin{pmatrix} n(x_1) * (x - x_2) & n(x_2) * (x - x_1) \\ d(x_1) * (x - x_2) & d(x_2) * (x - x_1) \end{pmatrix} * \left(\begin{pmatrix} a_1 * (x - x_2) & a_2 * (x - x_1) \\ a_3 * (x - x_2) & a_4 * (x - x_1) \end{pmatrix} * \begin{pmatrix} b_1 * (x - x_2) & b_2 * (x - x_1) \\ b_3 * (x - x_2) & b_4 * (x - x_1) \end{pmatrix}\right)^{\infty} * \begin{pmatrix} a_1 * (x - x_2) & a_2 * (x - x_1) \\ a_3 * (x - x_2) & a_4 * (x - x_1) \end{pmatrix} * \begin{pmatrix} n_{\delta} \\ d_{\gamma} \end{pmatrix}\right]$$

Thus these derivations can be used to expand any function into matrix multiplication series.