On Some Properties of a Generalization of Bessel-Maitland Function

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***Abstract*—The present paper is the investigation of certain properties of generalized Bessel-Maitland function, written in the form $J_{\nu,q}^{\mu,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{n!\Gamma(\mu n+\nu+1)}$, where $\mu,\nu,\gamma \in C; Re(\mu) \geq 0$, $Re(\nu) \geq -1, Re(\gamma) \geq 0$ and $q \in (0,1) \cup N$. For the function $J_{\nu,q}^{\mu,\gamma}(z)$, a number of results including differentiation and integration formulas, Mellin-Barnes integral representation, Laplace transform, Euler transform, k-transform, Varma transform, Mellin transform. Various relationship with other functions including Fox's H-function and Wright hypergeometric function were also established. In the end certain relations have been obtained by using the Riemann-Liouville fractional integrals and derivatives.**

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I. INTRODUCTION

The special function of the form defined by the series representation

$$J_{\nu}^{\mu}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n+\nu+1)} = \phi(\mu,\nu+1;-z) = H_{0,2}^{1,0}\left[z \,\middle|\, \begin{matrix} -\,-\,-\,-\,-; \\ (0,1),(-\nu,\mu); \end{matrix}\right] \quad (1.1)$$

is known as Bessel-Maitland function, or the Wright generalized function ([15], (8.3)). It has a wide application in the problem of physics, chemistry, biology, engineering and applied sciences. The theory of Bessel functions is intimately connected with the theory of certain types of differential equations. A detailed account of application of Bessel function is represented in the book of Watson [12].

In this paper, a generalization of Bessel-Maitland function is investigated and is defined as

$$J_{\nu,q}^{\mu,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{n!\,\Gamma(\mu n+\nu+1)} \quad (1.2)$$

where, $\mu,\nu,\gamma \in C; Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0$ and $q \in (0,1) \cup N$ and $(\gamma)_0 = 1$, $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$, denotes the generalized pochhammer symbol (see Rainville, [8]), which in particular reduces to $q^{qn}\prod_{j=1}^{q}\left(\frac{\gamma+j-1}{q}\right)_n$ if $q \in N$.

Some important special cases of this function are enumerated below:

(i) $J_{\nu,0}^{\mu,\gamma}(z) = J_{\nu}^{\mu}(z)$, defined by (1.1).

(ii) $J_{\nu-1,0}^{\mu,\gamma}(-z) = \phi(\mu,\nu;z)$, known as Wright function ([1], section 18.1) was introduced by Wright [9].

(iii) $\left(\frac{z}{2}J_{\nu,0}^{1,\gamma}\left(\frac{z^2}{4}\right)\right) = J_{\nu}(z)$, is the ordinary Bessel function (Rainville, [8], pp.109).

(iv) If $\mu = k \in N$ and $q \in N$,

$$J_{\nu,q}^{k,\gamma}(z) = \frac{1}{\Gamma(\nu+1)}\, {}_qF_k\left[\begin{matrix}\Delta(q;\gamma) & ; \\ \Delta(k;\nu+1); \end{matrix} -\frac{q^q z}{k^k}\right] \quad (1.3)$$

where, ${}_qF_k(.)$ is the generalized hypergeometric function and the symbol $\Delta(q;\gamma)$ is a q-tuple $\frac{\gamma}{q},\frac{\gamma+1}{q},\ldots,\frac{\gamma+q-1}{q}$; $\Delta(k;\nu+1)$ is a k-tuple $\frac{\nu+1}{k},\frac{\nu+2}{k},\ldots,\frac{\nu+k}{k}$.

Convergence criteria for the generalized hypergeometric function ${}_qF_k$.

(a) If $q \leq k$, the function ${}_qF_k$ converges for $|z| < \infty$.

(b) If $q = k+1$, the function ${}_qF_k$ converges for $|z| < 1$.

(c) If $q > k+1$, the function ${}_qF_k$ is divergent for $z \neq 0$.

(d) If $q = k+1$, the function ${}_qF_k$ is absolutely convergent on the unit circle $|z| = 1$, if

$$Re\left(\sum_{j=1}^{k} \frac{\nu + j}{k} - \sum_{i=1}^{q} \frac{\gamma + i - 1}{q}\right) > 0.$$

(v) $J_{\nu,q}^{\mu,\gamma}(z) = \frac{1}{\Gamma(\gamma)}\ {}_1\Psi_1\left[\begin{matrix}(\gamma, q) & ; \\ (\nu+1, \mu n); & \end{matrix} - z\right]$

$$= \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1}\left[\begin{matrix}(1-\gamma, q) & ; \\ (0,1), (-\nu, \mu); & \end{matrix}\ z\right] \quad (1.4)$$

where ${}_1\Psi_1(.)$ and $H_{1,2}^{1,1}(.)$ are respectively Wright generalized hypergeometric function [10] and H-function [6].

(vi) $J_{\nu-1,q}^{\mu,\gamma}(-z) = E_{\mu,\nu}^{\gamma,q}(z)$, where $\alpha, \beta, \gamma \in C$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma > 0)$ and $q \in (0,1) \cup N$, was given by Shukla and Prajapati [3].

$J_{\nu-1,1}^{\mu,\gamma}(-z) = E_{\mu,\nu}^{\gamma}(z)$, was introduced by Prabhakar [17].

$J_{\nu-1,1}^{\mu,1}(-z) = E_{\mu,\nu-1}(z)$, where $(\alpha, \beta \in C; Re(\alpha) > 0, Re(\beta) > 0)$, was studied by Wiman [2].

$J_{0,1}^{\mu,1}(-z) = E_{\mu}(z)$, where $z \in C$ and $Re(\alpha > 0)$, was introduced by Ghosta Mittag-Leffler [11].

(vii) $t^{\mu-1} J_{\mu-,1}^{\mu,1}(az^{\mu}) = F_{\mu}[-a, z], \mu > 0$, was studied by Robotnov [18], with respect to hereditary integrals for application to solid mechanics.

In the investigation of various properties and relations of the function $J_{\nu,q}^{\mu,\gamma}(z)$, we need the following well known fact.

Beta transform (Sneddon [13]): The Beta (Euler) transform of the function $f(z)$ is defined by

$$B\{f(z); a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)dz \quad (1.5)$$

where, $Re(a) > 0, Re(b) > 0$.

Laplace transform (Sneddon [13]): The Laplace transform of the function $f(z)$ is defined as

$$L\{f(z)\} = \int_0^{\infty} e^{-sz} f(z)dz. \quad (1.6)$$

K-Trasnform (Meijer [7]): The K-transform is defined by the following integral equation

$$R_\nu\{f(x); p\} = \int_0^{\infty} (px)^{\frac{1}{2}} K_\nu(px) f(x)dx \quad (1.7)$$

where p is a complex parameter and $K_\nu(z)$ represent a modified Bessel function of third kind defined by ([7], p.28, eq.1.168).

Varma Trasnform (Meijer [8]): The transform is defined by the integral equation

$$V(f, k, m; s) = \int_0^{\infty} (sx)^{m-\frac{1}{2}} \exp\left(-\frac{1}{2}sx\right) W_{k,m}(sx) f(x)dx, \quad (1.8)$$

where $W_{k,m}(z)$ represents a Wittaker function defined by ([5], p.55, eq.2.39).

Riemann-Liouville fractional derivative and integral (see, Samko, Kilbas and Marichev ([16], sect. 2)), for $\alpha \in C, Re(\alpha) > 0$:

The operators are defined by

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt; \quad (1.9)$$

$$(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}} dt; \quad (1.10)$$

$$(D_{0+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{0+}^{1-\{\alpha\}} f\right)(x)$$

$$= \frac{1}{\Gamma(1-\{\alpha\})} \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}} dt; \quad (1.11)$$

$$(D_{-}^{\alpha}f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{-}^{1-\{\alpha\}}f\right)(x)$$

$$= \frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{d}{dx}\right)^{[\alpha]+1} \int_{x}^{\infty} \frac{f(t)}{(t-x)^{\{\alpha\}}}dt, \quad (1.12)$$

where $[\alpha]$ means the integral part of number α and $\{\alpha\}$ means the fractional part of number α, $0 \le \{\alpha\} < 1$. The number $\alpha = \{\alpha\} + [\alpha]$.

II. BASIC PROPERTIES

In this section we derive several interesting properties of the function $J_{\nu,q}^{\mu,\gamma}(z)$.

Theorem(2.1): If $\mu, \nu, \gamma \in C; Re(\mu) \ge 0, Re(\nu) \ge -1, Re(\gamma) \ge 0$ and $q \in (0,1) \cup N$ is satisfied, then for $m \in N$

$$\left(\frac{d}{dz}\right)^{m} J_{\nu,q}^{\mu,\gamma}(z) = (-1)^m (\gamma)_{qm} J_{\nu+\mu m,q}^{\mu,\gamma+qm}(z) \quad (2.1.1)$$

$$J_{\nu,q}^{\mu,\gamma}(z) = (\nu+1)J_{\nu+1,q}^{\mu,\gamma}(z) + \mu z \frac{d}{dz} J_{\nu+1,q}^{\mu,\gamma}(z) \quad (2.1.2)$$

$$J_{\nu,q}^{\mu,\gamma}(z) - J_{\nu,q}^{\mu,\gamma-1}(z)$$
$$= -zq \sum_{n=0}^{\infty} \frac{(\gamma)_{qn+q-1}}{\Gamma(\mu n+\mu+\nu+1)} \frac{(-z)^n}{n!} \quad (2.1.3)$$

In particular,

$$J_{\nu,1}^{\mu,\gamma-1}(z) - J_{\nu,1}^{\mu,\gamma}(z) = z J_{\nu+\mu,1}^{\mu,\gamma}(z) \quad (2.1.4)$$

Proof: From (1.2),

$$\left(\frac{d}{dz}\right)^{m} J_{\nu,q}^{\mu,\gamma}(z) = \sum_{n=m}^{\infty} \frac{(-1)^n(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)} \frac{(z)^{n-m}}{(n-m)!}$$

$$= (-1)^m(\gamma)_{qm} \sum_{n=0}^{\infty} \frac{(\gamma+qm)_{qn}}{\Gamma(\mu n+\nu+\mu m+1)} \frac{(-z)^n}{n!}$$

$$= (-1)^m(\gamma)_{qm} J_{\nu+\mu m,q}^{\mu,\gamma+qm}(z)$$

which is a proof of (2.1.1);

$$(\nu+1)J_{\nu+1,q}^{\mu,\gamma}(z) + \mu z \frac{d}{dz} J_{\nu+1,q}^{\mu,\gamma}(z)$$

$$= (\nu+1)\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+2)} \frac{(-z)^n}{n!} + \mu z \frac{d}{dz}\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+2)} \frac{(-z)^n}{n!}$$

$$= (\nu+1)\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+2)} \frac{(-z)^n}{n!} + \mu n \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+2)} \frac{(-z)^n}{n!}$$

$$= J_{\nu,q}^{\mu,\gamma}(z)$$

which proves (2.1.2).

Now,

$$J_{\nu,q}^{\mu,\gamma}(z) - J_{\nu,q}^{\mu,\gamma-1}(z)$$
$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{n!\,\Gamma(\mu n+\nu+1)} - \sum_{n=0}^{\infty} \frac{(\gamma-1)_{qn}(-z)^n}{n!\,\Gamma(\mu n+\nu+1)}$$

$$= q\sum_{n=1}^{\infty} \frac{(\gamma)_{qn-1}}{\Gamma(\mu n+\nu+1)} \frac{(-z)^n}{(n-1)!}$$

$$= -qz\sum_{n=0}^{\infty} \frac{(\gamma)_{qn+q-1}}{\Gamma(\mu n+\mu+\nu+1)} \frac{(-z)^n}{n!}$$

which proves (2.1.3).

In particular, if $q = 1$ in (2.1.3), which at once yield (2.1.4).

Theorem(2.2): If $\mu, \nu, \gamma, \delta \in C; Re(\mu) \ge 0, Re(\nu) \ge -1, Re(\gamma) \ge 0, Re(\delta) \ge 0$ and $q \in (0,1) \cup N$ is satisfied, then

$$\frac{1}{\Gamma(\delta)}\int_{0}^{1} (\omega)^{\nu}(1-\omega)^{\delta-1} J_{\nu,q}^{\mu,\gamma}(z\omega^{\mu})d\omega$$
$$= J_{\nu+\delta,q}^{\mu,\gamma}(z), \quad (2.2.1)$$

If, $\mu, \nu, \gamma, \delta, \alpha \in C; Re(\mu) \ge 0, Re(\nu) \ge -1, Re(\gamma) \ge 0, Re(\delta) \ge 0$ and $q \in (0,1) \cup N$ is satisfi -ed, then

$$\frac{1}{\Gamma(\delta)}\int_{t}^{x} (x-s)^{\delta-1}(s-t)^{\nu} J_{\nu,q}^{\mu,\gamma}[\alpha(s-t)^{\mu}]d$$

$$= (x-t)^{\delta+\nu} J^{\mu,\gamma}_{\nu+\delta,q}[\alpha(x-t)^{\mu}], \quad (2.2.2)$$

If $\mu,\nu,\gamma,\delta,\lambda \in C; Re(\mu) \geq 0, Re(\nu) \geq -1,$ $Re(\gamma) \geq 0, Re(\delta), Re(\lambda) \geq 0$ and $q = 1$ is satisfied, then

$$\int_0^x (t)^{\lambda}(x-t)^{\nu} J^{\mu,\gamma}_{\nu,1}[\omega(x-t)^{\mu}] J^{\mu,\delta}_{\lambda,1}[\omega t^{\mu}]dt$$

$$= x^{\lambda+\nu+1} J^{\mu,\gamma+\delta}_{\lambda+\nu+1,1}(-\omega x^{\mu}), \quad (2.2.3)$$

If $\mu,\nu,\gamma,\delta \in C; Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0,$ $Re(\delta) \geq 0$ and $q \in (0,1) \cup N$ is satisfied, then

$$\int_0^z (t)^{\nu} J^{\mu,\gamma}_{\nu,q}(\omega t^{\mu})dt = z^{\nu+1} J^{\mu,\gamma}_{\nu+1,q}(\omega z^{\mu}). \quad (2.2.4)$$

Proof: By using the beta function, L.H.S. of (2.2.1) becomes

$$\frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)} \frac{(-z)^n}{n!} B(\mu n+\nu+1,\delta)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+\delta+1)} \frac{(-z)^n}{n!} = J^{\mu,\gamma}_{\nu+\delta,q}(z)$$

which is the proof of (2.2.1).

By changing the variable $s = t+\omega(x-t)$, the L.H.S. of (2.2.2) becomes

$$\frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-\alpha(x-t)^{\mu})^n}{n!\,\Gamma(\mu n+\nu+1)}$$

$$\times \int_0^1 (x-t)^{\delta+\nu}(1-\omega)^{\delta-1}(\omega)^{\mu n+\nu}d\omega$$

$$= \frac{(x-t)^{\delta+\nu}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-\alpha(x-t)^{\mu})^n}{n!\,\Gamma(\mu n+\nu+1)}$$

$$\times B(\mu n+\nu+1,\delta)$$

which yield (2.2.2).

Consider,

$$\int_0^x (t)^{\lambda}(x-t)^{\nu} J^{\mu,\gamma}_{\nu,q}[\omega(x-t)^{\mu}] J^{\mu,\delta}_{\lambda,q}[\omega t^{\mu}]dt$$

$$= \sum_{n,k=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{qk}}{\Gamma(\mu n+\nu+1)\Gamma(\mu k+\lambda+1)} \frac{(-\omega)^{n+k}}{n!\,k!}$$

$$\times \int_0^x (t)^{\mu k+\lambda}(x-t)^{\mu n+\nu}dt$$

$$= (x)^{\lambda+\nu+1} \sum_{n,k=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{qk}}{\Gamma(\mu n+\nu+1)\Gamma(\mu k+\lambda+1)}$$

$$\times \frac{(-\omega x^{\mu})^{n+k}}{n!\,k!} B(\mu k+\lambda+1, \mu n+\nu+1)$$

$$= (x)^{\lambda+\nu+1} \sum_{n,k=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{qk}}{\Gamma(\mu n+\mu k+\lambda+\nu+2)} \frac{(-\omega x^{\mu})^{n+k}}{n!\,k!}$$

$$= (x)^{\lambda+\nu+1} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(\gamma)_{q(n-k)}(\delta)_{qk}}{\Gamma(\mu n+\lambda+\nu+1)} \frac{(-\omega x^{\mu})^{n}}{n!}$$

substituting $q = 1$ and using the identity Carlitz [1],

$$(a+b)_m = \sum_{r=0}^{m} \binom{m}{r} (a)_r (b) m - r$$

the above equation subsequently yield (2.2.3).

Further,

$$\int_0^z (t)^{\nu} J^{\mu,\gamma}_{\nu,q}(\omega t^{\mu})dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-\omega)^n}{n!\,\Gamma(\mu n+\nu+1)} \int_0^z (t)^{\mu n+\nu}dt$$

$$= z^{\nu+1} J^{\mu,\gamma}_{\nu+1,q}(\omega z^{\mu})$$

which is a proof of a (2.2.4).

III. INTEGRAL TRANSFORM OF $J^{\mu,\gamma}_{\nu,q}(z)$

In this section, several integral transforms like Beta, Laplace, Varma, Mellin's and K-transform are discussed for the function $J^{\mu,\gamma}_{\nu,q}(z)$ under the following theorem.

Theorem(3.1)(Beta transform): By using the definition of Beta function, one obtain

$$\int_0^1 (z)^{\alpha-1}(1-z)^{\beta-1} J^{\mu,\gamma}_{\nu,q}(xz^{\delta})dz$$

$$= \frac{\Gamma(\beta)}{\Gamma(\gamma)}\ {}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\alpha,\delta) & ; \\ (\nu+1,\mu),(\alpha+\beta,\delta); & \end{matrix} -x\right] \quad (3.1.1)$$

where, $\mu,\nu,\gamma \in C; Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0, Re(\alpha) \geq 0, Re(\beta) \geq 0$ and $q \in (0,1) \cup N$

Proof: From (1.2) and (1.5), we get

$$\int_0^1 (z)^{\alpha-1}(1-z)^{\beta-1} J_{\nu,q}^{\mu,\gamma}(xz^{\delta})dz$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-x)^n}{n!\,\Gamma(\mu n+\nu+1)} B(\alpha+\delta n,\beta)$$

$$= \frac{\Gamma(\beta)}{\Gamma(\gamma)}\ {}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\alpha,\delta) & ; \\ (\nu+1,\mu),(\alpha+\beta,\delta); & \end{matrix} -x\right]$$

which is (3.1.1).

Particular Cases: If $\mu = \delta$ and $\alpha = \nu + 1$, then the relation (3.1.1) reduces to

$$\int_0^1 (z)^{\nu}(1-z)^{\beta-1} J_{\nu,q}^{\delta,\gamma}(xz^{\delta})dz$$

$$= \Gamma(\beta) J_{\nu+\beta,q}^{\delta,\gamma}(x) \quad (3.1.2)$$

If $\beta = \nu + 1$, $\delta = \mu$ and $z = (1-z)$, then the relation (3.1.1) reduces to

$$\int_0^1 (z)^{\alpha-1}(1-z)^{\nu} J_{\nu,q}^{\mu,\gamma}(x(1-z)^{\mu})dz$$

$$= \Gamma(\alpha) J_{\nu+\alpha,q}^{\mu,\gamma}(x) \quad (3.1.3)$$

Theorem(3.2)(Laplace transform): By using the definition of Laplace transform, one obtain

$$\int_0^{\infty} z^{\alpha-1} e^{-sz} J_{\nu,q}^{\mu,\gamma}(xz^{\delta})dz$$

$$= \frac{s^{-\alpha}}{\Gamma(\gamma)}\ {}_2\Psi_1\left[\begin{matrix}(\gamma,q),(\alpha,\delta); \\ (\nu+1,\mu);\end{matrix} -\frac{x}{s^{\delta}}\right] \quad (3.2.1)$$

where, $\mu,\nu,\gamma \in C; Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0, Re(\alpha) \geq 0, Re(s) \geq 0$ and $q \in (0,1) \cup N$.

Proof: In virtue of (1.2) and (1.6),

$$L\{z^{\alpha-1} J_{\nu,q}^{\mu,\gamma}(xz^{\delta})\}$$

$$= \int_0^{\infty} e^{-sz} z^{\alpha-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-xz^{\delta})^n}{n!\,\Gamma(\mu n+\nu+1)} dz$$

$$= \frac{s^{-\alpha}}{\Gamma(\gamma)_2}\ \Psi_1\left[\begin{matrix}(\gamma,q),(\alpha,\delta); \\ (\nu+1,\mu)\ ;\end{matrix} -\frac{x}{s^{\delta}}\right]$$

which is (3.2.1).

Particular Cases: If $q = 1, \alpha = \nu + 1, \delta = \mu$ in (3.2.1), reduces to

$$\int_0^{\infty} z^{\nu} e^{-sz} J_{\nu,1}^{\mu,\gamma}(xz^{\mu})dz$$

$$= s^{-\nu-1}(1+xs^{-\mu})^{-\gamma} \quad (3.2.2)$$

If $\alpha = \nu + 1, \delta = \mu, \gamma = q = 1, x = \pm t$, in (3.2.1), reduces to

$$\int_0^{\infty} z^{\nu} e^{-sz} J_{\nu,1}^{\mu,1}(\mp tz^{\mu})dz = s^{-\nu-1} \sum_{n=0}^{\infty} \left(\frac{\mp t}{s^{\mu}}\right)^n$$

$$= \frac{s^{-\nu-1}}{1-\frac{\mp t}{s^{\mu}}}, where \left|\frac{\mp t}{s^{\mu}}\right| < 1$$

$$= \frac{s^{\mu-\nu-1}}{s^{\mu} \pm t} \quad (3.2.3)$$

Theorem(3.3)(K-transform):

$$\int_0^{\infty} t^{\alpha-1} K_{\lambda}(st) J_{\nu,q}^{\mu,\gamma}(\omega t^{\rho})dt$$

$$= \frac{2^{\alpha-2}}{s^{\alpha}\Gamma(\gamma)}\ {}_3\Psi_1\left[\begin{matrix}(\gamma,q),(\alpha\pm\lambda,\rho); \\ (\nu+1,\mu)\quad ;\end{matrix} -\omega\left(\frac{2}{s}\right)^{\rho}\right] \quad (3.3.1)$$

where, $\mu,\nu,\gamma,\alpha,\rho \in C; Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0, Re(\alpha) \geq 0, Re(\rho) \geq 0$ and $q \in N$.

Proof: By changing the variable $st = z$ in L.H.S. of (3.3.1), we get

$$\int_0^{\infty} \left(\frac{z}{s}\right)^{\alpha-1} K_{\lambda}(z) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{n!\,\Gamma(\mu n+\nu+1)}$$

$$\times (-\omega)^n \left(\frac{z}{s}\right)^{\rho n} \left(\frac{1}{s}\right) dz$$

$$= s^{-\alpha} \sum_{n=0} \frac{(\gamma)_{qn} \left(-\frac{\omega}{s^\rho}\right)^n}{n!\,\Gamma(\mu n + \nu + 1)} \int_0^\infty (z)^{\alpha+\rho n-1} K_\lambda(z) dz$$

Now by using the formula (Mathai and Saxena [4], p.78):

$$\int_0^\infty x^{\rho-1} K_\nu(x) dx = 2^{\rho-2} \Gamma\left(\frac{\rho \pm \nu}{2}\right),$$

in the above equation, we obtain the relation (3.3.1).

Theorem(3.4)(Varma transform):

$$\int_0^\infty e^{-\frac{st}{2}} t^{\alpha-1} W_{\lambda,\delta}(st) J_{\nu,q}^{\mu,\gamma}(\omega t^\rho) dt$$

$$= \frac{s^{-\alpha}}{\Gamma(\gamma)}\, {}_3\Psi_2 \left[\begin{matrix} (\gamma, q), \left(\frac{1}{2} \pm \delta + \alpha + \rho\right); \\ (\nu+1, \mu), (1-\lambda+\alpha, \rho); \end{matrix} \; -\frac{\omega}{s^\rho} \right] \tag{3.4.1}$$

where, $\mu, \nu, \gamma, \alpha, \rho \in C; Re(\mu) \geq 0, Re(\nu) \geq -1,$ $Re(\gamma) \geq 0, Re(\alpha) \geq 0, Re(\rho) \geq 0$ and $q \in N$.

Proof: By changing the variable $st = z$ in L.H.S. of (3.4.1), we get

$$\int_0^\infty e^{-\frac{z}{2}} \left(\frac{z}{s}\right)^{\alpha-1} W_{\lambda,\delta}(z) \sum_{n=0}^\infty \frac{(\gamma)_{qn}}{n!\,\Gamma(\mu n + \nu + 1)}$$

$$\times (-\omega)^n \left(\frac{z}{s}\right)^{\rho n} \left(\frac{1}{s}\right) dz$$

$$= s^{-\alpha} \sum_{n=0}^\infty \frac{(\gamma)_{qn} \left(-\frac{\omega}{s^\rho}\right)^n}{n!\,\Gamma(\mu n + \nu + 1)}$$

$$\times \int_0^\infty e^{-\frac{z}{2}} (z)^{\alpha+\rho n-1} W_{\lambda,\delta}(z) dz$$

Now by using the formula (Mathai and Saxena [4], p.79):

$$\int_0^\infty e^{-\frac{t}{2}} t^{\nu-1} W_{\lambda,\mu}(t) dt$$

$$= \frac{\Gamma\left(\frac{1}{2} + \mu + \nu\right) \Gamma\left(\frac{1}{2} - \mu + \nu\right)}{\Gamma(1 - \lambda + \nu)}, Re(\nu \pm \mu) > -\frac{1}{2}$$

in the above equation, we obtain the relation (3.4.1).

IV. FRACTIONAL INTEGRATION AND DERIVATIVE

In this section, we establish several interesting properties of the function $J_{\nu,q}^{\mu,\gamma}(z)$ defined by (1.2) associated with the operator of Riemann-Liouville fractional integrals and derivatives.

Theorem(4.1): Let $\alpha, \mu, \nu, \lambda, \gamma, \omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\lambda) \geq -1, Re(\nu) \geq 0, Re(\gamma) \geq 0, Re(\delta) \geq 0$, and $q \in N$, then the left sided operator of Riemann-Liouville fractional integral I_{0+}^α is given for $x > 0$ by

$$\left(I_{0+}^\alpha \left[t^\nu J_{\lambda,q}^{\mu,\gamma}(\omega t^\delta)\right]\right)(x)$$

$$= \frac{x^{\lambda+\nu}}{\Gamma(\gamma)}\, {}_2\Psi_2 \left[\begin{matrix} (\gamma, q), (\nu+1, \delta) & ; \\ (\lambda+1, \mu), (\alpha+\nu+1, \delta); \end{matrix} - \omega x^\delta \right] \tag{4.1.1}$$

Proof: From the relation (1.2) and (1.9), we have

$$\left(I_{0+}^\alpha \left[t^\nu J_{\lambda,q}^{\mu,\gamma}(\omega t^\delta)\right]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \sum_{n=0}^\infty \frac{(\gamma)_{qn} (-\omega)^n (t)^{\delta n+\nu}}{n!\,\Gamma(\mu n + \lambda + 1)} dt$$

$$= \frac{x^{\lambda+\nu}}{\Gamma(\alpha)} \sum_{n=0}^\infty \frac{(\gamma)_{qn} (-\omega x^\delta)^n}{n!\,\Gamma(\mu n + \lambda + 1)} \int_0^1 (z)^{(\delta n+\nu+1)-1}$$

$$\times (1-z)^{\alpha-1} dz$$

evaluating the inner integral by beta-function formula, above relation reduces to (4.1.1).

Corollary(1.1): Let $\alpha, \mu, \nu, \gamma, \omega \in C; Re(\alpha) \geq 0,$ $Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0$ and $q \in N$, then there holds the relation

$$\left(I_{0+}^\alpha \left[t^\nu J_{\nu,q}^{\mu,\gamma}(\omega t^\mu)\right]\right)(x) = x^{\alpha+\nu} J_{\alpha+\nu,q}^{\mu,\gamma}(\omega x^\mu) \tag{4.1.2}$$

and

$$\left(I_{0+}^\alpha \left[t^\nu J_{\nu,1}^{\mu,\gamma}(\omega t^\mu)\right]\right)(x) = \frac{1}{\omega} x^{\alpha+\nu-\mu}$$

$$\times \left[J_{\alpha+\nu-\mu,1}^{\mu,\gamma-1}(\omega x^\mu) - J_{\alpha+\nu-\mu,1}^{\mu,\gamma}(\omega x^\mu)\right] \tag{4.1.3}$$

Proof: The proof of assertions (4.1.2), is similar to the theorem (4.1) and the proof of assertion (4.1.3) is obtained by virtue of the relation (2.1.4) and (4.1.2).

Theorem(4.2): Let $\alpha,\mu,\nu,\lambda,\gamma,\omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\lambda) \geq -1, Re(\nu) \geq 0, Re(\gamma) \geq 0, Re(\delta) \geq 0$, and $q \in N$, then the right sided operator of Riemann-Liouville fractional integral I_{-}^{α} is given for $x > 0$ by

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\nu-1}J_{\lambda,q}^{\mu,\gamma}(\omega t^{-\delta})\right]\right)(x) = \frac{x^{-\nu-1}}{\Gamma(\gamma)} \times {}_2\Psi_2\left[\begin{matrix}(\gamma,q)\ ,(\nu+1,\delta) & ; \\ (\lambda+1,\mu),(\alpha+\nu+1,\delta); \end{matrix} -\omega x^{-\delta}\right] \quad (4.2.1)$$

Proof: From the relation (1.2) and (1.10), we have

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\nu-1}J_{\lambda,q}^{\mu,\gamma}(\omega t^{-\delta})\right]\right)(x) = \frac{1}{\Gamma(\alpha)} \times \int_x^{\infty} (t-x)^{\alpha-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-\omega)^n (t)^{-\delta n-\alpha-\nu-1}}{n!\,\Gamma(\mu n+\lambda+1)} dt$$

$$= \frac{x^{-\nu-1}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-\omega x^{-\delta})^n}{n!\,\Gamma(\mu n+\lambda+1)} \times \int_0^1 (z)^{(\delta n+\nu+1)-1}(1-z)^{\alpha-1}dz$$

Evaluating the inner integral by beta-function formula, above relation reduces to (4.2.1).

Corollary(1.2): Let $\alpha,\mu,\nu,\gamma,\omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0$ and $q \in N$, then there holds the relation

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\nu-1}J_{\nu,q}^{\mu,\gamma}(\omega t^{-\mu})\right]\right)(x) = x^{-\nu-1}J_{\alpha+\nu,q}^{\mu,\gamma}(\omega x^{-\mu}) \quad (4.2.2)$$

and

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\nu-1}J_{\nu,1}^{\mu,\gamma}(\omega t^{-\mu})\right]\right)(x) = \frac{1}{\omega}x^{\mu-\nu-1}\left[J_{\alpha+\nu-\mu,1}^{\mu,\gamma-1}(\omega x^{-\mu}) - J_{\alpha+\nu-\mu,1}^{\mu,\gamma}(\omega x^{-\mu})\right] \quad (4.2.3)$$

Theorem(4.3): Let $\alpha,\mu,\nu,\lambda,\gamma,\omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\lambda) \geq -1, Re(\nu) \geq 0, Re(\gamma) \geq 0, Re(\delta) \geq 0$, and $q \in N$, then the left sided operator of Riemann-Liouville fractional derivative D_{0+}^{α} is given for $x > 0$ by

$$\left(D_{0+}^{\alpha}\left[t^{\nu}J_{\lambda,q}^{\mu,\gamma}(\omega t^{\delta})\right]\right)(x) = \frac{(x)^{\nu-\alpha}}{\Gamma(\gamma)} \times {}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\nu+1,\delta) & ; \\ (\lambda+1,\mu),(1+\nu-\alpha,\delta); \end{matrix} -\omega x^{\delta}\right] \quad (4.3.1)$$

Proof: By virtue of (1.2) and (1.11), we have

$$\left(D_{0+}^{\alpha}\left[t^{\nu}J_{\lambda,q}^{\mu,\gamma}(\omega t^{\delta})\right]\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}\left[t^{\nu}J_{\lambda,q}^{\mu,\gamma}(\omega t^{\delta})\right]\right)(x)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-\omega)^n}{n!\,\Gamma(\mu n+\lambda+1)\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1} \times \int_0^x t^{\delta n+\nu}(x-t)^{-\{\alpha\}}dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\delta n+\nu+1)(-\omega)^n}{n!\,\Gamma(\mu n+\lambda+1)\Gamma(\delta n+\nu-\{\alpha\}+2)} \times \left(\frac{d}{dx}\right)^{[\alpha]+1} (x)^{\delta n+\nu-\{\alpha\}+1}$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\delta n+\nu+1)}{n!\,\Gamma(\mu n+\lambda+1)\Gamma(\delta n+\nu-\alpha+1)} \times (-\omega)^n (x)^{\delta n+\nu-\alpha}$$

$$= \frac{1}{\Gamma(\gamma)}(x)^{\nu-\alpha} \times {}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\nu+1,\delta) & ; \\ (\lambda+1,\mu),(1+\nu-\alpha+,\delta); \end{matrix} -\omega x^{\delta}\right] (4.3.1)$$

Corollary(1.3): Let $\alpha,\mu,\nu,\gamma,\omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0$ and $q \in N$, then there holds the relation,

$$\left(D_{0+}^{\alpha}\left[t^{\nu}J_{\nu,q}^{\mu,\gamma}(\omega t^{\mu})\right]\right)(x) = x^{\nu-\alpha}J_{\nu-\alpha,q}^{\mu,\gamma}(\omega x^{\mu}) \quad (4.3.2)$$

and

$$\left(D_{0+}^{\alpha}\left[t^{\nu}J_{\nu,1}^{\mu,\gamma}(\omega t^{\mu})\right]\right)(x) = \frac{1}{\omega}x^{\nu-\mu-\alpha}\left[J_{\nu-\mu-\alpha,1}^{\mu,\gamma-1}(\omega x^{\mu}) - J_{\nu-\mu-\alpha,1}^{\mu,\gamma}(\omega x^{\mu})\right] \quad (4.3.3)$$

Theorem(4.4): Let $\alpha,\mu,\nu,\lambda,\gamma,\omega \in C; Re(\alpha) \geq 0, Re(\mu) \geq 0, Re(\lambda) \geq -1, Re(\nu) \geq 0, Re(\gamma) \geq 0, Re(\delta) \geq 0$, and $q \in N$, then the right sided operator of Riemann-Liouville fractional derivative D_{-}^{α} is given for $x > 0$ by

$$\left(D_{-}^{\alpha}\left[t^{\alpha-\nu-1}J_{\lambda,q}^{\mu,\gamma}(\omega t^{-\delta})\right]\right)(x) = \frac{1}{\Gamma(\gamma)}(x)^{-\nu-1}\ {}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\nu+1,\delta) \\ (\lambda+1,\mu),(1+\nu-\alpha+,\delta)\end{matrix};-\omega x^{-\delta}\right] \quad (4.4.1)$$

Proof: By virtue of (1.2) and (1.12), we have

$$\left(D_{-}^{\alpha}\left[t^{\alpha-\nu-1}J_{\lambda,q}^{\mu,\gamma}(\omega t^{-\delta})\right]\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{-}^{1-\{\alpha\}}\left[t^{\alpha-\nu-1}J_{\lambda,q}^{\mu,\gamma}(\omega t^{-\delta})\right]\right)(x)$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-\omega)^n}{n!\,\Gamma(\mu n+\lambda+1)\Gamma(1-\{\alpha\})} \times \left(-\frac{d}{dx}\right)^{[\alpha]+1}\int_0^{\infty} t^{-\delta n+\alpha-\nu-1}(x-t)^{-\{\alpha\}}dt$$

By changing the variable $t = \frac{x}{z}$, the above expression reduces into the form

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\nu-\alpha+\{\alpha\}+\delta n)(-\omega)^n}{n!\,\Gamma(\mu n+\lambda+1)\Gamma(1+\nu-\alpha+\delta n)} \times \left(-\frac{d}{dx}\right)^{[\alpha]+1}(x)^{-\delta n+\alpha-\nu-\{\alpha\}}$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\nu-\alpha+\{\alpha\}+\delta n)}{n!\,\Gamma(\mu n+\lambda+1)\Gamma(1+\nu-\alpha+\delta n)}\frac{\Gamma(1-\nu+\alpha-\{\alpha\}-\delta n)}{\Gamma(-\nu-\delta n)}(-\omega)^n(x)^{-\delta n-\nu-1} \quad (4.4.2)$$

By the reflection formula for the gamma-function (see, [16], (1.60)),

$$\frac{1}{\Gamma(-\nu-\delta n)} = \frac{\Gamma(1+\nu+\delta n)}{\Gamma(1+\nu+\delta n)\Gamma(-\nu-\delta n)}$$

$$= \Gamma(1+\nu+\delta n)\frac{Sin[\Gamma(1+\nu+\delta n)\pi]}{\pi}$$

and

$$\Gamma(\nu-\alpha+\{\alpha\}+\delta n)\Gamma(1-\nu+\alpha-\{\alpha\}-\delta n)$$

$$= \frac{\pi}{(-1)^{[\alpha]+1}Sin[\Gamma(1+\nu+\delta n)\pi]}$$

substituting these relations into (4.4.2), we obtain (4.4.1).

Corollary(1.4): Let $\alpha,\mu,\nu,\gamma,\omega \in C; Re(\alpha) \geq 0$, $Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0$ and $q \in N$, then there holds the relation

$$\left(D_{-}^{\alpha}\left[t^{\alpha-\nu-1}J_{\nu,q}^{\mu,\gamma}(\omega t^{-\mu})\right]\right)(x) = x^{-\nu-1}J_{\nu-\alpha,q}^{\mu,\gamma}(\omega x^{-\mu}) \quad (4.4.3)$$

and

$$\left(D_{-}^{\alpha}\left[t^{\alpha-\nu-1}J_{\nu,1}^{\mu,\gamma}(\omega t^{-\mu})\right]\right)(x) = \frac{1}{\omega}x^{\mu-\nu-1}\left[J_{\nu-\mu-\alpha,1}^{\mu,\gamma-1}(\omega x^{-\mu}) - J_{\nu-\mu-\alpha,1}^{\mu,\gamma}(\omega x^{-\mu})\right] \quad (4.4.4)$$

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