Amicable Numbers and Groups II

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Abstract- In this paper, we introduce the special amicable numbers. Afterward we extended the notion of special amicable numbers to finite groups. Also, we will compare amicable numbers (groups) and special amicable numbers (groups). We provide some general theorems and present examples of special amicable numbers and groups.

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1. Introduction

The study of numbers has been in progress for as long as many other important mathematical fields. ([9, 10, 16, 17]) It might be argued that elementary number theory began with Pythagoras. At this time in which mathematical Analysis has opened the way to many profound observations, those problems which have to do with the nature and properties of numbers seem almost completely neglected by Geometers, and the contemplation of numbers has been judged by many to add nothing to Analysis. Yet truly the investigation of the properties of numbers on many occasions requires more acuity than the subtlest questions of geometry, and for this reason it seems improper to neglect arithmetic questions for those. And indeed the greatest thinkers who are recognized as having made the most important contributions to Analysis have judged the affection of numbers as not unworthy, and in pursuing them have expended much work and study. Namely, it is known that Descartes, even though occupied with the most important meditations on both universal Philosophy and especially Mathematics, spent no little effort uncovering amicable numbers; this matter was then pursued even more by van Schooten. Through the centuries mathematicians tried to find other examples of amicable pairs, and they did indeed succeed. But is there a formula? Are there infinitely many?

Let $\sigma(n)$ denote the sum of the divisor of n. Two integers a, b are said to be an amicable (or friendly) pair if $\sigma(a)=\sigma(b)=a+b$. We say an integer n is amicable if it is a member of an amicable pair, or equivalently $\sigma(\sigma(n)-n)=\sigma(n)$. If m=n, they are called perfect numbers, otherwise they form an amicable pair. The first perfect numbers 6, 28, 496. The smallest amicable pair, consisting of the numbers 220 and 284, was known already to the Pythagoreans (ca. 500 BCE) because

$$\begin{cases}\sigma(m)=\sigma(220)=1+2+4+5+10+11+20+22+44+55+110+220=504\\ \sigma(n)=\sigma(284)=1+2+4+71+142+284=504\end{cases}. \qquad (1)$$

$\underrightarrow{(1)}\ \sigma(284)=\sigma(220)=220+284=504.$

Two further amicable pairs were discovered by medieval Islamic mathematicians, and rediscovered by Fermât and Descartes. All of these were even numbers. In fact, they were found by the famous rules given by Euclid for perfect, resp. by Thabit ibn Kurrah for amicable numbers (see, e.g., [1], [5] for a survey of this subject), and so were even by construction. L. Euler was the first to study systematically the question whether or not also odd numbers with these properties may be found. The existence of odd perfect numbers has remained a famous open problem in number theory, while the existence of odd amicable numbers was established by Euler. He described several methods to construct numerical examples, one of which is, for example, A=32×7×13× 5 ×7=69615, B=32×7×13×107=87633.

Since Euler's time, many more even and odd amicable pairs have been found and published. A superficial glance at the list of hitherto known odd amicable pairs illustrates the fact that the lack of two as a common factor has to be compensated by a sufficient amount of divisibility by the other small prime factors, like three, five. In fact, all odd amicable pairs that we know [2], [6], [7], [8] actually contain some power of three as a common factor. With some familiarity with the various known methods to find odd amicable pairs, it soon becomes clear, that it is actually very hard to avoid three as a common factor. Paul Bratley and John McKay even conjectured that all odd amicable numbers must be divisible by three, see [3], and also R. Guy's book on open problems in number theory. ([4])

Lemma 1.1. The function σ is multiplicative. ([22])

Notice 1.2. For $n > 1$, let $p_n=3\times 2^n-1$ and $q_n=9\times 2^{2n-1}-1$. If p_{n-1}, p_n and q_n are all primes then $a=2^n\times p_{n-1}\times p_n$ and $b=2^n\times q_n$ form a pair of amicable numbers.

His formula produced three pairs of amicable numbers. n = 2 produced the pair a=220, b=284, which were known. n=4 gave the pair a=17, 296, b=18, 416 and n=7 produced the pair a=9, 363, 584, b=9, 437, 056. Evidently, the calculation grew beyond Thabit's ability to continue. In seventeenth century Europe, Thabit's results were not known and in 1636 Fermat calculated the pair 17, 296, 18, 416. Since Fermat and Descartes were rather bitter rivals (some say enemies), Descartes decided that if Fermat found a pair of amicable numbers, he would have to find a pair also. In 1638 Descartes found the pair 9, 363, 584, 9, 437, 056. So almost 2000 years after the first pair of amicable numbers were known only two more pairs were found.

Euler's Rule for amicable pairs) Let n and m are two positive integers with $1\leq m\leq n-1$. If $\begin{cases} p=2^n\times(2^{n-m}+1)-1\\ q=2^m\times(2^{n-m}+1)-1\\ r=2^{n+m}\times(2^{n-m}+1)^2-1\end{cases}$ are all primes, then the pair $(2^n\times p\times q,\ 2^n\times r)$ is an amicable pair. Note that if n-m =1 in Euler's Rule, we get Thabit's Rule. Even though there are rules to generate amicable numbers, it is not known whether or not there are infinitely many amicable pairs. ([16, 17, 18, 24])

Theorem 1.3. The pair $(2^n\times p\times q, 2^n\times r)$ is amicable pair where $\begin{cases} p=3\times 2^{n-1}-1\\ q=3\times 2^{2n}-1\\ r=9\times 2^{2n-1}-1\end{cases}$ are prime. $(n>1)$

Theorem 1.4. (Euler's rule) The pair $(2^n\times p\times q, 2^n\times r)$ is amicable where $\begin{cases} p=2^m\times(2^{n-m}+1)-1\\ q=2^n\times(2^{n-m}+1)-1\\ r=2^{n+m}\times(2^{n-m}+1)-1\end{cases}$ are prime. $(1\leq m\leq n)$

Example 1.5. For n = 2, Thabits rule produces the cycle 220, 284. For more ways to compute amicable pairs, see [19].

Notice 1.6. The references [13, 15, 20, 23, 25] for further study amicable numbers are introduced. Now, let G be a finite group. Leinster in [11] extended the notion of perfect numbers to finite groups. He called a finite group is perfect if its order is equal to the sum of the orders of all normal subgroups of the group. In the other words, G is called perfect group if $\sigma(G) = \sum_{N \trianglelefteq G} |N| = 2|G|$. ([12, 21])

We use this model to describe the definition of amicable groups.

Proposition 1.7. Let n be a perfect number then the pair of (n, n) is amicable pair. ([14])

Proposition 1.8. Let m be a perfect number and n be a deficient number then the pair of (m, n) is not amicable pair. ([14])

Corollary 1.9. Similarly, we can show that if that m be a perfect number and n be a abundant number then the pair of (m, n) is not amicable pair. ([14])

Proposition 1.10. Let m and n are two deficient numbers then the pair of (m, n) is not amicable pair. ([14])

Proposition 1.11. Let σ(n) denote the sum of the divisor of n then σ(n) is a odd number if and only if n be a square or twice the square. ([9])

Theorem 1.12. Let m is an even number and n is an odd number such that (n, m) be the amicable pair. Then n is square. ([14])

Proposition 1.13. Let n be a natural number then $\sigma(2^n)$ is an odd number. ([10])

Theorem 1.14. If α be an even number and p is a prime then $\sigma(p^{\alpha})$ is an odd number. ([10])

Definition 1.15. Let G_1 and G_2 be finite groups. Then the pair of (G_1, G_2) is called amicable groups if $\sigma(G_1) = \sigma(G_2) = |G_1| + |G_2|$.

Example 1.16. The smallest pair of amicable groups is (C_{220}, C_{284}) because $\begin{cases} \sigma(C_{220}) - |C_{220}| = |C_{284}| \\ \sigma(C_{284}) - |C_{284}| = |C_{220}| \end{cases} \to \begin{cases} \sigma(220) - |220| = |284| \\ \sigma(284) - |284| = |220| \end{cases} \to \begin{cases} \sigma(220) = \sigma(4 \times 71) = 504 = 284 + 220 = 504 \\ \sigma(284) = \sigma(4 \times 5 \times 11) = 504 = 284 + 220 = 504 \end{cases}$.

Definition 1.17.(Extension of the Definition of Amicable Groups) Let $G_1, G_2, \dots, G_k$ be finite groups then $G_1, G_2, \dots, G_k$ are called k-amicable if $\sigma(G_1) = \sigma(G_2) = \cdots = \sigma(G_k) = |G_1| + |G_2| + \cdots + |G_{k-1}| + |G_k|$. For example, triplex $(C_{2^5 \times 3^3 \times 47 \times 109}, C_{2^5 \times 3^2 \times 7 \times 659}, C_{2^5 \times 3^2 \times 5279})$ is a 3-amicable groups because $\sigma(C_n) = \sigma(n)$. ([14])

2. Main Results

Theorem 2.1. Let C_n be the cyclic group of order n. We suppose that m≠n, if C_n and C_m are two perfect groups then the pair of (C_n, C_m) is not amicable pair.

Proof. Let the pair of (C_n, C_m) is an amicable pair, so $\begin{cases} \sigma(C_n) - |C_n| = |C_m| \\ \sigma(C_m) - |C_m| = |C_n| \end{cases} \xrightarrow{\sigma(C_n)=\sigma(n)} \begin{cases} \sigma(n) - n = m \\ \sigma(m) - m = n \end{cases}$. (*)

According to the assumptions of the theorem we have $\begin{cases} \sigma(C_n) = \sigma(n) = 2n \\ \sigma(C_m) = \sigma(m) = 2m \end{cases}$. (**)

Now, with replacement (**) in (*) we have $\begin{cases}\sigma(n)-n=m=n\\ \sigma(m)-m=n=m\end{cases}$.

But this is a contradiction. Thus the proof is finished.

Definition 2.2. Let G be a finite group then G is said to be a deficient group if σ(G) < 2|G| and G is said to be a abundant group if σ(G) > 2|G|.

Theorem 2.3. Let C_n be the cyclic group of order n. If C_n and C_m are two abundant groups then the pair of (C_n, C_m) is not amicable pair.

Proof. Let the pair of (C_n, C_m) is an amicable pair, so

$$\begin{cases}\sigma(C_n)-|C_n|=|C_m|\\ \sigma(C_m)-|C_m|=|C_n|\end{cases} \xRightarrow{\sigma(C_n)=\sigma(n)} \begin{cases}\sigma(n)-n=m\\ \sigma(m)-m=n\end{cases}.$$

According to the assumptions of the theorem we have $\begin{cases}\sigma(C_n)>2|C_n|\\ \sigma(C_m)>2|C_m|\end{cases} \to \begin{cases}\sigma(n)>2n\\ \sigma(m)>2m\end{cases}$. Therefore, we have

$\begin{cases}\sigma(n)-n=m>n\\ \sigma(m)-m=n>m\end{cases}$. But this is a contradiction. Thus the proof is finished.

Theorem 2.4. Let C_n be the cyclic group of order n. If the pair of (C_n, C_m) is an amicable pair and $m<n$ then n is deficient and m is abundant.

Proof. Since the pair of (C_n, C_m) is an amicable pair of groups, so

$$\begin{cases}\sigma(C_n)-|C_n|=|C_m|\\ \sigma(C_m)-|C_m|=|C_n|\end{cases} \xRightarrow{\sigma(C_n)=\sigma(n)} \begin{cases}\sigma(n)-n=m\\ \sigma(m)-m=n\end{cases} \to \begin{cases}\sigma(n)=n+m\\ \sigma(m)=m+n\end{cases}. \qquad (*)$$

By using (*) and $m<n$, we have $\begin{cases}\sigma(n)=m+n<n+n\\ \sigma(m)=n+m>m+m\end{cases} \to \begin{cases}\sigma(n)<2n\\ \sigma(m)>2m\end{cases}$.

Definition 2.5. Let m and n are two natural numbers then the pair of (m, n) is called **special** amicable if $\sigma(m)=\sigma(n)$. Similarly, let G_1 and G_2 are two finite groups then the pair of (G_1, G_2) is called **special** amicable if $\sigma(G_1)=\sigma(G_2)$.

Notice 2.6. The number of **special** amicable numbers (groups) is very important for classification of monotone arithmetic functions.

Example 2.7. The pair of (30, 46) is a special amicable pair because $\begin{cases}\sigma(30)=1+2+3+5+6+10+15+30=72\\ \sigma(46)=1+2+23+46=72\end{cases} \to$ σ30=σ46=76≠30+46.

Example 2.8. The pair of (C_6, C_{11}) is a special amicable pair because
$\begin{cases}\sigma(C_6)=\sigma(6)=1+2+3+6=12\\ \sigma(C_{11})=\sigma(11)=1+11=12\end{cases} \to \sigma(C_6)=\sigma(C_{11})=12.$

Theorem 2.9. There are infinitely many special amicable numbers.

Proof. To prove the theorem it is enough to show that there are infinite pairs like to (m, n) such that $\sigma(m)=\sigma(n)$. For this purpose, we introduce m=6(66k+1) and n=11(66k+1) where k is an integer. Therefore, $\sigma(m)=\sigma(6(66k+1))=\sigma(6)\sigma(66k+1)$. (1)

But, we well know that $\sigma(6) = \sigma(11)$. (2)

By using (1) and (2), we have $\sigma(m) = \sigma(6(66k+1)) = \sigma(6)\sigma(66k+1) = \sigma(11)\sigma(66k+1) = \sigma(11(66k+1)) = \sigma(n)$, so the proof is finished.

Corollary 2.10. There are infinitely many special amicable groups because $\sigma(C_n) = \sigma(n)$.

Theorem 2.11. (Another proof) There are infinitely many special amicable numbers.

Proof. To prove the theorem it is enough to show that there are infinite pairs like to (m, n) such that $\sigma(m) = \sigma(n)$.

For this purpose, we introduce m=$6t(66k+1)$ and n=$11t(66k+1)$ where k is an integer and (t, 6)=1, (t, 11)=1, (t, 66k+1)=1 and $t \neq 1$. Therefore, $\sigma(m) = \sigma(6t(66k+1)) = \sigma(6)\sigma(t)\sigma(66k+1)$. (1)

But, we well know that $\sigma(6) = \sigma(11)$. (2)

By using (1), (2), we have $\sigma(m) = \sigma(6t(66k+1)) = \sigma(6)\sigma(t)\sigma(66k+1) = \sigma(11)\sigma(t)\sigma(66k+1) = \sigma(11t(66k+1)) = \sigma(n)$, so the proof is finished.

Theorem 2.12. Let m, n are natural numbers and C_n be the cyclic group of order n then the pair of (m, n) is a amicable ↔ the pair of (C_m, C_n) is amicable.

Proof. The proof of the theorem is obvious because $\sigma(C_n) = \sigma(n)$.

Theorem 2.13. Let m, n are natural numbers and C_n be the cyclic group of order n then the pair of (m, n) is a special amicable ↔ the pair of (C_m, C_n) is special amicable.

Proof. The proof of the theorem is obvious because $\sigma(C_n) = \sigma(n)$.

Notice 2.14. Let G_1 and G_2 are two finite groups whose $G_1 \sim G_2$ then we know that $\sigma(G_1) = \sigma(G_2)$.

Therefore, we can say that the number of special amicable groups is greater than the number of special amicable numbers.

Question. Is it true to say that 'the number of amicable groups is greater than the number of amicable numbers'?

Theorem 2.15. Let G be a finite cyclic group then $(\sum_{N \trianglelefteq G} \frac{1}{|N|})^{-1} = \frac{|G|}{\sigma(G)}$.

Proof. We well know that $\sum_{N \trianglelefteq G} |N| = \sum_{N \trianglelefteq G} \frac{|G|}{|N|}$. Therefore, we have $\sum_{N \trianglelefteq G} |N| = |G| \sum_{N \trianglelefteq G} \frac{1}{|N|}$. This relationship allows us to conclude that $(\sum_{N \trianglelefteq G} \frac{1}{|N|})^{-1} = \frac{|G|}{\sigma(G)}$.

Theorem 2.16. Let G_1 and G_2 are two finite cyclic groups where G_1 and G_2 are two amicable groups then $(\sum_{N \trianglelefteq G_1} \frac{1}{|N|})^{-1} + (\sum_{M \trianglelefteq G_2} \frac{1}{|M|})^{-1} = 1$.

Proof. We assume that the two groups are amicable. Therefore, we have $\sigma(G_1) + \sigma(G_2) = |G_1| + |G_2|$. $(*)$

According to the previous theorem, we have $\begin{cases} (\sum_{N \trianglelefteq G_1} \frac{1}{|N|})^{-1} = \frac{|G_1|}{\sigma(G_1)} & (1) \\ (\sum_{M \trianglelefteq G_2} \frac{1}{|M|})^{-1} = \frac{|G_2|}{\sigma(G_2)} & (2) \end{cases}$. $\xrightarrow{(*),(1),(2)}$ We have

$(\sum_{N \trianglelefteq G_1} \frac{1}{|N|})^{-1} + (\sum_{M \trianglelefteq G_2} \frac{1}{|M|})^{-1} = \frac{|G_1|+|G_2|}{\sigma(G_1)} = 1$.

Notice 2.17. Is it true to say that 'the above theorem applies to all groups that are converse the Lagrange's theorem is confirmed'?

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