A Mathematical Model to study the effect of Renewal and Reversion of Inactive Christians on Church Growth

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Abstract

This paper examined the effects of reversion and renewal of inactive Christian on Church growth. A mathematical model for the problem was proposed and transformed into proportions in order to reduce model equations for easy analysis. Using the next-generation method, the basic reproduction number R_0 was computed in terms of the parameters of the reduced model equations. The Faith free equilibrium was obtained and found to be locally and globally asymptotically stable when $R_0 < 1$. Using the centre manifold theory approach, the Faithful equilibrium was showed to be locally asymptotically stable when $R_0 < 1$. Numerical simulation of the model was carried out to assess the effects of the effects of reversion and renewal of inactive Christian on Church growth. The result showed that regular revival, retreat and social welfare support programme for less privilege will help the church to grow.

Key words: renewal, reversion, the basic reproduction number, backward bifurcation, Church Growth, Inactive Christian.

INTRODUCTION

Church growth started in and around Judea over 2000 years ago and has exceeded one billion people, 28% of the world population [1].

The pioneer in the field of mathematical modeling of Church growth was Donald Mc Gavran [2]. He stated that church growth is a subject area that seeks to analyze why Christian churches at various levels of organization grow or decline. Until recently, Church growth studies have been confined to qualitative aspects of growth and the factors that help or hinder it [3].

However, an important part of this analysis is quantitative, which involves measuring the behavioral changes of Church growth over a period of time. Many researchers have studied this behavioral changes mathematically with constant population ([4],[5]) as well as of varying size ([6],[7],[8], [9]). The Mathematical model developed by [9] is divided into two populations, passive and active Christian populations. In their model, the passive Christians are not involve in evangelism and those who revert from active Christians become non-Christian immediately. The result shows that effectiveness of active Christians combined with a greater rate at which passive Christians become active Christians help the Church to grow.

Based on the work by [9], we propose a mathematical model to study the combine effect of reversion of inactive Christians to non-Christian and renewal of inactive Christians to active Christian in a varying size population.

MODEL FORMULATION OF CHURCH GROWTH

Let $N(t)$ be the total population of size at time t. The population $N(t)$ is divided into four classes, which are Non-Christian $S(t)$, Passive Christians $P(t)$, Active Christians $A(t)$ and Inactive Christians $I(t)$. Non-Christians $S(t)$ are those who belong to other Religions like Muslims and the atheisms. Passive Christian are the new converts into Christianity, who are yet to know full doctrine of the new faith in Christ and are involve in evangelization because of their new found Christ. Evangelization helps them to strengthen their faith. Active Christian are those fully in the church and know the doctrine of the church and involved in evangelization. Inactive Christian are those who are fully in the church and know the doctrine of the church but do not involve in evangelization due to incapacitation, redeployment, change of religion or loss of faith (backsliding).

We assume that Non-Christian are recruited at the rate bN with b as the birthrate that is birth of new born or immigrant in the population. They become Christian through contacts with the passive and active Christians $P(t)$ and $A(t)$ through evangelization either as a group or personal contact with contact rates as β_1 and β_2 respectively. It is assumed that $\beta_2 > \beta_1$ and all classes $S(t), P(t), A(t)$ and $I(t)$ are with natural death rate µ. The Passive Christian $P(t)$ progress to active Christian $A(t)$ after going through doctrine classes or induction class at the rate φ while some $A(t)$ progress to $I(t)$ class with the rate σ, this may be due to redeployment, loss of faith or incapacitation, which they need re-evangelization and revival to return to $A(t)$ at the rate γ known as the renewal rate. We assumed that inactive Christian may decide to return to Non-Christian class like Christian changing to another religion like Muslim at the rate λ known as the reversion rate.

We have the flow chart of the model as
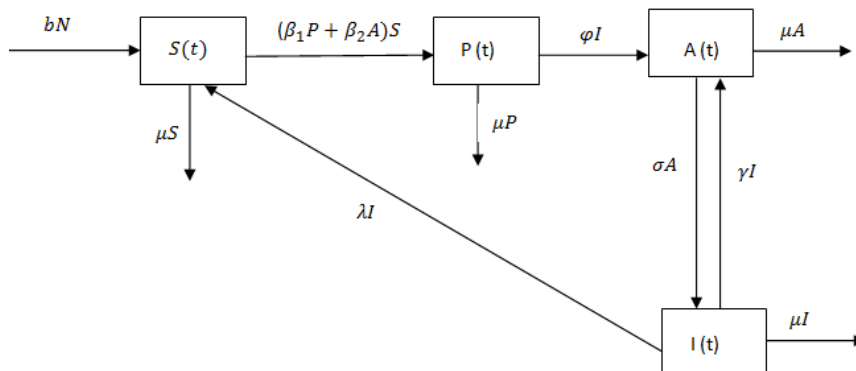
In view of the above flow chart and the assumption, the governing system of differentiate equation for the church growth will be

$$\frac{dS}{dt} = bN - \frac{(\beta_1 P + \beta_2 A)S}{N} - \mu S + \lambda I \qquad (1)$$

$$\frac{dP}{dt} = \frac{(\beta_1 P + \beta_2 A)S}{N} - \mu P - \varphi P \tag{2}$$

$$\frac{dA}{dt} = \varphi P - \sigma A - \mu A + \gamma I \tag{3}$$

$$\frac{dI}{dt} = \sigma A - (\mu + \lambda + \gamma)I \tag{4}$$

with initial conditions

$$S(0) = S_0, \ P(0) = P_0, \ A(0) = A_0 \ and \ I(0) = I_0 \ .$$

The total population N(t) is given as

$$N(t) = S(t) + P(t) + A(t) + I(t)$$

which implies that

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dP}{dt} + \frac{dA}{dt} + \frac{dI}{dt}$$

or $\quad \frac{dN}{dt} = bN - \mu N$

Now, in the above system (1-4), we use the following transformation:

$$s = \frac{S}{N} \ , \ p = \frac{P}{N} \ , \ a = \frac{A}{N} \quad and \ i = \frac{I}{N}$$

to get the following normalised system:

$$\frac{ds}{dt} = b - (\beta_1 p + \beta_2 a)s + \lambda i - bs \tag{5}$$

$$\frac{dp}{dt} = (\beta_1 p + \beta_2 a)s - \varphi p - bp \tag{6}$$

$$\frac{da}{dt} = \varphi p - \sigma a - ba + \gamma i \tag{7}$$

$$\frac{di}{dt} = \sigma a - \lambda i - \gamma i - bi \tag{8}$$

with the initial conditions

$$s(0) = s_0, \ p(0) = p_0, \ a(0) = a_0 \ and \ i(0) = i_0 \ .$$

such that

$$s + p + a + i = 1 \ .$$

POSITIVITY OF SOLUTIONS

For the model of the study of Christianity to be epidemiological meaningful and well posed, we need to say that all state variables for the model (5) – (8) are non-negative.

Theorem1.0: Let $\Omega = \{(s, p, a, i) \in \mathbb{R}_4^+ : s + p + a + i = 1\}$ then the solution $\{s(t), p(t), a(t), i(t)\}$ of the system (5) – (8) are all nonnegative.

Proof

Using equation (5) for $\frac{ds}{dt}$, we have

$\frac{ds}{dt} \leq b - bs$ or $\frac{ds}{dt} + bs \leq b$

By the integrating factor, we have

$s(t) \leq 1 + ce^{-bt}$

It is clear that , at $t = 0$

$s(0) \leq 1 + c$ or

$s(0) - 1 \leq c$

implying that

$s(t) \leq 1 + (s(0) - 1)e^{-bt}$

as $t \to \infty$, $s(t) \leq 1$

Therefore

$0 \leq s(t) \leq 1$

For $\frac{dp}{dt}$, we have

$\frac{dp}{dt} \geq -(\varphi + b)p$

or

$\frac{dp}{p} \geq -(\varphi + b)dt$

Integrating both sides, we have

$p(t) \geq Ae^{-(\varphi+b)t}$

Applying the initial condition $p(0) = p_0$, we obtain

$p(t) \geq p_0 e^{-(\varphi+b)t}$

and $p(t) \geq 0$ as $t \to \infty$.

Similarly,

$\frac{da}{dt} \geq -(\sigma + b)a$

from which we obtain

$$a(t) \geq a(0)e^{-(\sigma+b)t}$$

and

$a(t) \geq 0$ as t approaches infinity.

Furthermore,

$$\frac{di}{dt} \geq -(\lambda + \gamma + b)i$$

or

$$i(t) \geq i_0 e^{-(\lambda+\gamma+b)t},$$

which gives

$i(t) \geq 0$ as $t \to \infty$.

Clearly, this proves the above result in theorem 1.0 .

Since $s + p + a + i = 1$, we have

$s = 1 - p - a - i$

In order to reduce the equation (5)- (8) to

$$\frac{dp}{dt} = (\beta_1 p + \beta_2 a)(1 - p - a - i) - \varphi p - bp \tag{9}$$

$$\frac{da}{dt} = \varphi p - \sigma a - ba + \gamma i \tag{10}$$

$$\frac{di}{dt} = \sigma a - \lambda i - \gamma i - bi \tag{11}$$

FAITH – FREE EQUILIBRIUM

The Faith-free equilibrium (FFE) is the equilibrium when there is no Christian in the population.

At equilibrium point, $\frac{dp}{dt} = \frac{da}{dt} = \frac{di}{dt} = 0$, we have the system of equations (9)-(11) to be solved for equilibrium points.

Thus, the Faith-Free Equilibrium

$E_0 = (0,0,0)$

STABILITY OF FFE

We shall compute the basic reproduction number R_0 using the next generation method. In the terms of church growth, Basic reproduction number R_0 is define as the average number of Secondary

Christians produced when one Christian is introduced in a host population where everyone is non-Christian such that when $R_0 > 1$, the church will grow. By the method of next – generation method, Diekmann *et al.* [10] defined R_0 as the spectral radius (i.e. the dominant eigenvalue) of the next generation matrix. It is given as:

$$R_0 = \rho(GU^{-1})$$

Where $\rho(GU^{-1})$ is the spectral radius of the matrix GU^{-1}.

$$GU^{-1} = \left[\frac{\partial F_i(x_0)}{\partial x_j}\right]\left[\frac{\partial V_i(x_0)}{\partial x_j}\right]^{-1}$$

F_i is the rate of appearance of new Christian in compartment i

V_i is the transfer of individuals in and out of compartment i by another apart from new converts.

x_0 is the faith-free equilibrium.

Using (9) – (11), we have $F, V, G, U,$ and GU^{-1} given as

$$F = \begin{bmatrix} (\beta_1 p + \beta_2 a)(1 - p - a - i) \\ 0 \\ 0 \end{bmatrix}, \quad V = \begin{bmatrix} \varphi p + bp \\ -\varphi p + \sigma a + ba - \gamma i \\ -\sigma a + \lambda i + \gamma i + bi \end{bmatrix}$$

$$G = \begin{bmatrix} \frac{\partial F_1(x_0)}{\partial p} & \frac{\partial F_1(x_0)}{\partial a} & \frac{\partial F_1(x_0)}{\partial i} \\ \frac{\partial F_2(x_0)}{\partial p} & \frac{\partial F_2(x_0)}{\partial a} & \frac{\partial F_2(x_0)}{\partial i} \\ \frac{\partial F_3(x_0)}{\partial p} & \frac{\partial F_3(x_0)}{\partial a} & \frac{\partial F_3(x_0)}{\partial i} \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{\partial V_1(x_0)}{\partial p} & \frac{\partial V_1(x_0)}{\partial a} & \frac{\partial V_1(x_0)}{\partial i} \\ \frac{\partial V_2(x_0)}{\partial p} & \frac{\partial V_2(x_0)}{\partial a} & \frac{\partial V_2(x_0)}{\partial i} \\ \frac{\partial V_3(x_0)}{\partial p} & \frac{\partial V_3(x_0)}{\partial a} & \frac{\partial V_3(x_0)}{\partial i} \end{bmatrix} = \begin{bmatrix} \varphi + b & 0 & 0 \\ -\varphi & \sigma + b & -\gamma \\ 0 & -\sigma & (\lambda + \gamma + b) \end{bmatrix}$$

and

$$GU^{-1} = \begin{bmatrix} \frac{\beta_1 h + \beta_2 \varphi(\lambda + \gamma + b)}{(\varphi + b)h} & \frac{\beta_1(\lambda + \gamma + b)}{h} & \frac{\beta_2}{h} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$h = (\sigma + b)(\lambda + \gamma + b) - \sigma\gamma \qquad\qquad (*)$$

We find the eigenvalue of GU^{-1} as

$$|GU^{-1} - \lambda I| = 0$$

This gives

$$\lambda_1 = \lambda_2 = 0 \ \text{ or } \ \lambda_3 = \frac{\beta_1 h + \beta_2 \varphi(\lambda+\gamma+b)}{(\varphi+b)h}$$

Thus, the spectral radius of GU^{-1} is given by

$R_0 = max[|\lambda_1|, |\lambda_2|, |\lambda_3|]$, this implies

$$R_0 = \frac{\beta_1 h + \beta_2 \varphi(\lambda+\gamma+b)}{(\varphi+b)h} \qquad \text{or} \qquad R_0 = \frac{\beta_1(\sigma+b)\big((\sigma+b)(\lambda+\gamma+b)-\sigma\gamma\big) + \beta_2 \varphi(\lambda+\gamma+b)}{(\varphi+b)\big((\sigma+b)(\lambda+\gamma+b)-\sigma\gamma\big)}$$

LOCAL STABILITY OF THE FAITH-FREE EQUILIBRIUM

Theorem 2: The Faith-Free Equilibrium of the equations (9)-(11) is locally asymptotically stable if $R_0 < 1$ and unstable $R_0 > 1$.

The theorem 2 is prove using linearization method, the Jacobian matrix associated with the system (9) – (11) at the FFE $E_0 = (0,0,0)$ is

$$J(E_0) = \begin{bmatrix} \beta_1 - \varphi - b & \beta_2 & 0 \\ \varphi & -(\sigma + b) & \gamma \\ 0 & \sigma & -(\lambda + \gamma + b) \end{bmatrix} \qquad (12)$$

and the characteristics equation corresponding to $J(E_0)$ is given by

$$\rho(\lambda) = \lambda^3 + (d - a + c)\lambda^2 - (ac + ad - cd + \sigma\gamma + \beta_2 \varphi)\lambda - (acd - \sigma\gamma a + \beta_2 \varphi d) = 0$$

$$\rho(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C = 0$$

where

$$A = (d - a + c)$$

$$B = -(ac + ad - cd + \sigma\gamma + \beta_2 \varphi)$$

$$C = -(acd - \sigma\gamma a + \beta_2 \varphi d) \text{ and } a = \beta_1 - \varphi - b, c = (\sigma + b),$$

$$d = (\lambda + \gamma + b)$$

Using Routh–Hurwitz criteria, E_0 is locally asymptotically stable if $A > 0, B > 0, C > 0, and\ AB > C$.

We have

$$C = -(acd - \sigma\gamma a + \beta_2 \varphi d) > 0$$

This implies that

$$(acd - \sigma\gamma a + \beta_2 \varphi d) < 0$$

and obtain

$$\frac{\beta_1(\sigma+b)\big((\sigma+b)(\lambda+\gamma+b)-\sigma\gamma\big)+\beta_2\varphi(\lambda+\gamma+b)}{(\varphi+b)\big((\sigma+b)(\lambda+\gamma+b)-\sigma\gamma\big)} < 1$$

Therefore $R_0 < 1$. This proofs the theorem 2.

GLOBAL STABILITY OF THE DISEASE – FREE EQUILIBRIUM

Using the approach by Castillo – Chavez *et al.* [11], the system of equations (5)-(8) can be rewritten as

$$\frac{dI}{dt} = P(I), \qquad P(0) = 0 \tag{**}$$

where $I \in \mathbb{R}^3 = (p, a, i)$ denotes the proportion of Christians (passive, active and inactive) and $E_0 = (0,0,0)$ as FFE of this system.

The condition for global stability for E_0 is given by

$$P(I) = WI - \tilde{P}(I), \quad \tilde{P}(I) \geq 0 \quad for \quad I \in \Omega \tag{***}$$

where $W = D_I P(0)$ is an M-matrix (i.e. the off diagonal elements of W are nonnegative) and Ω is the region where the system of equations of the model makes epidemiological meaningful.

If the system (5)-(8) satisfies the above condition then the following theorem holds:

Theorem 3: The faith-free equilibrium $E_0 = (0,0,0)$ is globally asymptotically stable if $R_0 < 1$ and that condition (***) is satisfied.

From (**) and (***), we have

$$\tilde{P}(I) = (G - U)I - \frac{dI}{dt}, \quad \text{where } W = (G - U)$$

This gives

$$\tilde{P}(I) = \begin{bmatrix} (\beta_1 p + \beta_2 a)(p + a + i) \\ 0 \\ 0 \end{bmatrix}$$

By theorem 1, $\tilde{P}(I) \geq 0$ since $p(t), a(t), i(t)$ are all nonnegative. This implies that $E_0 = (0,0,0)$ is globally asymptotically stable for $R_0 < 1$.

Local Stability of Faithful Equilibrium

The local stability of Faithful equilibrium is determined by finding the eigenvalues of the Jacobian Matrix evaluated at the Faithful equilibrium. Sometimes, this approach can be mathematically complicated. Here, we recourse to the approach of centre manifold theory described by Chavez and Song [12] to investigate the stability of Faithful equilibrium. Centre Manifold theory is used to investigate the existence of backward and forward bifurcation at $R_0 = 1$ ([13],[14],[15]).When the

bifurcation is forward, it implies that Faith free equilibrium is locally asymptotically stable for $R_0 < 1$ and there is no Christian in the population and also Faithful equilibrium is locally asymptotically stable for $R_0 > 1$.Backward bifurcation occurs when the Faithful equilibrium exists for $R_0 < 1$ and Faith free equilibrium may exists when $R_0 > 1$.

Theorem 4: Centre manifold theory

Consider a general system of ODEs with the parameter β:

$$\frac{dx}{dt} = F(x, \beta) \tag{13}$$

$f: R \to R^n$ and $f \in C^2(R^2 \ x \ R)$

Where 0 is an equilibrium point for the system (9-11) for all values of the parameter β, that is $f(0, \beta) \equiv 0$ for all β and

$$A = D_*f(0,0) = \left[\frac{df_i}{dx_j}(0,0)\right]$$

is the linearization point 0 with β evaluate at 0. Zero is a simple eigenvalue of A and all other eigenvalues of A have negative real parts.

Matrix A has a non negative right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue.

Let f_k be the kth component of f and

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0)$$

$$b = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta}(0,0)$$

Then the local dynamics of the system (9 - 11) around the equilibrium point 0 is totally determined by the signs of a and b.

$a > 0$, $b > 0$ when $\beta < 0$ with $|\beta| \ll 1$, 0 is locally asymptotically stable and there exists a positive unstable equilibrium; when $0 < \beta \ll 1$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium.

$a < 0$, $b < 0$, with $|\beta| \ll 1$, 0 unstable; when $0 < \beta \ll 1$, asymptotically stable, and there exists a positive unstable equilibrium;

$a > 0$, $b < 0$, with $|\beta| \ll 1$, 0 unstable; and there exists a locally asymptotically stable negative equilibrium; when $0 < \beta \ll 1$, 0 is stable and a positive unstable equilibrium appears;

$a > 0$, $b < 0$, when β changes from negative equilibrium to positive, 0 changes its stability from stable to unstable, corresponding to a negative equilibrium becomes positive and locally asymptotically stable.

Particularly, if $a > 0$ and $b > 0$, then a subcritical (or backward) bifurcation occurs at $\beta = 0$.

Applying the theorem 3 involves the following changes of variables; Let

$$s = x_1, \quad a = x_2, \quad i = x_3,$$

with

$$x_1 + x_2 + x_3 < 1.$$

Let $X = (x_1, x_2, x_3)^T$ be the vector written so that the model can be re-written in the form $\frac{dX}{dt} = F(x)$, where $F = (f_1, f_2, f_3)^T$ as follows

$$\frac{dx_1}{dt} = f_1(x) = (\beta_1 x_1 + \beta_2 x_2)(1 - x_1 - x_2 - x_3) - \varphi x_1 - \gamma x_1 \tag{14}$$

$$\frac{dx_2}{dt} = f_2(x) = \varphi x_2 - b x_2 + \gamma x_3 - \sigma x_2 \tag{16}$$

$$\frac{dx_3}{dt} = f_3(x) = \sigma x_2 - \gamma x_3 - \lambda x_3 - b x_3 \tag{17}$$

The Jacobian matrix of the equations $(9) - (11)$ at the faith-free equilibrium $J(E_0)$ is defined in equation (12). Taking $\beta_1 = \beta$ and $\beta_2 = r\beta$, where β is chose as the bifurcation parameter and the bifurcation occurs at $R_0 = 1$, we consider the case $R_0 = 1$ and solve for the bifurcation parameter β.

We have

$$R_0 = \frac{\beta_1 h + \beta_2 \varphi(\lambda + \gamma + b)}{(\varphi + b)h} = 1$$

or

$$\frac{\beta_1 h + \beta_2 \varphi(\lambda + \gamma + b)}{(\varphi + b)h} = 1$$

from which we obtain

$$\beta = \frac{(\varphi + b)h}{h + r\varphi(\lambda + \gamma + b)}$$

where h is as defined in (*) in precious section.

The Linearized system of the transformed system $(9) - (11)$ with $\beta_1 = \beta$ and $\beta_2 = r\beta$ has a simple zero eigenvalue. Hence, we analyze the system at $\beta_1 = \beta$ and $\beta_2 = r\beta$ using the Centre Manifold theory. The Jacobian matrix of $(9) - (11)$ has right eigenvector associated with the zero eigenvalue as

$$\begin{bmatrix} \beta_1 - (\varphi + b) & \beta_2 & 0 \\ \varphi & -(\sigma + b) & \gamma \\ 0 & \sigma & -(\lambda + \gamma + b) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

where $w = (w_1, w_2, w_3)^T$ is the right eigenvetor.

Evaluating the system in (18) gives

$$w_1 = \frac{\beta_2}{(\varphi + b - \beta_1)} w_2 \qquad ; \quad w_3 = \frac{\sigma}{(\lambda + \gamma + b)} w_2$$

for which $w_2 > 0$.

The left eigenvector of the Jacobian $J(E_0)$ associated with the zero eigenvalue is given by $V = (v_1, v_2, v_3)^T$. Transposing Jacobian $J(E_0)$ first and multiply by V. We have

$$\begin{bmatrix} \beta_1 - (\varphi + b) & \varphi & 0 \\ \beta_2 & -(\sigma + b) & \sigma \\ 0 & \gamma & -(\lambda + \gamma + b) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

from which we get

$$v_1 = \frac{\varphi}{(\varphi + b - \beta_1)} v_2 \qquad ; \quad v_3 = \frac{\gamma}{(\lambda + \gamma + b)} v_2$$

for which $v_2 > 0$.

Computations of a and b

From the system (15-17), the associated non-zero partial derivative of F at FFE are given by

$$\frac{\partial^2 f_1}{\partial x_1^2} = -2\beta_1, \quad \frac{\partial^2 f_1}{\partial x_1 \partial x_2} = -\beta_2 - \beta_1, \quad \frac{\partial^2 f_1}{\partial x_2^2} = -2\beta_2, \quad \frac{\partial^2 f_1}{\partial x_1 \partial x_3} = -\beta_1,$$

$$\frac{\partial^2 f_1}{\partial x_2 \partial x_3} = -\beta_2, \quad \frac{\partial^2 f_1}{\partial x_1 \partial \beta} = 1, \quad \frac{\partial^2 f_1}{\partial x_2 \partial \beta} = r \qquad \qquad \text{with} \quad r > 0$$

It follows that

$$a = v_1 \left[w_1^2 \frac{\partial^2 f_1}{\partial x_1^2}(0,0) + w_1 w_2 \frac{\partial^2 f_1}{\partial x_1 \partial x_2}(0,0) + w_2^2 \frac{\partial^2 f_1}{\partial x_2^2} + w_1 w_3 \frac{\partial^2 f_1}{\partial x_1 \partial x_3} + w_2 w_3 \frac{\partial^2 f_1}{\partial x_2 \partial x_3} \right]$$

and

$$b = v_1 \left[w_1 \frac{\partial^2 f_1}{\partial x_1 \partial \beta}(0,0) + w_2 \frac{\partial^2 f_1}{\partial x_1 \partial \beta}(0,0) \right]$$

Taking $w_2 = 1$, $v_2 = 1$ as free eigenvectors and substituting $\beta_1 = \beta$, $\beta_2 = r\beta$, we have

$$a = -v_1 [2w_1^2 \beta_1 + w_1(\beta_2 + \beta_1) + 2\beta_2 + w_1 w_3 \beta_1 + w_3 \beta_2]$$

and

$b = v_1[w_1 + w_3 r]$.

If $\varphi + b > \beta_1$, then a forward bifurcation exists. This implies the following theorem.

Theorem 5: The Faithful equilibrium E_1 is locally asymptotically stable for $R_0 > 1$ when R_0 is close to one provided $\varphi + b > \beta_1$.

 Numerical Result

To examine the dynamics of the model numerically, the system is solved using the fourth-order Runge-Kutta method with the following values for the parameters $b = 0.04$, $\beta_1 = 0.35$, $\beta_2 = 0.7$, $\sigma = 0.30$, $\varphi = 0.30$, and initial conditions $p(0) = 0.30$, $a(0) = 0.15$, $i(0) = 0.05$ for the period of 30 years. The results are displayed graphically in figures $2(b) - 3(c)$. Figures $2(b) - 2(d)$ show the effect of the renewal rate of inactive Christian to active Christian into the population. As the inactive Christian decreases with increase in number of active Christian, the renewal rate increases. This simply means that a proportion of inactive Christian is becoming active Christian through revival, re-evangelism by some active members, social welfare support programme etc.

 Figures $3(a) - 3(c)$ show the effect of reversion rate of inactive Christian to non-Christian in the population. As the reversion rate of inactive Christian to non- Christian increases, the inactive Christian decreases likewise, this leads to a decrease in the proportion of active Christian. This increase in the reversion rate of inactive Christian will decline the growth of church.

Therefore, in order to increase the growth of the church, regular revival and retreat should be organize for members and also social welfare support programme to be put in place for those less privilege in the population.

CONCLUSION

A mathematical model of Church growth is proposed in order to study the effect of renewal of inactive Christian to active Christian and reversion of inactive Christian to non- Christian. The model is investigated to exhibit local and global asympotic stabilty at FFE provided $R_0 < 1$. Using the centre manifold theory, the Faithful equilibrium is showed be locally asympoptically stable when $R_0 > 1$ based on certain condition. Results from numerical result show that regular revival, retreat and social welfare support programme for less privilege will help the church to grow. We therefore suggest that concert efforts must be made to educate Christian on the need for renewal and the danger of reversion.
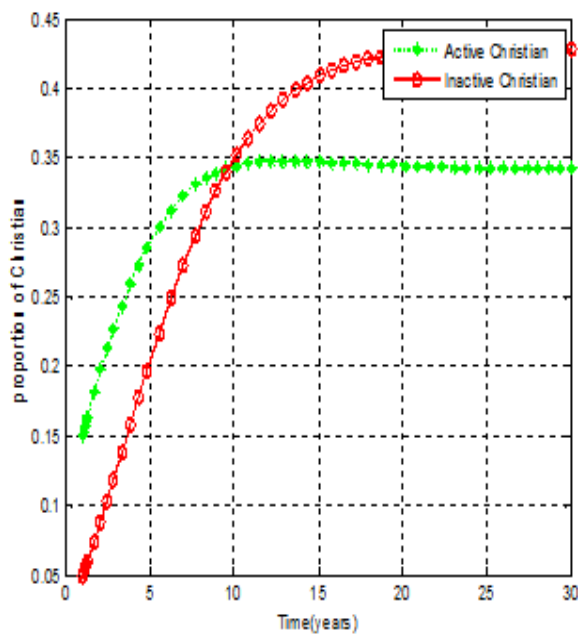
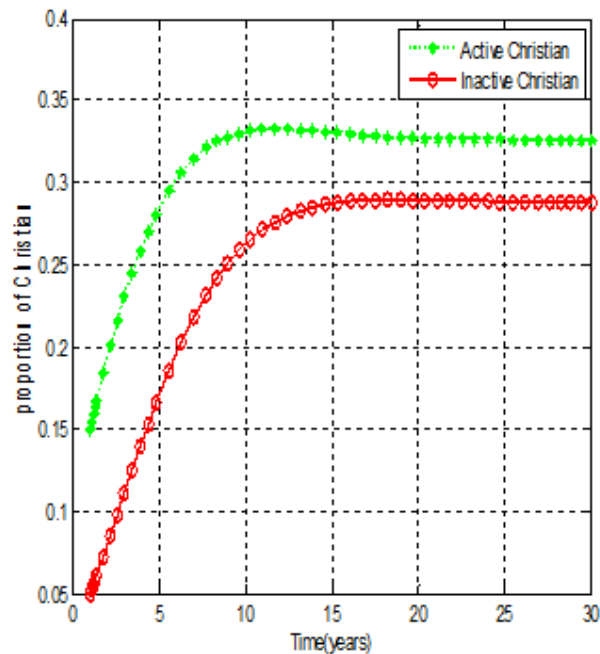Figure 2(b).Variation of population in different classes for $\gamma = 0.2$ and $\lambda = 0.00$

Figure 2(c).Variation of population in different classes for $\gamma = 0.2$ and $\lambda = 0.10$
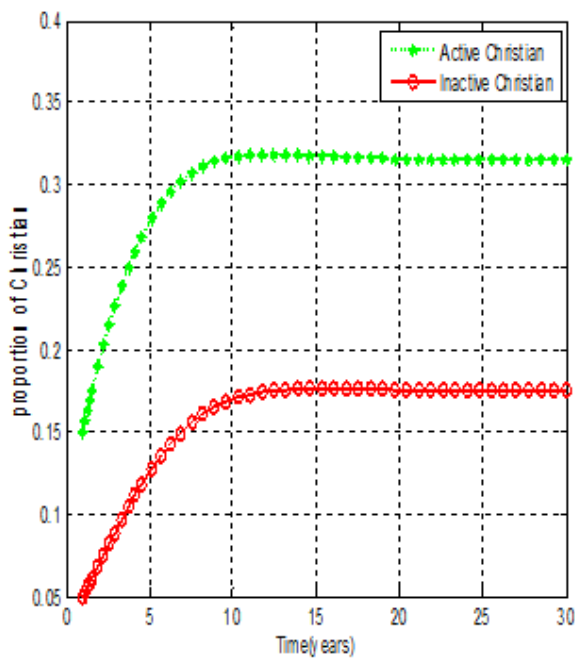

Figure 2(d).Variation of population in different classes for $\gamma = 0.2$ and $\lambda = 0.30$
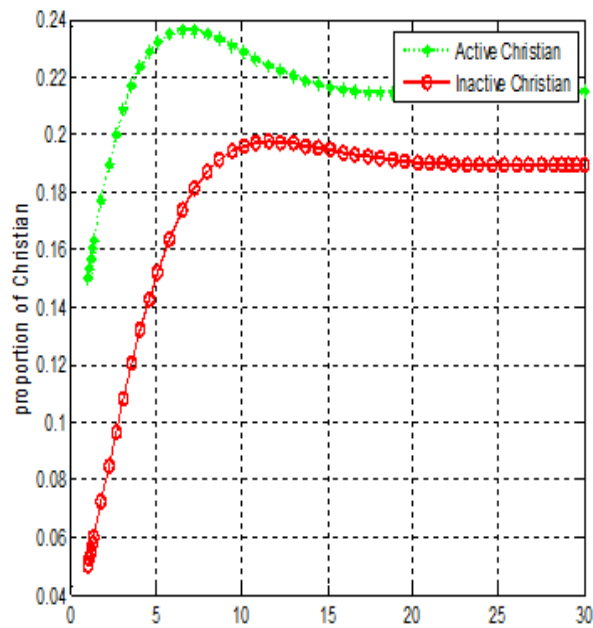

Figure 3(a).Variation of population in different classes for $\gamma = 0.3$ and $\lambda = 0.00$
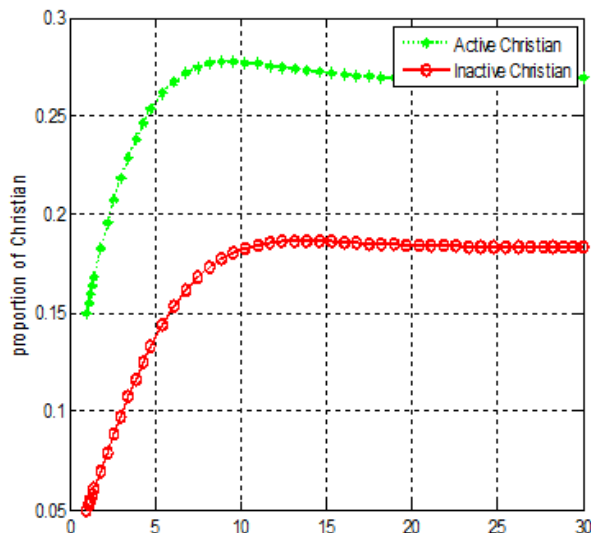
Figure 3(b).Variation of population in different classes for
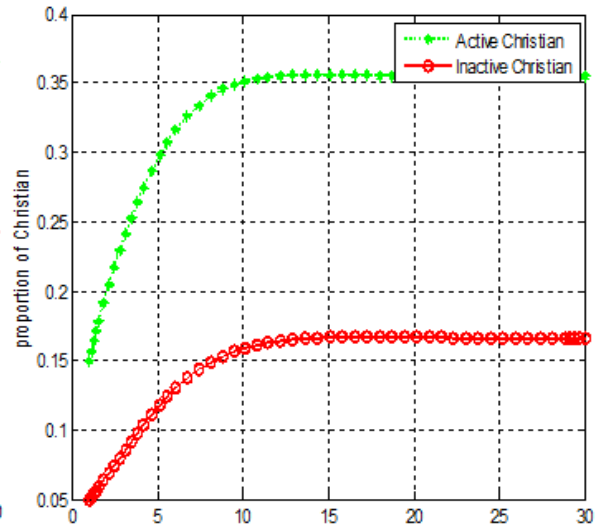$\gamma = 0.3$ and $\lambda = 0.10$

Figure 3(c).Variation of population in different classes for
$\gamma = 0.3$ and $\lambda = 0.30$

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