New Pathway Fractional Integral Operator Associated With Aleph - Function, Multivariable's General Class of Polynomial with H - Function

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ABSTRACT:

The aim of this paper is to a study of a pathway fractional integral operator associated with the pathway model and pathway probability density for the Aleph function with certain product of H-function and Multivariable's general class of polynomial.

KEY WORDS AND PHRASES: **Pathway Fractional integrals operator, Aleph function (χ-function), Fox'sH-function, general class of polynomials, Beta function and gamma function.**

INTRODUCTION

The fractional integral operator involving various special functions, have found Significant Importance and applications in various subfield of applicable mathematical analysis. Since last four decades, a number of workers like Mathai [1], Love [2], McBride [3],Saigo [4], etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration.

The Introduction of the Pathway Fractional integrals operator given by S.S. Nair [5] is

Let $f(x)\in L(a,b);\eta\in C,R(\eta)>0;a>0$ and let us take a "pathway parameter" $\alpha<1$.Then the pathway fractional integration operator is defined as follows

$$(P_{0+}^{(\eta,\alpha)}f)(x)=x^{\eta}\int_{0}^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}}f(t)\,dt \tag{1}$$

The pathway model is introduced by Mathai [6],[7] and discussed further by Mathai and Haubold [7],[1].For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p. d. f.):

$$f(x)=c|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} \tag{2}$$

Provide that

$-\infty<x<\infty;\delta>0;\beta\geq 0;\left[1-a(1-\alpha)|x|^{\delta}\right]>0;\gamma>0$,

where c is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as follows:

$$c=\frac{1}{2}\frac{\delta\left[a(1-\alpha)\right]^{\frac{\gamma}{\delta}}\Gamma(\frac{\gamma}{\delta}+\frac{\beta}{1-\alpha}+1)}{\Gamma(\frac{\gamma}{\delta})\Gamma(\frac{\beta}{1-\alpha}+1)}\text{, for } \alpha<1 \tag{3}$$

$$c=\frac{1}{2}\frac{\delta\left[a(1-\alpha)\right]^{\frac{\gamma}{\delta}}\Gamma(\frac{\beta}{1-\alpha})}{\Gamma(\frac{\gamma}{\delta})\Gamma(\frac{\beta}{1-\alpha}-\frac{\gamma}{\delta})}\text{ ,for }\frac{1}{\alpha-1}-\frac{\gamma}{\delta}>0,\alpha>0 \tag{4}$$

$$c = \frac{1}{2} \frac{\delta [\alpha\beta]^{\frac{\gamma}{\delta}}}{\Gamma(\frac{\gamma}{\delta})} \text{, for } \alpha \to 1 \quad (5)$$

Observe that for α<1 it is a finite range density with $\left[1 - a(1-\alpha)|x|^{\delta}\right] > 0$ and (2) remains in the extended generalized type-1 beta family. The pathway density in (3), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha > 0$, writing 1-α = - (α-1) we have

$$f(x) = c|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{\frac{-\beta}{\alpha-1}} \quad (6)$$

Provided $-\infty < x < \infty; \delta > 0; \beta \geq 0, \gamma > 0$ that, which is the extended generalized type-2 beta model for real *x*. It includes the type-2 beta density, the *F*-density, the Student-*t* density, the Cauchy density and many more.

Here we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (2) and (6) take the exponential form, since

$$\lim_{\alpha \to 1} c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} = \lim_{\alpha \to 1} c|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{\frac{-\beta}{\alpha-1}} = c|x|^{\gamma-1} e^{a\eta|x|^{\delta}} \quad (7)$$

This includes the generalized gamma, the Weibull, the chi-square, the Laplace, Maxwell-Boltzmann and other related densities.

For more details on the pathway model, the reader is referred to the recent papers of Mathai and Haubold [3], [1].

The Aleph (χ)-function, introduced by Sudland [8], however the notation and complete definition is presented here in the following manner in terms on the Mellin- Barnes type integrals

$$\chi[z] = \chi^{m,n}_{x_i, y_i, \tau_i; r}\left[z \,\middle|\, \begin{matrix} (a_j, A_j)_{1,n} \, [\tau_i(a_{ji}, A_{ji})]_{n+1, x_i, r} \\ (b_j, B_j)_{1,m} \, [\tau_i(b_{ji}, B_{ji})]_{m+1, y_i, r} \end{matrix} \right] = \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{x_i, y_i, \tau_i; r}(-s)\, z^{(s)} ds \quad (8)$$

For all $z \neq 0$ where $\omega = \sqrt{(-1)}$ and

$$\Omega^{m,n}_{x_i, y_i, \tau_i; r}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r} \tau_i \prod_{j=1+n}^{x_i} \Gamma(a_{ji} + A_{ji} s) \prod_{j=1+m}^{y_i} \Gamma(1 - b_{ji} - B_{ji} s)} \quad (9)$$

The integration path $L = L_{i\gamma\infty}, \gamma \in R$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles, assumed to be simple of $\Gamma(1 - a_j - A_j s)$, j=1,...,n do not coincide with the pole of $\Gamma(b_j + B_j s)$, j=1,...,m the parameter p_i, q_i are non-negative integers satisfying: $0 \leq n \leq x_i, 0 \leq m \leq y_i, \tau_i > 0$ for i= 1,..., r. and $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$. The empty product in (2) is interpreted as unity. The existence conditions for the defining integral (1) are giving below

$$\phi_l > 0, |\arg(z)| < \frac{\pi}{2}\phi_l, \; l = 1, ..., r \quad (10)$$

$$\phi_l \geq 0, |\arg(z)| < \frac{\pi}{2}\phi_l \text{ and } R\{\xi_l\} < 0 \quad (11)$$

Where

$$\phi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l \left(\sum_{j=n+1}^{x_l} A_{jl} + \sum_{j=m+1}^{y_l} B_{jl} \right) \quad (12)$$

$$\xi_l = \sum_{j=1}^{n} b_j + \sum_{j=1}^{m} a_j + \tau_l \left(\sum_{j=n+1}^{x_l} b_{jl} + \sum_{j=m+1}^{y_l} a_{jl} \right) + \frac{1}{2}(x_l - y_l), l = 1, ..., r \quad (13)$$

For detailed account of Aleph (χ)-function see [8] and [9].The general polynomials of R variables given by Srivasthava [10] defined and represented as:

$$S^{m_1, ..., m_R}_{n_1, ..., n_R}\left[x_1, ..., x_R\right] =$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \prod_{i=1}^{R} \left\{ \frac{(-n_i)_{m_i s_i}}{\lfloor s_i} x^{s_i} \right\} A(n_1, s_1; \ldots; n_R s_R) \tag{14}$$

Where $n_i, m_i = 1, \ldots; m_i \neq 0, \forall i \in 1, 2, \ldots, R$ the coefficient $A(n_1, s_1; \ldots; n_R s_R), (s_i \geq 0)$ are arbitrary constant, real or complex. The general class of polynomials [10] is capable of reducing to a number of familiar multivariable polynomials by suitable specializing the arbitrary coefficients $A(n_1, s_1; \ldots; n_R s_R), (s_i \geq 0)$.

Fox H-function in series representation is given in [11], [12] is as follows:

$$H_{P,Q}^{M,N}[Z] = H_{P,Q}^{M,N}\left[z \middle| \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right] =$$

$$\sum_{h=1}^{N} \sum_{\upsilon=0}^{\infty} \frac{(-1)^{\upsilon} X(\xi)}{\Gamma(\upsilon+1) E_h} \left(\frac{1}{z} \right)^{\xi} \tag{15}$$

Where $\xi = \dfrac{(e_h - 1 - h)}{E_h}$ and $h = 1, 2, \ldots, N$

And

$$X(\xi) = \frac{\{\prod_{j=1}^{M} \Gamma(f_j + F_j \xi)\}\{\prod_{j=1}^{N} \Gamma(1 - e_j - E_j \xi)\}}{\{\prod_{j=M+1}^{Q} \Gamma(1 - f_j - F_j \xi)\}\{\prod_{j=N+1}^{P} \Gamma(e_j + E_j \xi)\}}$$

Theorem (1):

With the set of sufficient conditions (10), (11), (12) and (13), let (η, u,u_1,…,u_R, ϵ C), $R(\delta) > 0$, $\mathrm{Re}\left(1 + \frac{\eta}{(1-\alpha)}\right) > 0$, $\mathrm{Re}\left(\eta, u, u_1, \ldots, u_R, \beta\right) > 0$ and m_i is an arbitrary positive integral and coefficients $\left(n_1, s_1; \ldots; u_R s_R\right)$ are arbitrary constant ,real or complex.

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}\left[x^{u_i}, \ldots, x^{u_R} \right] \chi_{p_i,q_i,\tau_i,r}^{m,n}\left[dx^{\beta} \right] H_{P,Q}^{M,N}\left[cx^{\lambda} \middle| \begin{matrix} (e_p, E_P) \\ (f_p, F_P) \end{matrix} \right] \right]$$

$$= \frac{x^{\eta+u+u_i s_i+\ldots+u_R s_R}}{[a(1-\alpha)]^{u+u_i s_i+\ldots+u_R s_R}} \Gamma\left(\frac{\eta}{1-\alpha} + 1 \right)$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \prod_{i=1}^{R} \left\{ \frac{(-n_i)_{m_i s_i}}{\lfloor s_i} x^{s_i} \right\} A(n_1, s_1; \ldots; n_R s_R)$$

$$H_{P,Q}^{M,N}\left[\frac{cx^{\lambda}}{[a(1-\alpha)]^{\lambda}} \middle| \begin{matrix} (e_p, E_P) \\ (f_p, F_P) \end{matrix} \right]$$

$$\chi_{p_i+1,q_i+1,\tau_i,r}^{m,n+1}\left[\frac{dx^{\beta}}{[a(1-\alpha)]^{\beta}} \middle| \begin{matrix} (1-u-u_i s_i-\ldots-u_R s_R-\lambda\xi, \beta), (a_j, A_j)_{1,n}, \ldots, \left[\tau_i(a_j, A_j)\right]_{n+1,p_i} \\ (b_j, B_j)_{1,m}, \ldots, \left[\tau_i(b_j, B_j)\right]_{m+1,q_i}, \left(-\frac{\eta}{(1-\alpha)} - u - u_i s_i - \ldots - u_R s_R + \lambda\xi, \beta\right) \end{matrix} \right] \tag{16}$$

Proof:

Using the definitions (1), (8), (14) and (15) then by interchange the order of integrations and summations (which is permissible under the conditions stated above), evaluate inner integral by making use of beta and gamma function formula, we arrive at the desired results.

Special cases:

1. If we have putting $\tau_1 = \tau_2 = \ldots = \tau_r = 1$ in equation (16) then Aleph –function reduce to

I-function [13] .

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}\left[x^{u_i}, \ldots, x^{u_R} \right] \chi_{p_i,q_i,1,r}^{m,n}\left[dx^{\beta} \right] H_{P,Q}^{M,N}\left[cx^{\lambda} \middle| \begin{matrix} (e_p, E_P) \\ (f_p, F_P) \end{matrix} \right] \right]$$

$$= \frac{x^{\eta+u+u_i s_i+\ldots+u_R s_R}}{[a(1-\alpha)]^{u+u_i s_i+\ldots+u_R s_R}} \Gamma\left(\frac{\eta}{1-\alpha} + 1 \right)$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \prod_{i=1}^{R} \left\{ \frac{(-n_i)_{m_i s_i}}{\lfloor s_i} x^{s_i} \right\} A(n_1, s_1; \ldots; n_R s_R)$$

$$H_{P,Q}^{M,N}\left[\frac{cx^{\lambda}}{[a(1-\alpha)]^{\lambda}} \left| \begin{matrix} (e_p, E_P) \\ (f_p, F_P) \end{matrix} \right. \right]$$

$$I_{p_i+1, q_i+1, r}^{m, n+1}\left[\frac{dx^{\beta}}{[a(1-\alpha)]^{\beta}} \left| \begin{matrix} (1-u-u_i s_i - \ldots - u_R s_R - \lambda\xi, \beta), (a_j, A_j)_{1,n}, \ldots, [(a_j, A_j)]_{n+1, p_i} \\ (b_j, B_j)_{1,m}, \ldots, [(b_j, B_j)]_{m+1, q_i}, (-\frac{\eta}{(1-\alpha)} - u - u_i s_i - \ldots - u_R s_R + \lambda\xi, \beta) \end{matrix} \right. \right]$$

(17)

2 If we choosing

$\tau_1 = \tau_2 = \ldots = \tau_r = 1$ and r=1 in equation (16) then Aleph –function reduce to H-function[9]

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}\left[x^{u_i}, \ldots, x^{u_R} \right] \chi_{p_i, q_i, 1, 1}^{m,n}\left[dx^{\beta} \right] H_{P,Q}^{M,N}\left[cx^{\lambda} \left| \begin{matrix} (e_p, E_P) \\ (f_p, F_P) \end{matrix} \right. \right] \right]$$

$$= \frac{x^{\eta+u+u_i s_i + \ldots + u_R s_R}}{[a(1-\alpha)]^{u+u_i s_i + \ldots + u_R s_R}} \Gamma\left(\frac{\eta}{1-\alpha} + 1 \right)$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \prod_{i=1}^{R} \left\{ \frac{(-n_i)_{m_i s_i}}{\lfloor s_i} x^{s_i} \right\} A(n_1, s_1; \ldots; n_R s_R)$$

$$H_{P,Q}^{M,N}\left[\frac{cx^{\lambda}}{[a(1-\alpha)]^{\lambda}} \left| \begin{matrix} (e_p, E_P) \\ (f_p, F_P) \end{matrix} \right. \right]$$

$$H_{p+1, q+1}^{m, n+1}\left[\frac{dx^{\beta}}{[a(1-\alpha)]^{\beta}} \left| \begin{matrix} (1-u-u_i s_i - \ldots - u_R s_R - \lambda\xi, \beta), (a_p, A_p) \\ (b_q, B_q), (-\frac{\eta}{(1-\alpha)} - u - u_i s_i - \ldots - u_R s_R + \lambda\xi, \beta) \end{matrix} \right. \right]$$

(18)

3 If we choosing $M = N = P = Q = 1$ and $\lambda = 1$ In equation (16) then we get equation (2.8) in paper [9].

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}\left[x^{u_i}, \ldots, x^{u_R} \right] \chi_{p_i, q_i, \tau_i, r}^{m,n}\left[dx^{\beta} \right] \right]$$

$$= \frac{x^{\eta+u+u_i s_i + \ldots + u_R s_R}}{[a(1-\alpha)]^{u+u_i s_i + \ldots + u_R s_R}} \Gamma\left(\frac{\eta}{1-\alpha} + 1 \right)$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \prod_{i=1}^{R} \left\{ \frac{(-n_i)_{m_i s_i}}{\lfloor s_i} x^{s_i} \right\} A(n_1, s_1; \ldots; n_R s_R)$$

$$\chi_{p_i+1, q_i+1, \tau_i, r}^{m, n+1}\left[\frac{dx^{\beta}}{[a(1-\alpha)]^{\beta}} \left| \begin{matrix} (1-u-u_i s_i - \ldots - u_R s_R; \beta), (a_j, A_j)_{1,n}, \ldots, [\tau_i (a_j, A_j)]_{n+1, p_i} \\ (b_j, B_j)_{1,m}, \ldots, [\tau_i (b_j, B_j)]_{m+1, q_i}, (-\frac{\eta}{(1-\alpha)} - u - u_i s_i - \ldots - u_R s_R; \beta) \end{matrix} \right. \right]$$

(19)

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