A Block Bootstrap Procedure for Long Memory Processes

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Abstract— **Long range dependence (LRD) or long memory, initially observed in data analysis in hydrology, is established in diverse fields, for example in financial economics, networks traffic, psychology, cardiology, etc. The presence of long memory in a stochastic process has important consequences in statistical inferences such is a slower rate of decaying of the variance of the sample mean than the typical rate under short dependence.**

The block bootstrap has been largely developed for weakly dependent time series. Much of research has focused on the large properties of block bootstrap inference about sample means. The block bootstrap for time series consists in randomly resampling blocks of consecutive values of the given data and aligning these blocks into a bootstrap sample.

In this paper, we consider a bootstrap technique, which uses blocks composed of cycles, to estimate the variance of the sample mean. The blocks are compounded by a fix number of cycles. The total number of cycles and their lengths are random. The finite sample properties of the method are investigated by means of Monte Carlo experiments and the results indicate that it can be used as an alternative to other block bootstrap methods.

Keywords— **Long range dependence, time series, block bootstrap, variance estimation.**

I. INTRODUCTION

Long memory or long range dependence (LRD) has been observed in natural, economic and financial time series as well as in medical records. Long memory is a characteristic of a stationary process in which the underlying time series realizations display significant temporal dependence at very distant observations. The autocorrelation function of a process with long memory takes far longer to decay than that of the common processes modeled by the vast class of ARMA (Auto Regression Moving Average) models usually called as short memory processes. Long memory was noticed in some processes because the decaying rate of the variance of the sample mean was proportional to $n^{-\alpha}$ with $\alpha \in (0,1)$, while the usual rate for a sample of independent and identically distributed (i.i.d.) observations or weakly correlated data is n^{-1}, where n is the series length. On the other hand, confidence intervals for the sample mean based on the Normal approximation are too short.

The bootstrap methodology, proposed originally by [1] is an effective technique to present solutions when the parametric methods and statistical theory do not work. The block bootstrap for time series consists in randomly resampling blocks of consecutive values of the given data and aligning these blocks into a bootstrap sample. Additionally, the type of block resampling flexibly allows for different block bootstraps, such as the moving block bootstrap (MBB) of [2] and [3], and the non-overlapping block bootstrap (NBB) of [4] among several others. However, most developments for the block bootstrap have treated only weakly dependent data.

Refernce [5] showed that the MBB could fail in approximating sample means for a category of strongly or long-range dependent processes generated by transformations of Gaussian series. This finding appears to have largely deflated confidence in the block bootstrap for LRD. The matched-block bootstrap of [6] samples blocks dependently, attempting to follow each block with one that might realistically follow it in the underlying process, to better match the dependence structure of the data.

This paper aims to provide a new block bootstrap procedure that can be applied in long memory processes. In this paper, we consider a bootstrap technique, which uses blocks composed of cycles. A cycle is defined as a pair of alternating high and low data that is created when the terms of the time series cross the sample mean. Then we randomly resample blocks composed of a fixed number of consecutive cycles and concatenate them to form the bootstrap series. A simulation Monte Carlo study is conducted to empirically estimate the performance of this bootstrap method for estimation of variance of the sample mean.

The remainder of the paper is organized as follows. Section 2 introduces notation and the class of ARFIMA processes. Some techniques of the block bootstrap methods, as well as the bootstrap method with blocks composed of cycles, are described in section 3. Section 4 presents the results of a simulation study of the finite sample properties of the proposed bootstrap method. Conclusions are reported in the last section.

II. LONG MEMORY AND ARFIMA MODELS

An important consequence of the presence of the long memory in a time series is a slower rate of decaying of the variance of the sample mean than the typical rate under short range dependence. In this section, we briefly describe the LRD notion and its impact on the behavior of the variance of the sample mean. Also, the ARFIMA model is defined.

A. Long Memory and the Variance of the Sample Mean

Long-memory or long-range dependence (LRD) refers to the property of a time series to exhibit a significant dependence between very distant observations. Let $\{X_t, t \in Z\}$ be a stationary time series with real values. It is intuitively expected that the autocorrelation function vanishes when the distance between the data becomes large. So we can suppose that the data are asymptotically independent and the autocorrelations are absolutely summable. It is the case in most of stationary time series included the vast class of Autoregressive Integrated Moving Average (ARMA) models, which are in general classified as short-memory processes ([7], [8]). In contrast, LRD is generally defined by the fact that the autocorrelations are absolutely non-summable ([9]). The variance of the sample mean of correlated data, such as the time series, depends also on the autocorrelations. Let $\{x_1, x_2, ..., x_n\}$ be a realization of a stochastic process $\{X_t, t \in Z\}$ with constant and finite mean and variance, and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ be the sample mean. Then the variance of the sample mean is equal to

$$\mathrm{var}(\bar{X}_n) = \frac{\sigma^2}{n^2}\sum_{i,j=1}^{n} \rho(i,j) = \sigma^2 \frac{1+\delta_n(\rho)}{n} \qquad (1)$$

This formula differs to the usual normal formula $\mathrm{var}(\bar{X}_n) = \frac{\sigma^2}{n}$, by non-zero correction term

$$\delta_n(\rho) = \frac{1}{n}\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \rho(i,j) \qquad (2)$$

Here we have made the following notations:

$$\mu = E(X_t),\ \sigma^2 = \mathrm{var}(X_t),\ \gamma(i,j) = E[(X_i - \mu)(X_j - \mu)],\ \rho(i,j) = \frac{\gamma(i,j)}{\sigma^2} \qquad (3)$$

If the correlations $\rho(i,j)$ depend only on the lag $|i-j|$, than the above equation (2) can be simplified to

$$\delta_n(\rho) = 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho(k) \qquad (4)$$

where $\rho(k) = \rho(t, t+k)$ does not depend on the time t.

The correction term, $\delta_n(\rho)$, is zero for non-correlated sequences. For weakly correlated data this term is approximately constant for large samples ([10]). However, if the underline process exhibits LRD, the correction term $\delta_n(\rho)$ increases with the sample size, affecting the decaying rate of the variance of the sample mean. In these cases the usual variance of the sample mean, i.e. the variance of the single observation divided by the sample size, is too small. On the other hand, confidence intervals for the sample mean based on the Normal approximation are too short.

In the last twenty years a wide literature has been dedicated to the study of this type of memory structure, especially since the two papers of [11] and [12] introduced, separately, the concept of fractional integration in time series analysis, by allowing the parameter d in an ARIMA(p, d, q) to assume non-integer real values. Before them fractional Brownian motion and fractional Gaussian noise had been introduced by [13]. However, there are several possible definitions of the property of LRD ([14]). Most of the definitions appearing in the literature are based on the second order properties of a stochastic process (for more details see [10]).

In fact, LRD processes are characterized by slowly decaying autocorrelations or by a spectral density function exhibiting a pole at the origin.

Let $\{X_t, t \in Z\}$ be a stationary process with autocorrelation function $\rho(k)$, and spectral density function

$$f(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \gamma(k) e^{ik\omega} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \gamma(k)\cos(k\omega) \qquad (5)$$

In general, long memory is defined by the fact that the autocorrelations of the process are absolutely non-summable ([9]) i.e.

$$\sum_{k=-\infty}^{\infty} |\rho(k)| = \lim_{n\to\infty}\sum_{k=-n}^{n} |\rho(k)| = \infty \qquad (6)$$

However, there are alternative definitions. According to [10] two common definitions of long memory are given as follows

Definition in time domain:

Suppose that there exists a real number $\alpha \in (0,1)$ and a constant $c_\rho > 0$ not depending on *k*, such that

$$\lim_{k\to\infty} \frac{\rho(k)}{c_\rho k^{-\alpha}} = 1 \qquad (7)$$

Then X_t is called a stationary process with long memory or long-range dependence or strong dependence, or a stationary process with slowly decaying or long-range correlations.

Definition in spectral domain:

Suppose that there exists a real number $\beta \in (0,1)$ and a constant $c_f > 0$ not depending on ω such that

$$\lim_{\omega\to 0} \frac{f(\omega)}{c_f |\omega|^{-\beta}} = 1 \qquad (8)$$

Then X_t is called a stationary process with long memory or long-range dependence or strong dependence.

B. The ARFIMA(p, d, q) Process

One of the most popular long memory processes is the auto regression fractionally integrated moving average (ARFIMA) process, independently introduced by [11] and [12]. This process simply generalizes the usual ARIMA(*p*, *d*, *q*) process by allowing the parameter *d* to assume any real value.

The process $\{X_t, t\in Z\}$ is said to be a canonical ARFIMA(*p*, *d*, *q*) process with $d\in(-0.5,0.5)$, if it is a stationary solution of the difference equation

$$\Phi(B)(1-B)^d (X_t - \mu) = \Theta(B)\varepsilon_t, \qquad (9)$$

where $\Phi(B) = 1-\phi_1 B - ... - \phi_p B^p$ and $\Theta(B) = 1+\theta_1 B + ... + \theta_q B^q$ are the AR and MA polynomials of degree *p* and *q*, respectively, *B* is the back shift operator, $\mu = E(X_t)$, ε_t is a white noise process and

$$(1-B)^d = \sum_{k=0}^{\infty} \pi_k B^k \qquad (10)$$

with $\pi_k = \dfrac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)}$ and $\Gamma(\cdot)$ is the gamma function.

If $p=q=0$, the process $\{X_t, t\in Z\}$ is called Fractionally Integrated Noise and denoted by FI(*d*). When $d\in(0,0.5)$ the ARFIMA(*p*, *d*, *q*) process is stationary and the autocorrelation function decays to zero hyperbolically and we have

$$\rho(k) \approx C|k|^{2d-1} \text{ for } |k|\to\infty, \qquad (11)$$

where the constant *C* does not depend on lag *k*. Also, [10] shows that

$$\operatorname{var}(\bar{X}_n) \approx cn^{2d-1} \text{ for } n\to\infty, \qquad (12)$$

with *c* not depending on *n*. In this case, we say that the process has a long memory behavior. When $d\in(-0.5,0)$ the ARFIMA(*p*, *d*, *q*) process is a stationary process with a strong mean reversion; we say that the process has intermediate memory or the process is anti-persistent.

In the following, we will concentrate on FI(*d*) processes with $d\in(0,0.5)$. For this range of values of *d* the process is stationary, invertible and possesses long range dependence.

III. The Bootstrap Methods for Time Series

Bootstrap method, originally introduced by [1], is a resample technique that provides better estimations in small sample sizes than the classical methods. The classical bootstrap uses single observations as the resampling units and requires independence among the data, which is rarely to get in time series. When the data on the hand are not i.i.d., such as a time series, the resample technique must be carried out in such a way that the dependence structure of the original time series to be preserved in the bootstrap time series. The most common are the block bootstrap methods.

Block bootstrap methods for dependent data were introduced by [2], [4], [15], among others. These methods are highly adaptive, or nonparametric, in the spirit of bootstrap methods. In these methods, the data is divided in blocks which are approximately independent and the joint distribution of the variables in different blocks is about to the same due to stationarity. Different methods differ in the way used for block constructing.

The basic block bootstrap procedure can be described as following. Given data $\mathbf{X} = \{X_i, i=1,2,...,n\}$ from a stationary time series, prepare *b* blocks $B_1,...,B_b$ of *l* consecutive observations each. The blocks may be overlapping with $b = n-l+1$, or non-overlapping, with b the integer part of $\frac{n}{l}$. Then, a bootstrap series is generated by randomly sampling blocks of observations, and concatenating them to form a series of length n. If n is not a multiple of l, then the last block is truncated so the series is the right length. The overlapping block technique is commonly abbreviated to MBB (moving block bootstrap), while the second one to NBB (non-overlapping block bootstrap). For the MBB to resample all blocks uniformly, [16] suggested wrapping the data around in a circle before the blocks are created. Their procedure is called the circular block bootstrap (CBB). Reference [17]) proposed another alternative, the stationary bootstrap (SB), which works by selecting blocks formed with random starting points and random length. They established that the SB estimate for the sample mean is unbiased even for finite samples. They also showed that the SB estimate of the variance of the sample mean is consistent.

Block bootstrap methods have been used for the estimation of parameters of ARFIMA(p, d, q) model (see for example [18]). Reference [5] showed that in general, the MBB procedure fails to provide a valid approximation to the distribution of normalized sample mean under LRD. One of the reasons behind this is that joining independent bootstrap blocks to form the bootstrap series destroys the strong dependence of the underlying observations. Reference [6] sampled blocks dependently, attempting to follow each block with one that might realistically follow it in the underlying process, to better match the dependence structure of the data. This procedure is called matched-block bootstrap. Reference [19] used the matched-block bootstrap for LRD processes, investigating block matching rules, based on linear combinations of observations in the block.

A crucial issue to the block bootstrap techniques is the optimal choice of block length. Reference [20] indicated that the optimal block length depends significantly on context. Reference [21] and [22] established a data-dependent method that successfully provides the optimal block length or expected block length for the CBB and SB method, respectively.

In this paper, we consider blocks that are composed of one or more consecutive cycles. Reference [23] used this bootstrap technique for estimating the memory parameter d, of an FI(d) process. Reference [24] applied this bootstrap procedure for correcting the size distortion of asymptotic tests for LRD detection. In this paper we focus on FI(d) processes and consider the point estimation of the variance of the sample mean as well as constructing confidence intervals for the mean.

Let $\{X_i, i = 1,2,...,n\}$ be the observed time series. A cycle is defined as a pair of alternating high and low runs (or vice versa) of the data that are created when the terms of the time series cross the sample mean. Then we define a block composed of a fixed number of consecutive cycles. Let $C_1, C_2,..., C_k$ denote the created cycles and let $n_i, i = 1,2,...,k$ be the number of terms of the cycle C_i. Then, the created cycles would be written as $C_1 = \{X_1, X_2,..., X_{n_1}\}$, $C_2 = \{X_{n_1+1}, X_{n_1+2},..., X_{n_1+n_2}\}$, …, $C_k = \{X_{n_1+n_2+...+n_{k-1}+1},..., X_n\}$. The cycle lengths and k are random variables so long as they are determined automatically by the data. The number of cycles of a block is a tuning parameter and is analogous with the block length in MBB method. If the number of cycles per block is set to be s, then the blocks would be defined by $B_i = \{C_i, C_{i+1},..., C_{i+s-1}\}$ for $i = 1,2,..., k-s+1$.

We consider the circular moving-block bootstrap approach, which amounts to wrapping the data around in a circle before the blocks are created as in [16]. In this case there are exactly k (the number of cycles) blocks for any value of s. We treat a cycle as an inseparable observation. Since the cycles, consequently the blocks, are created automatically by the series' crossings of the sample mean, we expect that the transition across joint points of the blocks in the bootstrap pseudo series to be more realistic.

Table 1 illustrates the steps involved in bootstrap technique with circular blocks composed of cycles. The data are simulated from an FI(d) process with $d = 0.3$ for a sample of $n = 15$ observations. The "level" is defined to be the plus sign if the respective observation is greater than the sample mean and the minus sign otherwise. Number of created cycles is $k = 4$ with lengths $n_1 = 3$, $n_2 = 4$, $n_3 = 6$ and $n_4 = 2$ respectively. The number of cycles per block is chosen to be $s = 3$. The observations, required for the time series to be wrapped up to complete all blocks, are obtained by "*Copy-Pasting*" from starting values. In the following we will use the CBBC (Circular Block Bootstrap with Cycles) abbreviation for the proposed technique.

IV. SIMULATION STUDY

This section investigates the performance of the CBBC method under LRD. We implement the CBBC technique to draw inferences about of the sample mean under LRD. For comparison to other block bootstrap methods, we also examine the MBB and SB methods. In the following simulation study, we consider data from (mean-zero) fractionally integrated FI(d) processes defined by equation (9) for $p = q = 0$. We focus on a variety of long-memory parameters $d \in \{0.1, 0.2, 0.3, 0.4\}$, and sample sizes $n \in \{100, 200, 500, 1000\}$, which are expected to more critically impact on resampling performance than the innovation type or a further filter ([25]). In each case, the distribution of innovations ε_t is standard normal and $\mu = 0$. For each model and sample size we estimated the variance of the sample mean and computed 90% confidence intervals for the process mean. The bootstrap point estimates are computed as averages over 1000 Monte Carlo trials of the variances of 1000 bootstrap replicates for each bootstrap method. The bootstrap confidence intervals are constructed by percentile bootstrap method ([26]) using 999 bootstrap replications. Then the empirical coverage probability is computed over 1000 Monte Carlo simulations. Also, the average interval length is computed. The block length for MBB or expected block length for SB is chosen by the data driven method proposed in [21] and [22]. Approximately the same block length is chosen for CBBC method.

TABLE I

ILLUSTRATION OF BOOTSTRAP TECHNIQUE WITH CIRCULAR BLOCKS COMPOSED OF CYCLES

(DATA AFTER TIME 15 ARE OBTAINED BY WRAPPING THE TIME SERIES. IN PARENTHESIS IS THE OLD CYCLE)

Time	Data	Wrapped data	Level	Cycle	Overlapping circular blocks (with $s = 3$ cycles)			
					B_1	B_2	B_3	B_4
1	-0.16302	-0.16302	-	1	1			
2	0.445951	0.445951	+	1	1			
3	0.771359	0.771359	+	1	1			
4	-1.14752	-1.14752	-	2	1	2		
5	-0.13973	-0.13973	-	2	1	2		
6	-1.4016	-1.4016	-	2	1	2		
7	0.281973	0.281973	+	2	1	2		
8	-0.36006	-0.36006	-	3	1	2	3	
9	-0.50011	-0.50011	-	3	1	2	3	
10	-0.19678	-0.19678	-	3	1	2	3	
11	0.928974	0.928974	+	3	1	2	3	
12	0.231137	0.231137	+	3	1	2	3	
13	0.111818	0.111818	+	3	1	2	3	
14	-0.7839	-0.7839	-	4		2	3	4
15	1.012432	1.012432	+	4		2	3	4
		-0.16302	-	5 (1)			3	4
		0.445951	+	5 (1)			3	4
		0.771359	+	5 (1)			3	4
		-1.14752	-	6 (2)				4
		-0.13973	-	6 (2)				4
		-1.4016	-	6 (2)				4
		0.281973	+	6 (2)				4
Mean:	-0.0606							

Simulation results for estimating the variance of the sample mean are shown in figure 1. All estimations are downward biased. The bias increases significantly for increasing value of d and decreases slightly for increasing value of d. However, even the CBBC procedure that performed the best, substantially underestimates the real variance. Reference [19] has noted that estimating the variance of the sample mean for LRD processes is substantially more difficult than the same problem for a short memory process. In practice this would cause confidence intervals to be too narrow.

Table 2 provides empirical coverage for 90% standard and bootstrap confidence intervals while their average lengths are shown in table 3. All bootstrap intervals provide improvement of the coverage probability of the standard intervals. This improvement is more notable for large values of memory parameter and for large sample sizes. The CBBC intervals provide the greatest empirical coverage in almost all cases having only slightly larger average lengths.

From all results of the simulation study we may conclude that the applying the CBBC provides is a simple effective technique for the interval estimation of the variance of the sample mean in ARFIMA(0, d, 0) models. At least in our conducted models, CBBC performs better than the other block bootstrap methods.

V. CONCLUSIONS

Estimating the variance of the sample mean is a difficult issue in presence of long memory. In this paper we have proposed a new bootstrap method for time series, the circular bootstrap with blocks composed of cycles (CBBC), which seems to be promising for replicating the dependence structure of long memory processes. The Monte Carlo experiments have shown that the CBBC is better than some existing methods. The CBBC procedure outperformed the CBB and SB methods in terms of reduction of bias of the estimates of variance of the sample mean for FI(d) processes. Also, the CBBC seems to be the best solution to build confidence intervals for the sample mean for FI(d) processes amongst the standard and bootstrap methods considered in this paper.

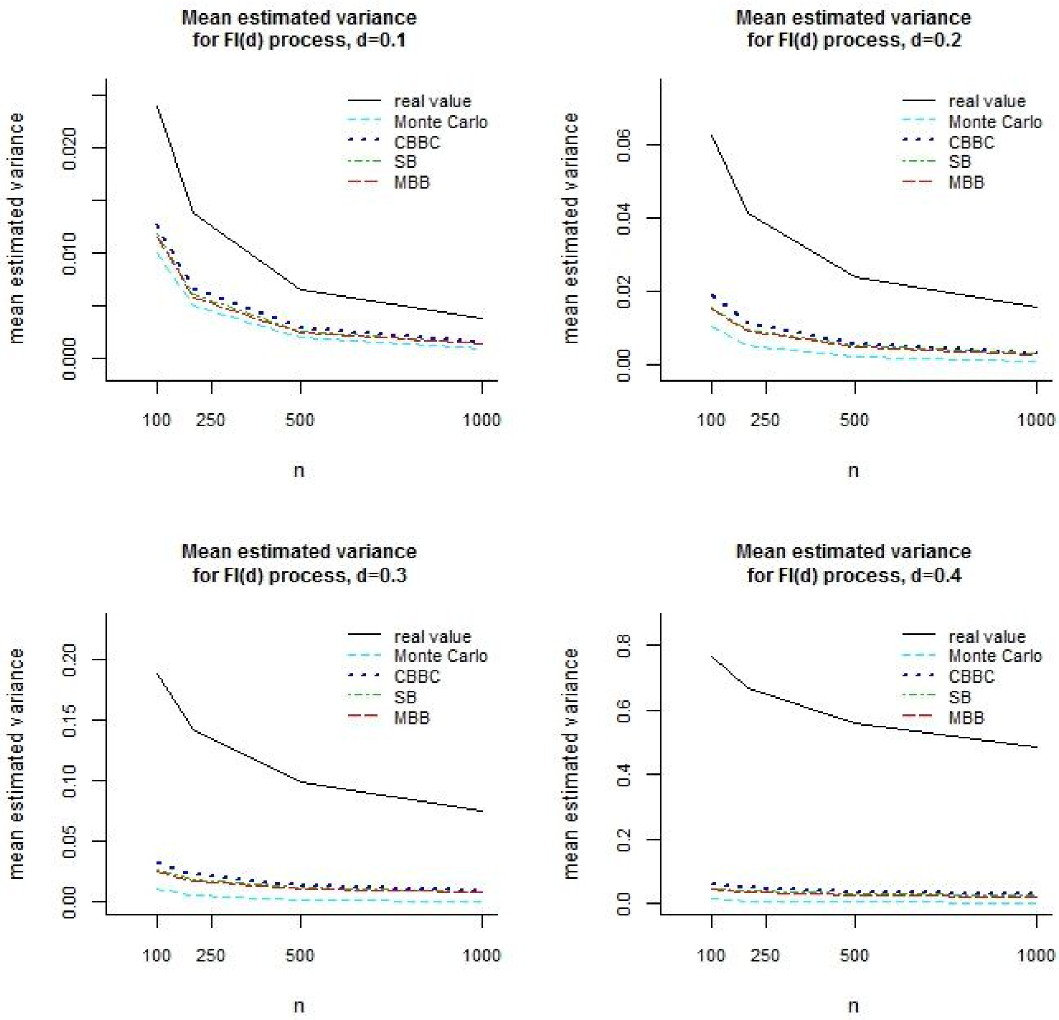

Figure 1. Monte Carlo and bootstrap estimates of the variance of the sample mean

TABLE II

EMPIRICAL COVERAGE (IN PERCENTAGE) FOR 90% CONFIDENCE INTERVALS FOR THE SAMPLE MEAN

d	*n*	Standard	CBBC	SB	MBB
0.1	100	72.1	74.3	75.3	75.0
	200	66.0	71.5	69.7	69.6
	500	64.6	72.8	70.7	69.5
	1000	63.1	73.2	71.1	70.8
0.2	100	50.2	62.4	58.2	57.6
	200	47.1	63.8	59.4	57.9
	500	36.2	57.3	54.7	53.9
	1000	35.8	53.7	52.5	50.4
0.3	100	33.5	51.7	45.4	44.3
	200	23.7	48.5	42.1	41.3
	500	20.5	49.0	43.2	42.0
	1000	17.9	45.5	41.0	39.7
0.4	100	17.8	39.5	30.2	30.3
	200	16.7	43.6	32.0	31.8
	500	10.5	41.3	26.7	26.1
	1000	9.0	42.1	27.5	25.4

TABLE III

AVERAGE INTERVAL LENGTH FOR 90% CONFIDENCE INTERVALS FOR THE MEAN

d	*n*	Standard	CBBC	SB	MBB
0.1	100	.3317	.3646	.3545	.3517
	200	.2340	.2663	.2555	.2510
	500	.1481	.1768	.1678	.1643
	1000	.1051	.1276	.1232	.1207
0.2	100	.3376	.4418	.4088	.4009
	200	.2402	.3431	.3186	.3106
	500	.1528	.2430	.2353	.2266
	1000	.1083	.1870	.1850	.1775
0.3	100	.3542	.5716	.5160	.5073
	200	.2537	.4812	.4393	.4283
	500	.1628	.3791	.3597	.3442
	1000	.1159	.3146	.3047	.2900
0.4	100	.3795	.7481	.6623	.6574
	200	.2762	.6828	.6113	.6003
	500	.1812	.6029	.5445	.5214
	1000	.1308	.5320	.4861	.4577

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