On Generalized Concircular ϕ-Recurrent $N(k)$-Contact Metric Manifold

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Abstract: The object of the present paper is to study generalized concircular ϕ-recurrent $N(k)$-contact metric manifold and obtained some important results.

Key Words: $N(k)$-contact metric manifold, η-Einstein manifold, Generalized Concircular ϕ-recurrent manifold, constant curvature.

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1. Introduction

In 1988, S. Tanno [12] introduced the notion of k-nullity distribution of a Contact metric manifold as a distribution such that the characteristic vector field ξ of the Contact metric manifold belongs to the distribution. The Contact metric manifold with ξ belonging to the k-nullity distribution is called $N(k)$-Contact metric manifold and such a manifold is also studied by various authors. In 2008, De, Gazi [6] studied ϕ-recurrent N(k)-Contact metric manifold.

In this paper we study Generalized Concircular ϕ-recurrent N(k)-Contact metric manifold. Here we show that Generalized Concircular ϕ-recurrent N(k)-Contact metric manifold is an η-Einstein manifold, and we find a relation between the associated 1-forms A and B. We also prove that the characteristic vector field ξ and the vector field ρ associated to the 1-forms A and B are co-directional. Finally we prove that a generalized Concircular ϕ-recurrent N(k)-Contact metric manifold is of constant curvature.

2. Contact Metric Manifold

A (2n+1)-dimensional manifold M^{2n+1} is said to admit an almost Contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

(2.1) $(a)\ \phi^2(X) = -X + \eta(X)\xi,\ (b)\ \eta(\xi) = 1,\ (c)\ \eta\circ\phi = 0,\ (d)\ \phi\xi = 0.$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1}\times \boldsymbol{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of $\boldsymbol{R}$ and f is a smooth function on $M \times \boldsymbol{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is

(2.2) $$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then M becomes an almost contact metric manifold equipped with an almost contact structure (ϕ, ξ, η, g). From (2.1) it can be easily seen that

(2.3) $$(a)\ g(X, \phi Y) = -g(\phi X, Y), (b)\ g(X, \xi) = \eta(X),$$

for all vector fields X, Y . An almost contact metric structure becomes a contact metric structure if

(2.4) $$g(X, \phi Y) = d\eta(X, Y),$$

for all vector fields X, Y. The 1-form η is then a contact form and ξ is its characteristic vector field. The k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by[12]

$$N(k): p \rightarrow Np(k) = \{Z \in TpM : R(X, Y)Z = g(Y, Z)X - g(X, Z)Y\},$$

k being a constant. If the characteristic vector $\xi \in N(k)$, then we call a Contact metric manifold an $N(k)$-Contact metric manifold.

In $N(k)$-Contact metric manifold the following relations hold [6]:

(2.5) $$h^2 = (k-1)\phi^2, \qquad k \leq 1,$$

(2.6) $$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

(2.7) $$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X],$$

(2.8) $$S(X, \xi) = 2nk\eta(X),$$

(2.9) $$S(X, Y) = 2(n-1)g(X, Y) + 2(n-1)g(hX, Y) + [2(1-n) + 2nk]\eta(X)\eta(Y), \qquad n \geq 1,$$

$$r = 2n(2n-2+k),$$

(2.10) $$\nabla_X \xi = -\phi X - \phi hX,$$

(2.11) $$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX, Y),$$

(2.12) $$(\nabla_X \eta)(Y) = g(X + hX, \phi Y),$$

(2.13) $$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],$$

(2.14) $$\eta(R(X, Y)Z) = k(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].$$

Definition 2.1. ([6]) A $N(k)$-Contact metric manifold is said to be locally concircular ϕ-symmetric if

(2.15) $$\phi^2((\nabla_W \bar{C})(X, Y)Z) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.2. ([6]) A $N(k)$-Contact metric manifold is said to be concircular ϕ- recurrent if there exists a non-zero 1-form A such that

(2.16) $$\phi^2((\nabla_W \bar{C})(X, Y)Z) = A(W)\bar{C}(X, Y)Z,$$

for arbitrary vector fields X, Y, Z and W, where $\bar{C}$ is a Concircular curvature tensor given by[4]

$$\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.17}$$

where R is the curvature tensor, and r is the scalar curvature.

If the 1-form A vanishes, then the manifold reduces to locally concircular ϕ-symmetric manifold.

Definition 2.3. A $N(k)$-Contact metric manifold is said to be generalized concircular ϕ-recurrent if its curvature tensor $\bar{C}$ satisfies the condition

$$\phi^2\big((\nabla_W \bar{C})(X,Y)Z\big) = A(W)\bar{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{2.18}$$

where A and B are two 1-forms, B is non-zero and these are defined by

$$A(W) = g(W,\rho_1), \qquad B(W) = g(W,\rho_2),$$

and ρ_1, ρ_2 are vector fields associated with 1-forms A and B, respectively.

3. Generalized Concircular ϕ-Recurrent $N(k)$-Contact Metric Manifold

Let us consider a Generalized Concircular ϕ-recurrent $N(k)$ −Contact metric manifold. Then by virtue of 2.1(a) and (2.18) we have

$$\begin{aligned} &-((\nabla_W \bar{C})(X,Y)Z) + \eta((\nabla_W \bar{C})(X,Y)Z)\xi \\ &= A(W)\bar{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

from which it follows that,

$$\begin{aligned} &-g((\nabla_W \bar{C})(X,Y)Z,U) + \eta((\nabla_W \bar{C})(X,Y)Z)\eta(U) \\ &= A(W)g(\bar{C}(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{3.2}$$

Let $\{ei\}$, $i = 1,2,\ldots,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = \{e_i\}$ in (3.2) and taking summation over i, $1 \le i \le 2n+1$, we get

$$\begin{aligned} (\nabla_W S)(Y,Z) = {} & \frac{dr(W)}{2n+1} g(Y,Z) - \frac{dr(W)}{2n(2n+1)}[g(Y,Z)\eta(Y)\eta(Z)] \\ & -A(W)\left[S(Y,Z) - \frac{r}{2n+1} g(Y,Z)\right] - 2nB(W)g(Y,Z). \end{aligned} \tag{3.3}$$

Replacing Z by ξ in (3.3) and using (2.12), we have

$$(\nabla_W S)(Y,\xi) = \frac{dr(W)}{2n+1}\eta(Y) - A(W)\eta(Y)\left[2nk - \frac{r}{2n+1}\right] - 2nB(W)\eta(Y). \tag{3.4}$$

Now we have,

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).$$

Using (2.8) and (2.10) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = -2nkg(\phi W + \phi hW,\ Y) + S(Y,\ \phi W + \phi hW). \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$S(Y,\phi W + \phi hW) = 2nkg(\phi W + \phi hW, Y) + \frac{dr(W)}{2n+1}\eta(Y) - A(W)\eta(Y)\left[2nk - \frac{r}{2n+1}\right] - 2nB(W)\eta(Y). \tag{3.6}$$

Replacing Y by ϕY in (3.6), and after a brief simplification, we get

$$S(Y,W) = 2[(n + k - 1) + n(k-1)(nk+n-1)]g(Y,W) + 2[(n-1)(k-1) - n(k-1)(nk+n-1)]\eta(Y)\eta(W),$$

or,

$$S(Y,W) = ag(Y,W) + b\eta(Y)\eta(W), \tag{3.7}$$

Where $a = 2[(n + k - 1)] + n(k-1)(nk+n-1)$,
$b = 2[(n-1)(k-1) - n(k-1)(nk+n-1)]$ are constants.
Therefore we state the following:

Theorem 3.1. A Generalized Concircular ϕ-recurrent $N(k)$-Contact metric manifold is an η-Einstein manifold.

Now putting $Y = Z = ei$ in (3.2) and taking summation over i, $i = 1,2,\dots,2n+1$, we get

$$-(\nabla_W S)(X,U) + \frac{dr(w)}{2n+1}g(X,U) + (\nabla_W S)(X,\xi)\eta(U) - \frac{dr(W)}{2n+1}\eta(X)\eta(U) = A(W)[S(X,U) - \tfrac{r}{2n+1}g(X,U)] + 2nB(W)g(X,U). \tag{3.8}$$

Putting $U = \xi$ in (3.8), we have

$$A(W)\eta(X)\left[2nk - \tfrac{r}{2n+1}\right] + 2nB(W)\eta(X) = 0. \tag{3.9}$$

Putting $X = \xi$ in (3.9) we have,

$$B(W) = \left[\tfrac{r}{2n(2n+1)} - k\right]A(W). \tag{3.10}$$

Hence we state the following theorem:

Theorem 3.2. *In a generalized Concircularly ϕ-recurrent N(k)-Contact metric manifold , the 1-forms A and B are related as in (3.10).*

Now from (3.1) we have

$$(\nabla_W \bar{C})(X,Y)Z = \eta\big((\nabla_W \bar{C})(X,Y)Z\big)\xi - A(W)\bar{C}(X,Y)Z - B(W)[g(Y,Z)X - g(X,Z)Y]. \tag{3.11}$$

This implies

$$(\nabla_W R)(X,Y)Z = \eta\big((\nabla_W R)(X,Y)Z\big)\xi - A(W)R(X,Y)Z$$
$$+\frac{dr(W)}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi\,]$$
$$+\frac{r}{2n(2n+1)}A(W)[g(Y,Z)X - g(X,Z)Y\,] - B(W)[g(Y,Z)X - g(X,Z)Y]. \tag{3.12}$$

From (3.12) and the Bianchi identity we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z)$$
$$= \left[\frac{r}{2n(2n+1)}A(W) - B(W)\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$$
$$+ \left[\frac{r}{2n(2n+1)}A(X) - B(X)\right][g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]$$
$$+ \left[\frac{r}{2n(2n+1)}A(Y) - B(Y)\right][g(X,Z)\eta(W) - g(W,Z)\eta(X)]. \tag{3.13}$$

By virtue of (2.14), we obtain from (3.13) that

$$A(W)k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + A(X)k[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]$$
$$+ A(Y)k[g(X,Z)\eta(W) - g(W,Z)\eta(X)]$$
$$= \left[\frac{r}{2n(2n+1)}A(W) - B(W)\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$$
$$+ \left[\frac{r}{2n(2n+1)}A(X) - B(X)\right][g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]$$
$$+ \left[\frac{r}{2n(2n+1)}A(Y) - B(Y)\right][g(X,Z)\eta(W) - g(W,Z)\eta(X)]. \tag{3.14}$$

Putting $Y = Z = e_i$ in (3.14) and taking summation over i, $1 \le i \le 2n+1$, we get

$$(a)\ A(W)\eta(X) = A(X)\eta(W),$$
$$(b)\ B(W)\eta(X) = B(X)\eta(W) \tag{3.15}$$

for all vector fields X, W.

Replacing X by ξ in (3.15) we get

$$(a) A(W) = \eta(W)\eta(\rho_1)$$
$$(b) B(W) = \eta(W)\eta(\rho_2), \tag{3.16}$$

From (3.15) and (3.16), we can state the following theorem:

Theorem 3.3. *In a generalized concircular ϕ-recurrent N(k)-contact metric manifold, the characteristic field ξ and the vector fields ρ_1 and ρ_2 associated to the 1-forms A and B respectively are co-directional and the 1-forms A and B are given by (3.16).*

4. 3-dimensional Generalized Concircular ϕ-Recurrent $N(k)$ –Contact Metric Manifold

In a 3-dimensional $N(k)$ –Contact metric Manifold (M^3, g), we have

$$R(X,Y)Z = \left(\frac{r}{2}-2k\right)[g(Y,Z)X - g(X,Z)Y] + \left(3k-\frac{r}{2}\right)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \tag{4.1}$$

and

$$S(X,Y) = \left(\frac{r}{2}-k\right)g(X,Y) + \left(3k-\frac{r}{2}\right)\eta(X)\eta(Y). \tag{4.2}$$

Using (4.1) in (2.17), we get

$$\bar{C}(X,Y)Z = \left[\frac{r}{2}-2k-\frac{r}{2n(2n+1)}\right][g(Y,Z)X - g(X,Z)Y] + \left[3k-\frac{r}{2}\right][g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \tag{4.3}$$

Differentiating the equation (4.3) covariantly, we get

$$(\nabla_W\bar{C})(X,Y)Z = \left[\frac{10dr(W)}{21}\right][g(Y,Z)X - g(X,Z)Y] - \left[\frac{dr(W)}{2}\right][g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] + \left[3k-\frac{r}{2}\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)](\nabla_W\xi) + \left[3k-\frac{r}{2}\right][\eta(Y)X - \eta(X)Y](\nabla_W\eta)(Z) + \left[3k-\frac{r}{2}\right][g(Y,Z)\xi - \eta(Z)Y](\nabla_W\eta)(X) - \left[3k-\frac{r}{2}\right][g(X,Z)\xi - \eta(Z)X](\nabla_W\eta)(Y). \tag{4.4}$$

Noting that we may assume that all vector fields X, Y, Z, W are orthogonal to ξ and using (2.1), we get

$$(\nabla_W\bar{C})(X,Y)Z = \left[\frac{10dr(W)}{21}\right][g(Y,Z)X - g(X,Z)Y] + \left[3k-\frac{r}{2}\right][g(Y,Z)(\nabla_W\eta)(X) - g(X,Z)(\nabla_W\eta)(Y)]\xi. \tag{4.5}$$

Applying ϕ^2 on both sides of (4.5) and using (2.1), we get

$$\phi^2(\nabla_W\bar{C})(X,Y)Z = \frac{10dr(W)}{21}[g(X,Z)Y - g(Y,Z)X]. \tag{4.6}$$

Using (2.18), the equation (4.6) reduces to,

$$A(W)\bar{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y] = \frac{10dr(W)}{21}[g(Y,Z)X - g(X,Z)Y]. \tag{4.7}$$

Putting $W = \{ei\}$, where $\{ei\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

$$\bar{C}(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y], \tag{4.8}$$

where $\lambda = \left[\frac{10dr(e_i)}{21A(e_i)} + \frac{B(e_i)}{A(e_i)}\right]$ is a scalar, since A and B are non-zero 1-forms. Then by Schur's theorem λ will be a constant on the manifold. Therefore, we state the following theorem:

Theorem 4.4. *A 3-dimensional Generalized Concircular ϕ-recurrent N(k)-Contact metric manifold is of constant curvature.*

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