On The Quintic Equation with Three Unknowns $a(x^2+y^2)-(2a-1)xy = (k^2+(4a-1)s^2)^n z^5$

S.Vidhyalakshmi[1] A.Kavitha[*2] M.A.Gopalan[3]

Department of Mathematics,Shrimati Indira Gandhi College,Trichy-620002.

Abstract-**We obtain infinitely many non-zero integer triples (x, y, z) satisfying the quintic equation $a(x^2+y^2)-(2a-1)xy=(k^2+(4a-1)s^2)^n z^5$. A few interesting relations between the solutions and special numbers are presented.**

Keywords - **Quintic equation with three unknowns, integral solutions.**
MSC 2000 Mathematics subject classification: 11D41.

NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

CP_n^m : Centered Pyramidal number of rank n with size m.

$F_{4,n,6}$: Four dimension hexagonal figurate number of rank n

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-3]. For illustration, one may refer [4-8] for quintic equation with three unknowns. This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous quintic equation with three unknowns given by $a(x^2+y^2)-(2a-1)xy=(k^2+(4a-1)s^2)^n z^5$. A few relations between the solutions and special numbers are presented.

II. METHOD OF ANALYSIS

The non-homogeneous quintic equation to be solved for its distinct non-zero integral solutions is

$$a(x^2+y^2)-(2a-1)xy=(k^2+(4a-1)s^2)^n z^5 \qquad (1)$$

Introduction of the linear transformations

$$x=u+v \qquad y=u-v \qquad (2)$$

in (1) leads to

$$u^2+(4a-1)v^2=(k^2+(4a-1)s^2)^n z^5 \qquad (3)$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

Pattern:1

Let $z=\alpha^2+(4a-1)\beta^2$ (4)

Using (4) in (3) and applying the method of factorization, define

$$\left(u+i\sqrt{4a-1}v\right)=(k+i\sqrt{4a-1}s)^n(\alpha+i\sqrt{4a-1}\beta)^5 \qquad (5)$$

Since the complex number raised to any integer power is also a complex number, we write

$$(k+i\sqrt{4a-1}s)^n = c+i\sqrt{4a-1}d \tag{6}$$

where
$$c=\frac{1}{2}[(k+i\sqrt{4a-1}s)^n+(k-i\sqrt{4a-1}s)^n]$$
$$d=\frac{1}{2i\sqrt{4a-1}}[(k+i\sqrt{4a-1}s)^n-(k-i\sqrt{4a-1}s)^n]$$

Using (6) in (5) and equating the real and imaginary parts, we have

$$\left.\begin{aligned} u &= cf-(4a-1)dg \\ v &= cg+fd \end{aligned}\right\} \tag{7}$$

where, $f=\alpha^2-10\alpha^3\beta^2(4a-1)+5\alpha\beta^4(4a-1)^2$
$g=5\alpha^4\beta-10\alpha^2\beta^3(4a-1)+\beta^5(4a-1)^2$

Using (7) in (2), it is seen that

$$\left.\begin{aligned} x &= c(f+g)-d[(4a-1)g-f] \\ y &= c(f-g)-d[(4a-1)g+f] \end{aligned}\right\} \tag{8}$$

Thus (4) and (8) represent the non-zero distinct integral solutions to (1)

Pattern:2

Consider (3) as

$$u^2+(4a-1)v^2=(k^2+(4a-1)s^2)^n z^5 * 1 \tag{9}$$

Write 1 as

$$1=\frac{[(2a-1)+i\sqrt{4a-1}][(2a-1)-i\sqrt{4a-1}]}{4a^2} \tag{10}$$

Substituting (4) and (10) in (9) and employing the factorization method, define

$$u+i\sqrt{4a-1}v=[cf-(4a-1)dg+i\sqrt{4a-1}(cg+df)][\frac{(2a-1)+i\sqrt{4a-1}}{2a}] \tag{11}$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} u &= \frac{2a-1}{2a}[cf-(4a-1)dg]-\frac{1}{2a}[(4a-1)(cg+df)] \\ v &= \frac{1}{2a}[cf-(4a-1)dg+(2a-1)(cg+df)] \end{aligned}\right\} \tag{12}$$

As our interest is on finding integer solutions, we choose α and β suitably so that u and v are integers. Replacing α by $a\alpha$ and β by $a\beta$ in (12) and using (2) the corresponding integer solutions to (1) are found to be

$x=a^5[cf-(4a-1)dg]-a^5[cg+df]$
$y=a^4(a-1)[cf-(4a-1)dg]-(3a-1)a^4[cg+df]$
$z=a^2[\alpha^2+(4a-1)\beta^2]$

For simplicity and clear understanding, we exhibit below the integer solutions and their corresponding properties when

$$a=3,\ k=3,\ s=1,\ n=1 \tag{13}$$

For this choice, (1) and (3) simplify respectively to

$$3(x^2+y^2)-5xy=20z^5 \tag{14}$$
$$u^2+11v^2=20z^5 \tag{15}$$

Let
$$z=a^2+11b^2 \tag{16}$$

Write 20 as

$$20=(3+i\sqrt{11})(3-i\sqrt{11}) \tag{17}$$

Using (16) and (17) in (15) and applying the methods of factorization, define

$$u+i\sqrt{11}v=(3+i\sqrt{11})(a+i\sqrt{11}b)^5 \tag{18}$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} u &= 3[a^5 - 110a^3b^2 + 605ab^4] - 11[5a^4b - 110a^2b^3 + 121b^5] \\ v &= [a^5 - 110a^3b^2 + 605ab^4] + 3[5a^4b - 110a^2b^3 + 121b^5] \end{aligned}\right\} \quad (19)$$

Using (18) in (2) we have

$$\left.\begin{aligned} x &= 4[a^5 - 110a^3b^2 + 605ab^4] - 8[5a^4b - 110a^2b^3 + 121b^5] \\ y &= 2[a^5 - 110a^3b^2 + 605ab^4] - 14[5a^4b - 110a^2b^3 + 121b^5] \end{aligned}\right\} \quad (20)$$

Thus, (16) and (20) represent the integer solutions to the equation (14)

Properties

$(i)\ x(n,1) - y(n,1) + 2t_{4,n}CP_n^6 - 10\left(24F_{4,n,6} - 6CP_n^{28} - 6CP_n^8 - t_{152,n} + 44t_{3,n} - 22t_{4,n}\right) \equiv 0(mod756)$

$(ii)\ x(n,1) + y(n,1) + 6CP_n^6(t_{4,n} - 110) - 7260t_{3,n} + 3630t_{4,n} + 22t_{4,n}(5t_{4,n} - 110) \equiv 0(mod2)$

$(iii) x(n,1) + 220t_{4,n}\left(2 - 11t_{4,n}\right) + 88CP_n^6\left(10 - 11t_{4,n}\right) + 80t_{3,n} - 40t_{4,n}$ is a perfect square

Note1:

It is seen that in addition to (17), 20 may also be written in two different ways as

$$\left.\begin{aligned} 20 &= \frac{(2+i4\sqrt{11})(2-i4\sqrt{11})}{9} \\ 20 &= \frac{(13+i\sqrt{11})(13-i\sqrt{11}}{9} \end{aligned}\right\} \quad (21)$$

Following the procedure similar to the above the corresponding integer solutions for the above two cases are as follows:

Solutions for (i):

$$x = 486[A^5 - 110A^3B^2 + 605AB^4] - 3402[5A^4B - 110A^2B^3 + 121B^5]$$
$$y = -162[A^5 - 110A^3B^2 + 605AB^4] - 3726[5A^4B - 110A^2B^3 + 121B^5]$$
$$z = 9A^2 + 99B^2$$

Solutions for (ii):

$$x = 1134[A^5 - 110A^3B^2 + 605AB^4] + 162[5A^4B - 110A^2B^3 + 121B^5]$$
$$y = 972[A^5 - 110A^3B^2 + 605AB^4] - 1944[5A^4B - 110A^2B^3 + 121B^5]$$
$$z = 9A^2 + 99B^2$$

Note 2:

Consider (14) as

$$3(x^2 + y^2) - 5xy = 20z^5 * 1 \quad (22)$$

Write 1 as

$$1 = \frac{[(2n^2-2n-5)+i(2n-1)\sqrt{11}][(2n^2-2n-5)-i(2n-1)\sqrt{11}]}{(2n^2-2n+6)^2} \quad (23)$$

Considering (16), (17) (23) and following the analysis presented above, the corresponding integer solutions to (14) are found to be

$$x = (6n^2 - 28n - 4)(2n^2 - 2n + 6)^4(a^5 - 110a^3b^2 + 605ab^4 + 5a^4b - 110a^2b^3 + 121b^5) + (2n^2 + 4n - 8)(2n^2 - 2n + 6)^4[a^5 - 110a^3b^2 + 605ab^4 - 11[5a^4b - 110a^2b^3 + 121b^5]$$

$y = (6n^2 - 28n - 4)(2n^2 - 2n + 6)^4(a^5 - 110a^3b^2 + 605ab^4 - 5a^4b + 110a^2b^3 - 121b^5) - (2n^2 + 4n - 8)(2n^2 - 2n + 6)^4[a^5 - 110a^3b^2 + 605ab^4 + 11[5a^4b - 110a^2b^3 + 121b^5]$

$z = (2n^2 - 2n + 6)^2(a^2 + 11b^2)$

Conclusion:

It is worth to mention here that, instead of (17), one may also consider (21) to represent (20) in (22) and thus obtain two more patterns of integer solutions to (14). Thus, in this paper, we have analysed various patterns of non-zero distinct integer solutions to (1) and in-particular to (14). As the quintic equations are rich in variety, one may attempt to obtain integer solutions to quintic equations with three or more variables along with the corresponding properties.

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