Queue Length Analysis of a Repairable Bulk Arrival Queuing System under Double Threshold Policy, Repeated Vacations and Single SOS Facility

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***Abstract*-This paper analyses Bi-level Threshold policy for a bulk arrival queueing system with single second optional service facility and multiple vacation. In the present model the server is subject to break down during the service. Then it is sent for repair immediately and the customer just being served, waits in the service facility for the server to return from repair facility and then to complete the remaining service. The Probability Generating Functions of the system is derived using supplementary variable technique. Moreover, some important performance measures of this model are also presented with some numerical examples.**

***Keywords*-Bi-level threshold policy, Batch arrival, Second optional service facility, Multiple vacation, Supplementary variable technique**

I. INTRODUCTION:

Queueing systems in which the server is sometimes inactive (taking vacations or working elsewhere) while customers are distribution. Because of the complexity, the most general models are investigated under the assumption that the server is perfect and never fails. However in practice, the server may breakdown at any time and need to be repaired. The phenomena of the server breakdowns, can be encountered in the area of computers, communication networks, flexible manufacturing systems etc. The performance of the system may be affected heavily by these breakdowns and limited service capacity. Queuing systems with server unreliable situations are the topic of worth investigating from the performance point of view. Recently Julia Rose Mary (2011) analysed a more general batch arrival repairable queueing system in which the server provides c-kinds of general heterogeneous service and the arriving customer has the option of choosing any one of the c-kinds of service. In the present model, the author analyses a batch arrival queueing system $M^X/G_{SOS}/1$ with Second Optional Service facility in which the server is subject to breakdown during the service and take multiple vacations whenever the system becomes empty. The system is studied under steady state and the total PGF of the system size is derived through different partial generating functions. Various performance and measures are waiting for service may find many applications in the performance modelling of computer and communication systems. Vacation concept was introduced by Levi and Yechalli in 1975 and two standard policies, single and multiple vacations are defined. Lee et al. (2003) analyzed a batch arrival systems M^X / G / 1 under bilevel threshold with early setup and with / without server vacation. In the present model the bi-level threshold policy is analyzed under multiple vacation policy. Single vacation models represent a machine maintenance or postproduction operation, while multiple vacation models may correspond to an efficient utilization of servers for secondary jobs. Queuing models with server interruptions have proved to be a useful abstraction in situations, where a single server operates more than one service to the arriving customers. Wang and Ke (2002) analyzed the control policy for M / G / 1 queueing system with an unreliable server. Later, Ke (2004b) generalizes the previous model to M^X / G / 1 queue with startup time where the server breakdowns occur according to Poisson process and the repair time has general

calculated to analyze the system analytically as well as numerically. Some particular cases are also deduced.

II. MODEL DESCRIPTION:

The customers arrive in batches according to the time homogeneous Compound Poisson process with group arrival rate λ .

A. *Idle period:*

The server is turned off and leaves the system for a vacation of random length V_1 as soon as the system empties. After returning from the vacation, if the server finds **m** or more customers in the system, then he immediately starts a setup operation of random length D. Otherwise he takes repeated number of vacations V_2 ,V_3 … until he finally finds at least **m** customers accumulated in the system. The random variables V_1,V_2,… are assumed to be independently and identically distributed with generic representation V. At the end of the setup operation , if the queue length is greater than or equal to N, then the server begins to serve the customers one by one exhaustively. Otherwise, the server remains dormant in the system waiting for the queue length to reach or exceed N. Here the idle period is made up of vacation period, setup period and dormant period. The vacation time (V) and setup time (D) assumed to follow general distributions V(x) and D(x) with finite moments.

B. Busy period:

During busy period, the server provides two kinds of general heterogeneous service to the customers. The customers are served, one at a time according to the order of their arrivals. Each customers undergoes the First Essential Service(FES). After completing the FES, the customer may leave the system with probability(1-r) or may offer for Second Optional Service (SOS) in an additional channel with probability r ($0 \le r \le 1$). It is assumed that the service times of FES and SOS respectively follow heterogeneous general distributions $S_i(x)$, i=1,2 with density function $s_i(x)$ and finite moments $E(S_i^k)$, i,k=1,2.

C. Breakdowns and repairs:

The server is subjected to breakdowns at any time while serving customers. It is assumed that the life time of the server follows exponential distribution with parameter a_i (i=1,2) according as the breakdowns occur during the FES or SOS of the server. Whenever the breakdowns occur, the server is sent for repair immediately and the customer just being served, waits in the service facility for the server to return from repair facility and then to complete the remaining service. The repair times R_i(i=1,2) of the server follow general distributions $R_i(y)$ with density functions $r_i(y)$ and finite k^{th} moments $E(R_i^k)$,k=1,2. Immediately after the server is fixed , the customer waiting in the service facility is taken up for service. Further it is assumed that the service time for a customer is cumulative and after repair, the server is as good as new. The server continues this type of service until the system becomes empty. The arriving customers always join the system and form a single waiting line based on the order of the batches. It is further assumed that the customer with in a batch are pre-ordered for service. The customers are served one by one according to the order in the queue. i.e., the server is turned off only when the system becomes empty and leaves the system for vacation. Thus busy period and breakdown period constitute completion period .The system will be turned on again for setup only when the server turns from the vacation. Thus the cycle is made up of completion period and idle period. Finally, various stochastic processes involved in the system are assumed to be independent of each other. This model is denoted by $M^X_{(m,N)}/G_{SOS}/1/MV$

III. SYSTEM SIZE DISTRIBUTION AT RANDOM EPOCH

For the multiple vacation model, the buildup period is 0. The state of the system is denoted by Y(t)=0,1,2,3,4,5 and 6 when the server is on vacation , doing setup work, in dormant state, busy with FES,SOS and in repair mode due to FES and SOS respectively.

The definitions of the state dependent probabilities are explained below:

Let Z(t)=j (j=1,2,…) denote that the server is on j^{th} vacation at time t counting from the idle period initiation point.

$Q_{n,j}(x,t)dt = \Pr(N(t)=n, x \le V_0(t) \le x+dt, Y(t)=0, Z(t)=j, j \ge 1)$

$$D_n(x,t)dt = \Pr(N(t)=n, x \le D^0(t) \le x+dt, Y(t)=1), \quad n \ge m$$

$$U_n(t) = \Pr(N(t)=n,\ Y(t)=2), \quad m \le n \le N-1$$

$$P_{n,1}(x,t)dt = \Pr(N(t)=n,\ x \le S_1{}^0(t) \le x+dt,\ Y(t)=3), \quad n \ge 1$$

$$P_{n,2}(x,t)dt = \Pr(N(t)=n,\ x \le S_2{}^0(t) \le x+dt,\quad Y(t)=4), \quad n \ge 1$$

$B_{n,i}(y)$ denotes the case where Y(t) =5,6

$$B_{n,1}(x,y,t)dt = \Pr(N(t)=n, S_1^0(t)=x, y \le R_1^0(t) \le Y+dt, Y(t)=5), \quad n \ge 1$$

$$B_{n,2}(x,y,t)dt = \Pr(N(t)=n, S_2^0(t)=x, y \le R_2^0(t) \le Y+dt, Y(t)=6), \quad n \ge 1$$

$U_n(t)$ denotes the probability that there are **n** customers in the system at time t, when the system is in dormant state. $D_n(x,t)$ *and* $P_{n,i}(x,t)\ i=1,2$ denote the probability that there are n customers in the system at arbitrary epoch with the remaining setup time and service time lie in the interval [x,x+ Δt].
$Q_{n,j}(x,t)$denotes the joint probability that at time t, there are **n** customers in the system, the server is in the j^{th} vacation and the remaining vacation time lies in the interval (x,x+dt).

$B_{n,i}(x, y, t)$ dt, i = 1, 2 is the joint probability that at time t, there are **n** customers in the system, the remaining service time for the customer under service is equal to x, and the server is being repaired with the remaining repair time between y and y + dt, where x = 0, n = 0.
Further $Q_n(0), D_n(0), P_{n,i}(0)\ i=1,2$ denote the probability that there are **n** customers in the system at the

termination of vacation period, setup period and service time respectively.

Following the arguments of Cox (1955) and observing the changes of states during the interval (t,t+dt) for any time t, the steady state equations are obtained:

A. Vacation state

$$\frac{-d}{dx}Q_{0,1}(x) = -\lambda Q_{0,1}(x) + (P_{1,1}(0)(1-r) + P_{1,2}(0))v(x)$$

$$\frac{-d}{dx}Q_{n,1}(x) = -\lambda Q_{n,1}(x) + \lambda\sum_{k=1}^{n} Q_{n-k,1}(x)g_k\,,$$
$n \geq 1$

$$\frac{-d}{dx}Q_{0,j}(x) = -\lambda Q_{0,j}(x) + Q_{0,j-1}(0)v(x)\,,$$
$j \geq 2$

$$\frac{-d}{dx}Q_{n,j}(x) = -\lambda Q_{n,j}(x) + Q_{n,j-1}(0)v(x)$$

$$+\lambda\sum_{k=1}^{n} Q_{n-k,j}(x)g_k$$

$j \geq 2,\ 1 \leq n \leq m\text{-}1$

$$\frac{-d}{dx}Q_{n,j}(x) = -\lambda Q_{n,j}(x) + Q_{n-k,j}(x)g_k\,,$$
$j \geq 2, n \geq m$

B. Setup state:

$$\frac{-d}{dx}D_m(x) = -\lambda D_m(x) + \sum_{j=1}^{\infty} Q_{m,j}(0)d(x)$$

$$\frac{-d}{dx}D_n(x) = -\lambda D_n(x) + \lambda\sum_{k=1}^{n-m} D_{n-k}(x)g_k + \sum_{j=1}^{\infty} Q_{n,j}(0)d(x),$$
$n \geq m+1$

C. Dormant State

$$\lambda U_m = D_m(0)$$

$$\lambda U_n = D_n(0) + \lambda\sum_{k=1}^{n-m} U_{n-k,}g_k \quad m+1 \leq n \leq N\text{-}1$$

D. Busy with FES

$$\frac{-d}{dx}P_{1,1}(x) = -(\lambda + a_1)P_{1,1}(x) + (1-r)P_{2,1}(0)s_1(x)$$
$$+ P_{2,2}(0)s_1(x) + B_{1,1}(x,0)$$

$$\frac{-d}{dx}P_{n,1}(x) = -(\lambda + a_1)P_{n,1}(x) + (1-r)P_{n+1,1}(0)s_1(x)$$
$$+ P_{n+1,2}(0)s_1(x) + B_{n,1}(x,0) + \lambda\sum_{k=1}^{n-1} P_{n-k,1}(x)g_k$$
$$2 \leq n \leq N\text{-}1$$

$$\frac{-d}{dx}P_{n,1}(x) = -(\lambda + a_1)P_{n,1}(x) + (1-r)P_{n+1,1}(0)s_1(x)$$
$$+ P_{n+1,2}(0)s_1(x) + B_{n,1}(x,0) + D_n(0)s_1(x)$$
$$+ \lambda s_1(x)\sum_{k=n-N+1}^{n-m} U_{n-k}g_k + \lambda\sum_{k=1}^{n-1} P_{n-k,1}(x)g_k$$

$$n \geq N$$

E. Busy with SOS

$$\frac{-d}{dx}P_{1,2}(x) = -(\lambda + a_2)P_{1,2}(x) + rP_{1,1}(0)s_2(x) + B_{1,2}(x,0)$$

$$\frac{-d}{dx}P_{n,2}(x) = -(\lambda + a_2)P_{n,2}(x) + rP_{n,1}(0)s_2(x)$$
$$+ B_{n,2}(x,0) + \lambda\sum_{k=1}^{n-1} P_{n-k,2}(x)g_k$$

$$n \geq 2$$

F. Breakdown in FES:

$$\frac{-\partial}{\partial x}B_{1,1}(x,y) = -\lambda B_{1,1}(x,y) + a_1 P_{1,1}(x)r_1(y)$$

$$\frac{-\partial}{\partial x}B_{n,1}(x,y) = -\lambda B_{n,1}(x,y) + a_1 P_{n,1}(x)r_1(y)$$
$$+ \lambda\sum_{k=1}^{n-1} B_{n-k,1}(x,y)g_k$$

$$n \geq 2$$

G. Breakdown in SOS:

$$\frac{-\partial}{\partial x}B_{1,2}(x,y)=-\lambda B_{1,2}(x,y)+a_2P_{1,2}(x)r_2(y)$$

$$\frac{-\partial}{\partial x}B_{n,2}(x,y)=-\lambda B_{n,2}(x,y)+a_2P_{n,2}(x)r_2(y)+\lambda\sum_{k=1}^{n-1}B_{n-k,2}(x,y)g_k \qquad n\geq 2$$

Taking the LST of the steady state equations with respect to x, we have

$$\theta Q^*_{0,1}(\theta)-Q_{0,1}(0)=\lambda Q^*_{0,1}(\theta)-(P_{1,1}(0)(1-r)+P_{1,2}(0)V^*(\theta)) \qquad (1)$$

$$\theta Q^*_{n,1}(\theta)-Q_{n,1}(0)=\lambda Q^*_{n,1}(\theta)-\lambda\sum_{k=1}^{n}Q^*_{n-k}(\theta)g_k \qquad n\geq 1 \qquad (2)$$

$$\theta Q^*_{0,j}(\theta)-Q_{0,j}(0)=\lambda Q^*_{0,j}(\theta)-Q_{0,j-1}(0)V^*(\theta) \qquad j\geq 2 \qquad (3)$$

$$\theta Q^*_{n,j}(\theta)-Q_{n,j}(0)=\lambda Q^*_{n,j}(\theta)-Q_{n,j-1}(0)V^*(\theta)-\lambda\sum_{k=1}^{n}Q^*_{n-k,j}(\theta)g_k \qquad j\geq 2, 1\leq n\leq m-1 \qquad (4)$$

$$\theta Q^*_{n,j}(\theta)-Q_{n,j}(0)=\lambda Q^*_{n,j}(\theta)-\lambda\sum_{k=1}^{n}Q^*_{n-k,j}(\theta)g_k \qquad j\geq 2,\ n\geq m \qquad (5)$$

$$\theta D^*_m(\theta)-D_m(0)=\lambda D^*_m(\theta)-\sum_{j=1}^{\infty}Q_{m,j}(0)D^*(\theta) \qquad (6)$$

$$\theta D^*_n(\theta)-D_n(0)=\lambda D^*_n(\theta)-\sum_{j=1}^{\infty}Q_{n,j}(0)D^*(\theta)-\lambda\sum_{k=1}^{n-m}D^*_{n-k}(\theta)g_k \qquad n\geq m+1 \qquad (7)$$

$$\lambda U_m=D_m(0) \qquad (8)$$

$$\lambda U_n=D_n(0)+\lambda\sum_{k=1}^{n-m}U_{n-k}g_k \qquad m+1\leq n\leq N-1 \qquad (9)$$

$$\theta P^*_{1,1}(\theta)-P_{1,1}(0)=(\lambda+a_1)P^*_{1,1}(\theta)-(1-r)P_{2,1}(0)S^*_1(\theta)-P_{2,2}(0)S^*_1(\theta)-B^*_{1,1}(\theta,0) \qquad (10)$$

$$\theta P^*_{n,1}(\theta)-P_{n,1}(0)=(\lambda+a_1)P^*_{n,1}(\theta)-(1-r)P_{n+1,1}(0)S^*_1(\theta)-P_{n+1,2}(0)S^*_1(\theta)-\lambda\sum_{k=1}^{n-1}P^*_{n-k,1}(\theta)g_k \qquad 2\leq n\leq N-1 \qquad (11)$$

$$\theta P^*_{n,1}(\theta)-P_{n,1}(0)=(\lambda+a_1)P^*_{n,1}(\theta)-(1-r)P_{n+1,1}(0)S^*_1(\theta)-P_{n+1,2}(0)S^*_1(\theta)-D_n(0)S^*_1(\theta)-B^*_{n,1}(\theta,0)-\lambda S^*_1(\theta)\sum_{k=n-N+1}^{n-m}U_{n-k}g_k-\lambda\sum_{k=1}^{n-1}P^*_{n-k,1}(\theta)g_k \qquad n\geq N \qquad (12)$$

$$\theta P^*_{1,2}(\theta)-P_{1,2}(0)=(\lambda+a_2)P^*_{1,2}(\theta)-rP_{1,1}(0)S^*_2(\theta)-B^*_{1,2}(\theta,0) \qquad (13)$$

$$\theta P^*_{n,2}(\theta)-P_{n,2}(0)=(\lambda+a_2)P^*_{n,2}(\theta)-rP_{n,1}(0)S^*_2(\theta)-B^*_{n,2}(\theta,0)-\lambda\sum_{k=1}^{n-1}P^*_{n-k,2}(\theta)g_k \qquad n\geq 2 \qquad (14)$$

$$\frac{-\partial}{\partial y}B^*_{1,1}(\theta,y)=-\lambda B^*_{1,1}(\theta,y)+a_1P^*_{1,1}(\theta)r_1(y) \qquad (15)$$

$$\frac{-\partial}{\partial y}B^*_{n,1}(\theta,y)=-\lambda B^*_{n,1}(\theta,y)+a_1P^*_{n,1}(\theta)r_2(y)+\lambda\sum_{k=1}^{n-1}B^*_{n-k,1}(\theta,y)g_k \qquad (16)$$

$$\frac{-\partial}{\partial y} B^*_{1,2}(\theta, y) = -\lambda B^*_{1,2}(\theta, y) + a_2 P^*_{1,2}(\theta) r_2(y) \quad (17)$$

$$\frac{-\partial}{\partial y} B^*_{n,2}(\theta, y) = -\lambda B^*_{n,2}(\theta, y) + a_2 P^*_{n,2}(\theta) r_2(y) + \lambda \sum_{k=1}^{n-1} B^*_{n-k,2}(\theta, y) \quad (18)$$

Taking LST w.r.to y, equations (15) to (18), imply

$$\theta_1 B^{**1}_{1,1}(\theta,\theta_1) - B^*_{1,1}(\theta,0) = \lambda B^{**1}_{1,1}(\theta,\theta_1) - a_1 P^*_{1,1}(\theta) R_1^{*1}(\theta_1) \quad (19)$$

$$\theta_1 B^{**1}_{n,1}(\theta,\theta_1) - B^*_{n,1}(\theta,0) = \lambda B^{**1}_{n,1}(\theta,\theta_1) - a_1 P^*_{n,1}(\theta) R_1^{*1}(\theta_1) - \lambda \sum_{k=1}^{n-1} B^{**1}_{n-k,1}(\theta,\theta_1) g_k \qquad n \geq 2 \quad (20)$$

$$\theta_1 B^{**1}_{1,2}(\theta,\theta_1) - B^*_{1,2}(\theta,0) = \lambda B^{**1}_{1,2}(\theta,\theta_1) - a_2 P^*_{1,2}(\theta) R_2^{*1}(\theta_1) \quad (21)$$

$$\theta_1 B^{**1}_{n,2}(\theta,\theta_1) - B^*_{n,2}(\theta,0) = \lambda B^{**1}_{n,2}(\theta,\theta_1) - a_2 P^*_{n,2}(\theta) R_2^{*1}(\theta_1) - \lambda \sum_{k=1}^{n-1} B^{**1}_{n-k,2}(\theta,\theta_1) g_k \qquad n \geq 2 \quad (22)$$

IV. PROBABILITY GENERATING FUNCTIONS

To obtain the partial probability generating functions of the number of customers in the system, the following Probability Generating Functions are defined,

$$Q^*_j(z,\theta) = \sum_{n=0}^{\infty} Q^*_{n,j}(\theta) z^n \; ;$$

$$Q_j(z,0) = \sum_{n=0}^{\infty} Q_{n,j}(0) z^n \quad j \geq 1 \quad (23)$$

$$D^*(z,\theta) = \sum_{n=m}^{\infty} D^*_n(\theta) z^n$$

$$D(z,0) = \sum_{n=m}^{\infty} D_n(0) z^n \quad (24)$$

$$P^*_i(z,\theta) = \sum_{n=1}^{\infty} P^*_{n,i}(\theta) z^n \, ; P^*_i(z,0) = \sum_{n=1}^{\infty} P_{n,i}(0) z^n \quad i=1,2 \quad (25)$$

$$B^{**1}_i(z,\theta,\theta_1) = \sum_{n=1}^{\infty} B^{**1}_{n,i}(\theta,\theta_1) z^n \, ;$$

$$B^*_i(z,\theta,0) = \sum_{n=1}^{\infty} B^*_{n,i}(\theta,0) z^n \quad i=1,2 \quad (26)$$

$$U(z) = \sum_{n=m}^{N-1} U_n z^n \, ; \quad (27)$$

Following the algebraic manipulation , the expressions for the partial PGFs of the system size for the present model are listed below

Let α_n denote the probability that n-customers arrive during the vacation time V(t), then

$$V^*(w_x(z)) = \sum_{n=0}^{\infty} \alpha_n z^n \quad \text{where} \quad w_x(z) = \lambda(1 - X(z))$$

$$\sum_{j=1}^{\infty} Q^*_j(z,0) = P_1(0) \frac{(1 - V^*(w_x(z)))}{w_x(z)} \sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)}$$

$$= \frac{P_1(0)(1 - V^*(w_x(z)))}{w_x(z)} \beta(z) \quad (28)$$

where $P_1(0) = P_{11}(0)(1-r) + P_{12}(0)$, $\beta_0 = 1$,

$\beta_n = \sum_{j=1}^{n} \frac{\alpha_j \beta_{n-j}}{1-\alpha_0}$, for $1 \leq n \leq$ m-1 and

$$\beta(z) = \sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)}$$

$$D^*(z,0) = \frac{P_1(0)(1 - D^*(w_x(z)))}{w_x(z)} [(V^*(w_x(z)) - 1)\beta(z) + 1] \quad (29)$$

$$U(z) = \frac{P_1(0)}{\lambda}\sum_{n=m}^{N-1}\varphi_n^R z^n = P_1(0)\varphi^R(z) \text{ where}$$

$$\varphi^R(z) = \sum_{n=m}^{N-1}\frac{\varphi_n^R z^n}{\lambda} \qquad (30)$$

$$P_1^*(z,0) = \frac{z(1-S_1^*(h_{a_1}(w_x(z))))}{h_{a_1}(w_x(z))}\frac{(P_1(0)w_x(z)I_{(m,N)}^R(z))}{(z-H_R^*(h_a(w_x(z))))} \qquad (31)$$

where $h_{a_1}(w_x(z)) = a_1(1-R_1^{*1}(w_x(z))) + w_x(z)$,

$$h_{a_2}(w_x(z)) = w_x(z) + a_2(1-R_2^{*1}(w_x(z)))$$

$$H_R^*(h_a(w_x(z))) = S_1^*(h_{a_1}(w_x(z)))(1-r+rS_2^*(h_{a_2}(w_x(z)))) \qquad (31.a)$$

$$I_{(m,N)}^R(z) = D^*(w_x(z))\left[\frac{1-V^*(w_x(z))}{w_x(z)}\right]\sum_{n=0}^{m-1}\frac{\beta_n z^n}{(1-\alpha_0)} + \left(\frac{1-D^*(w_x(z))}{w_x(z)}\right) + \sum_{n=m}^{N-1}\frac{\varphi_n^R z^n}{\lambda}$$

$$P_2^*(z,0) = \frac{zrS_1^*(h_{a_1}(w_x(z)))(S_2^*(h_{a_2}(w_x(z)))-1)}{(h_{a_2}(w_x(z)))}\frac{(P_1(0)w_x(z)I_{(m,N)}{}^R(z))}{(z-H_R^*(h_a(w_x(z))))} \qquad (32)$$

$$B_1^{**1}(z,\theta,\theta_1) = \frac{a_1P_1^*(z,0)(1-R_1^{*1}(w_x(z)))}{w_x(z)} \qquad (33)$$

$$B_2^{**1}(z,\theta,0) = \frac{a_2P_2^*(z,0)(1-R_2^{*1}(w_x(z)))}{w_x(z)} \qquad (34)$$

To derive the total PGF of the system size distribution, the following generating functions are considered.

Let $P_I(z)$ = Probability generating function of the system size when the server is

Idle (vacation + setup + dormant) state

ie $P_I(z) = \sum_{j=1}^{\infty}Q_j^*(z,0) + D^*(z,0) + U(z)$

$$= P_1(0)I_{(m,N)}^R(z) \qquad (35)$$

(adding equation (28) to (30))

$P_{comp}(z) =$ the PGF of the system size when the server is busy or in breakdown state

$$= \sum_{i=1}^{2}(P_i^*(z,0) + B_i^{**1}(z,0,0))$$

$$= \frac{zP_1(0)I_{(m,N)}^R(z)(H_R^*(h_a(w_x(z)))-1)}{z-H_R^*(h_a(w_x(z)))} \qquad (36)$$

(adding equation (31) to (34))

Thus the total PGF of the system size distribution is given by

$$P_{SOS(m,N)}^R(z) = P_I(z) + P_{comp}(z)$$

$$= \frac{P_1(0)(z-1)H_R^*(h_a(w_x(z)))I_{(m,N)}^R(z)}{z-H_R^*(h_a(w_x(z)))}$$ (adding equation (35) and (36)) (37)

The constant $P_1(0)$ can be calculated by using the normalizing condition

$P_{(m,N)}^R(1) = 1$ and found to be $P_1(0) = \dfrac{1-\rho_R}{d_{(m,N)}^R}$ (38)

where $\rho_R = \lambda E(x)E(H_R)$ (39)

$$E(H_R) = E(S_1)(1+a_1E(R_1)) + rE(S_2)(a_2E(R_2))$$

$$d_{m,N}^R = I_{m,N}^R(1) = E(D) + E(V)\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)} + (1/\lambda)\sum_{n=m}^{N-1}\varphi_n^R \qquad (40)$$

Substituting for $P_1(0)$ in (equation (37)) the total PGF= $P_{m,N}^R(z)$ is given by

$$P^R_{(m,N)}(z) = \frac{(1-\rho_R)(z-1)H^*_R(h_a(w_x(z)))}{(z-H^*_R(h_a(w_x(z))))}\frac{I^R_{(m,N)}(z)}{I^R_{(m,N)}(1)} \quad (41)$$

Equation (41) justifies the decomposition property.

V. DECOMPOSITION PROPERTY

The PGF of the system size distribution of the (m,N) policy ($M^X_{(m,N)}$)/G_{SOS}/1/MV Breakdown queueing model is decomposed into a product of two random variables one of which is the PGF of the system size of the classical SOS model ($M^X_{(m,N)}$)/G_{SOS}/1 with server breakdown without vacation and without N-policy namely $\frac{(1-\rho_R)(z-1)H^*_R(h_a(w_x(z)))}{(z-H^*_R(h_a(w_x(z))))}$ and the other is $\frac{I^R_{(m,N)}(z)}{I^R_{(m,N)}(1)}$, which gives the PGF of the conditional system size distribution during the idle period under the steady state condition $\rho_R < 1$

VI. PERFORMANCE MEASURES

System size probabilities and mean system size:

In this section, the steady-state system size probabilities and the expected number of customers in the system, when the server is in different states are calculated.

A. The server in idle state

Let P_V, P_{set} and P_{dor} denote the steady state system size probabilities and L_V, L_{set} and L_{dor} denote the average number of customers, present in the system when the system is in vacation state , doing setup operation and in dormant state respectively. Then the measures can be calculated from the partial PGFs of the system size given in equations (28) to (30)

$$P_V = \lim_{z\to 1}\sum_{j=1}^{\infty} Q^*_j(z,0) = P_1(0)E(V)\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)} \quad (42)$$

$$L_V = \left[\frac{d}{dz}\sum_{j=1}^{\infty} Q^*_j(z,0)\right]_{z=1} = P_1(0)\left[\frac{\lambda E(X)E(V^2)}{2}\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)} + E(V)\sum_{n=0}^{m-1}\frac{n\beta_n}{(1-\alpha_0)}\right] \quad (43)$$

$$P_{set} = \lim_{z\to 1} D^*(z,0) = P_1(0)E(D) \quad (44)$$

$$L_{set} = \left[\frac{d}{dz}D^*(z,0)\right]_{z=1} = P_1(0)\lambda E(X)\left[\frac{E(D^2)}{2} + E(D)E(V)\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)}\right] \quad (45)$$

$$P_{dor} = \lim_{z\to 1} U(z) = \frac{P_1(0)}{\lambda}\sum_{n=m}^{N-1}\varphi^R_n \quad (46)$$

$$L_{dor} = \left[\frac{d}{dz}U(z)\right]_{z=1} = \frac{P_1(0)}{\lambda}\sum_{n=m}^{N-1} n\varphi^R_n \quad (47)$$

Thus the probability that the server is idle (P_I) and the mean number of customers accumulated in the system when the server is idle (L_I) are given by

$$P_I = P_V + P_{set} + P_{dor}$$

By adding equation(42), (44) and (46)

$$P_I = P_1(0)I^R_{(m,N)}(1) = P_1(0)d^R_{(m,N)} = (1-\rho_R) \quad (48)$$

$$L_I = L_V + L_{set} + L_{dor} = \frac{L^R_{(m,N)}}{d^R_{(m,N)}}(1-\rho_R)$$

where $L^R_{(m,N)}$

$$= \lambda E(X)\left[\left(\frac{E(V^2)}{2} + E(D)E(V)\right)\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)} + \frac{E(D^2)}{2}\right]$$

$$+E(V)\sum_{n=0}^{m-1}\frac{n\beta_n}{(1-\alpha_0)}+(1/\lambda)\sum_{n=m}^{N-1}n\varphi_n^R$$

$$d_{(m,N)}^R=I_{(m,N)}^R(1)=E(D)$$

$$+E(V)\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)}+(1/\lambda)\sum_{n=m}^{N-1}\varphi_n^R$$

B. The server in busy state:

The probability that the server is busy with FES(P_{FES}) and SOS(P_{sos})and the expected number of customers in the system (L_{FES} and L_{SOS}) when the server is in the respective states are obtained by using the equations (31) and (32).

Thus P_{FES}=probability that the server is busy in FES

$$=\lim_{z\to 1}P_1^*(z,0)=\lambda E(X)E(S_1) \qquad (49)$$

$$L_{FES}=\left[\frac{d}{dz}P_1^*(z,0)\right]_{z=1}$$

$$=P_{FES}+\frac{\lambda E(X(X-1))+(\lambda E(X))^3E(H_R^2)}{2(1-\rho_R)}E(S_1)$$

$$+\lambda E(X)\left[\frac{{}^{R}_{(m,N)}}{d_{(m,N)}^R}E(S_1)+\lambda E(X)(E(S_1{}^2))(1+a_1E(R_1))\right]$$

(50)

$$E(H_R)=E(S_1)[1+a_1E(R_1)]+rE(S_2)(1+a_2E(R_2))$$

$$E(H_R^2)=a_1E(S_1)E(R_1^2)+a_2rE(S_2)E(R_2^2)$$
$$+E(S_1^2)(1+a_1E(R_1)^2)+[rE(S_2^2)(1+a_2E(R_2))^2$$
$$+2rE(S_1)E(S_2)(1+a_1E(R_1))+(1+a_2E(R_2))]$$

P_{SOS} = probability that the server is busy in SOS

$$=\lim_{z\to 1}P_2^*(z,0)=r\lambda E(X)E(S_2)$$

(51)

$$L_{SOS}=\left[\frac{d}{dz}P_2^*(z,0)\right]_{z=1}$$

$$=P_{SOS}+\frac{\lambda E(X(X-1))+(\lambda E(X))^3E(H_R^2)}{2(1-\rho_R)}rE(S_2)$$

$$+r\lambda E(X)[\frac{L_{(m,N)}^R}{d_{(m,N)}^R}E(S_2)+\lambda E(X)\frac{E(S_2^2)}{2}(1+a_2E(R_2))$$

$$+\lambda E(X)E(S_2)e(S_1)(1+a_1E(R_1))]$$

(52)

P_{busy} = probability that the server is in busy state

$$=\lim_{z\to 1}(P_1^*(z,0)+P_2^*(z,0))$$

$$=\lambda E(X)[E(S_1)+rE(S_2)]=\rho_{SOS}$$

(53)

$$L_{busy}=L_{FES}+L_{SOS}$$

C. The server is in breakdown state:

The probabilities that the server is in breakdown states due to FES (P_{BR1}) and (P_{BR2}) and the expected number of customers in the system (L_{BR1} and L_{BR2}) in the corresponding states are obtained by using the equations (33) and (34).

P_{BR1}=probability that the server is in breakdown state due to FES

$$=\lim_{z\to 1}(B_1^{**1}(z,0,0)$$

$$=\lambda E(X)a_1E(S_1)E(R_1)$$

(54)

$$L_{BR1}=\frac{d}{dz}\left[B_1^{**1}(z,0,0)\right]_{z=1}$$

$$=L_{FES}(a_1E(R_1))+P_{FES}\frac{\lambda E(X)}{2}(a_1E(R_1^2))$$

(55)

P_{BR2} = probability that the server is in breakdown state due to SOS

$$=\lim_{z\to 1}\left[B_2^{**1}(z,0,0)\right]_{z=1}$$

$$=\lambda E(X)ra_2E(S_2)E(R_2) \qquad (56)$$

$$L_{BR2}=\frac{d}{dz}\left[B_2^{**1}(z,0,0)\right]_{z=1}$$

$$= a_2\left(L_{SOS}E(R_2) + P_{SOS}\frac{\lambda E(X)E(R_2^2)}{2}\right) \quad (57)$$

Thus the probability that the server is in breakdown state and the corresponding system size are given by

$$P_{Br} = P_{BR1} + P_{BR2}$$

$$L_{BR} = L_{BR1} + L_{BR2}$$

D. Mean system size

Thus the expected number of customers waiting in the system for the present model is given by

$$L^R_{(m,N)} = L_I + L_{busy} + L_{Br} \qquad = L_{SOS} + \frac{l^R_{(m,N)}}{d^R_{(m,N)}} \quad (58)$$

$$L^R_{(m,N)} = L^{Br}_{SOS} + \frac{L^R_{(m,N)}}{D^R_{(m,N)}}$$

where

$$L^{Br}_{SOS} = \rho_R + \frac{\lambda E(X(X-1))E(H_R) + (\lambda E(X))^2 E(H_R^2)}{2(1-\rho_R)} \quad (59)$$

gives the mean system size $M^X/G/1$ queueing model with server breakdown without N-policy and without vacation.

VII. PARTICULAR CASE

(1) If r=1, then all the customers are allowed to undergo both types of services one followed by the other, thus the equation (31.a) at r=1 and r=0 respectively imply,

$$H_R^*(h_a(w_x(z))) = S_1^*(h_{a_1}(w_x(z)))(S_2^*(h_{a_2}(w_x(z))))$$

and

(2) by $H_R^*(h_a(w_x(z))) = S_1^*(h_{a_1}(w_x(z)))$

(3) The results of the corresponding reliable server queueing models canbe obtained by putting $a_i = 0$, for i = 1, 2.

(4) The PGF for the corresponding N-policy queueing models can be obtained by putting m = N, so that $\varphi_n^R = 0$ $\forall$n and

$$I^R_{(m,N)}(z) = I^R_{(N)}(z) =$$

$$\frac{D^*(w_x(z))(1-V^*(w_x(z)))}{w_x(z)}\sum_{n=0}^{m-1}\frac{\beta_n z^n}{(1-\alpha_0)} + \frac{(1-D^*(w_x(z)))}{w_x(z)}$$

(5) The total PGF of the corresponding non-vacation models can be derived by putting $V^*(w_x(z)) = 0$ and for the model without setup operation $D^*(w_x(z)) = 0$

(6) By the suitable selection of the parameters, r, a_i (i = 1, 2), m, N, gk, E(V) and E(D), it is verified that the results of Lee et al. (1994b, 1997, 2003), Ke (2004a) can be deduced from the model discussed here.

VIII. NUMERICAL ANALYSIS

In this section some numerical results are presented to study the effects of various parameters. The effects of the batch arrival rate (λ),the setup parameter 'nu' and vacation parameter 'ita' on the expected queue size is pictorically represented by means of graphs in figures (1) and (2) for the given set of parameters.Figures (1) and (2) show that the queue length increases as the arrival rate increases and decreases with the setup parameter in figure(1) and vacation parameter in figure(2).Figure (3) deals with the effect of arrival rate ('la') on the system size probabilities.

For the computation purpose the following distributions are assumed for different random variables.

Random variables	Distribution	Mean	Second order moments
FES S_1	Two stage hyper exponential	$E(S_1) = \frac{a_1}{\mu_{11}} + \frac{1-a_1}{\mu_{1,2}}$ $0 \le a_1 \le 1$	$E(S_1^2) = 2(\frac{a_1}{\mu_{11}^2} + \frac{1-a_1}{\mu_{12}^2})$
SOS S_2	Two stage hyper exponential	$E(S_2) = \frac{b_1}{\mu_{21}} + \frac{1-b_1}{\mu 22}$ $0 \le b_1 \le 1$	$E(S_2^2) = 2(\frac{b_1}{\mu_{21}^2} + \frac{1-b_1}{\mu_{22}^2})$

Repair time in FES (SOS)	Exponential	$\frac{1}{\beta_1}\ (\frac{1}{\beta_2})$	$\frac{2}{\beta_1^{\ 2}}\ (\frac{2}{\beta_2^{\ 2}})$
Setup time D	Erlang 3 type	$E(D)=\frac{1}{\nu}$	$E(D^2)=\frac{4}{3\nu^2}$
Vacation V	Erlang 3 type	$E(V)=\frac{1}{\eta}$	$E(V^2)=\frac{4}{3\eta^2}$
Batch size X	Geometric	$E(X)=\frac{1}{(1-p)}$	$E(X(X-1))=\frac{2p}{(1-p)^2}$

(r1=.2, β_1=.6, β_2 =.7,p=.75,m=2,N=5, μ_{11}=1.5, μ_{12}=2, μ_{21} =6, μ_{22}=.5,a=.5,b=.8, a_1=.3, a_2=.5)

Figure (1) η =.5

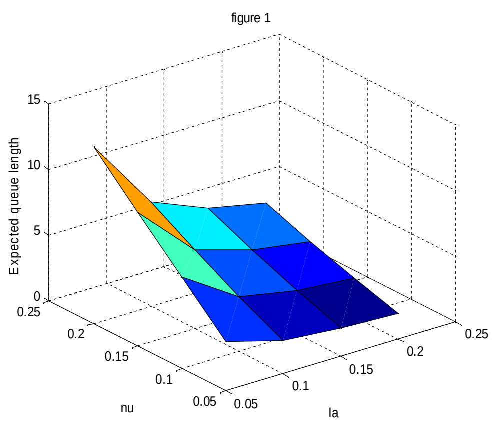
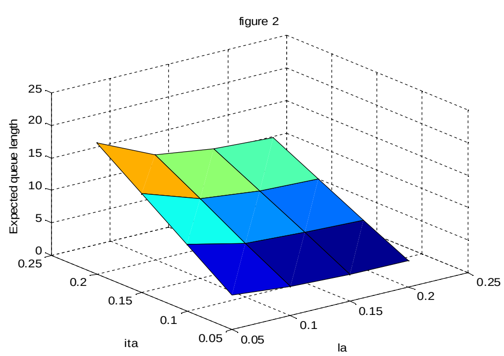
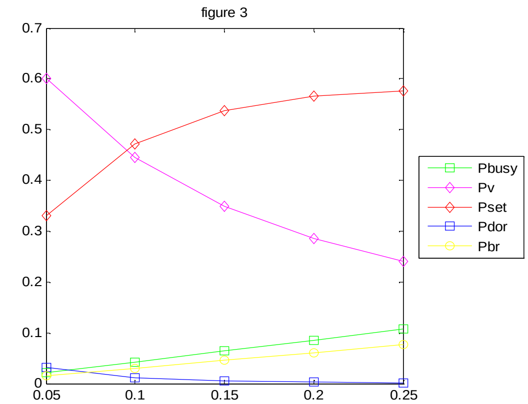


Figure (2) ν =.05



Figure(3) ν =.05, η =.5



IX. CONCLUSIONS

In this paper we have derived analytic steady-state results for the (m,N) policy $M^X/G/1$ queueing system with repeated vacations and single SOS fecility.The numerical results of various system performance measures such as system size probabilities and mean system size are providedin a closed form.The results of various models are deduced as particular cases of the existing model

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