

# Observations on the Bi-quadratic Equation

$$xy + (k^2 + 1)z^2 = 5w^4$$

M. A. Gopalan<sup>#1</sup>, B. Sivakami<sup>#2</sup>,

<sup>#1</sup>Department of Mathematics, Shrimati Indira Gandhi College, Tiruchirapalli-620002.

<sup>#2</sup>Department of Mathematics, Chettinad College of Engineering and Technology, Karur-639114.

**Abstract-** Three different patterns of non-zero integral solutions to the bi-quadratic equation with four unknowns given by  $xy + (k^2 + 1)z^2 = 5w^4$  are obtained. A few interesting relations between the solutions and special polygonal numbers are exhibited.

**Keywords-** Bi-quadratic equation with four unknowns, Integral solutions. Special polygonal numbers.

## I. INTRODUCTION

Diophantine equations have an unlimited field for research by reason of their variety. In particular, the quartic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context one may refer [6-18] for various problems on the quartic diophantine equations with 2,3,4 variables. This paper concerns with the problem of determining non-trivial integral solutions of the bi-quadratic equation with four unknowns given by  $xy + (k^2 + 1)z^2 = 5w^4$ . Explicit integral solutions of the above equation are presented in different patterns. In each of these patterns, a few interesting relations among the solutions and special polygonal numbers are obtained.

## II. NOTATIONS

| Polygonal Numbers           | Notations for rank 'n' | Definitions                 |
|-----------------------------|------------------------|-----------------------------|
| Triangular number           | $T_n$                  | $\frac{1}{2} n(n+1)$        |
| Pentagonal number           | $Pen_n$                | $\frac{1}{2} (3n^2 - n)$    |
| Hexagonal number            | $Hex_n$                | $2n^2 - n$                  |
| Octagonal number            | $Oct_n$                | $3n^2 - 2n$                 |
| Decagonal number            | $Dec_n$                | $4n^2 - 3n$                 |
| Tetradecagonal number       | $TED_n$                | $\frac{1}{2} (12n^2 - 10n)$ |
| Centered Square number      | $CS_n$                 | $n^2 + (n-1)^2$             |
| Gnomonic number             | $Gno_n$                | $2n - 1$                    |
| Pronic number               | $Pro_n$                | $n(n+1)$                    |
| Octahedral number           | $OH_n$                 | $\frac{1}{3} n(2n^2 + 1)$   |
| Stella Octangula number     | $SO_n$                 | $n(2n^2 - 1)$               |
| Star number                 | $Star_n$               | $6n(n-1) + 1$               |
| Pentagonal Pyramidal number | $PP_n$                 | $\frac{1}{2} n^2(n+1)$      |

|                         |                    |                            |
|-------------------------|--------------------|----------------------------|
| Tetrahedral number      | Tetra <sub>n</sub> | $\frac{1}{6}n(n+1)(n+2)$   |
| Octahedral number       | OH <sub>n</sub>    | $\frac{1}{3}n(2n^2 + 1)$   |
| Stella Octangula number | SO <sub>n</sub>    | $n(2n^2 - 1)$              |
| Icositetragonal number  | ICT <sub>n</sub>   | $\frac{1}{2}(22n^2 - 20n)$ |

Using these values of u and v in (2), the integral solutions to (1) are obtained as,

$$\left. \begin{aligned} x(a,b,k) &= (2+k)a^4 - 4(1-2k)(2+k)a^3b \\ &\quad - 6(2+k)a^2b^2 + 4(1-2k)ab^3 + (2+k)b^4 \\ y(a,b,k) &= (2-k)a^4 - 4(1+2k)a^3b \\ &\quad - 6(2-k)a^2b^2 + 4(1+2k)ab^3 + (2-k)b^4 \\ z(a,b) &= a^4 + 8a^3b - 6a^2b^2 - 8ab^3 + b^4 \\ w(a,b) &= a^2 + b^2 \end{aligned} \right\} \dots\dots\dots(7)$$

### III. METHOD OF ANALYSIS

The Diophantine equation representing the bi-quadratic equation under consideration is

$$xy + (k^2 + 1)z^2 = 5w^4 \quad \dots\dots\dots(1)$$

Different solution patterns to (1) are presented below

#### Pattern 1:

Introducing the transformations

$$x = u + kv, y = u - kv, z = v \quad \dots\dots\dots(2)$$

in (1), it simplifies to

$$u^2 + v^2 = 5w^4 \quad \dots\dots\dots(3)$$

Substituting  $w = a^2 + b^2$  in (3) and choosing

$$5 = (2+i)(2-i), \text{ it becomes}$$

$$(u + iv)(u - iv) = (2+i)(2-i)(a^2 + b^2)^4 \quad \dots\dots\dots(4)$$

$$\text{Let us define } u + iv = (2+i)(a + ib)^4 \quad \dots\dots\dots(5)$$

By equating real and imaginary parts, we get

$$\left. \begin{aligned} u &= 2a^4 - 4a^3b - 12a^2b^2 + 4ab^3 + 2b^4 \\ v &= a^4 + 8a^3b - 6a^2b^2 - 8ab^3 + b^4 \end{aligned} \right\} \quad \dots\dots\dots(6)$$

#### Properties:

$$\begin{aligned} (i) \quad &4(2+k)x(a,1,k) + 16(1-2k)(2+k)[2PP_a - Obl_a] \\ &+ 2(2+k)^2[3CS_a + 12T_a - 5] \end{aligned}$$

is a perfect square.

$$\begin{aligned} (ii) \quad &y(a,1,k) = (2-k)[Star_a^2 - 10Hex_a^2 + 5Oct_a^2] \\ &- (1+2k)[3OH_a + SO_a - 2Gno_a - 2] \end{aligned}$$

$$(iii) \quad (2-k)x(a,1,k) - (2+k)y(a,1,k) = -16kT_a(3Dec_a - 8Pen_a + 5)$$

$$(iv) \quad 2[x(a,1,k) + y(a,1,k) + z(a,1,k)] + 20 = 5CS_a^2 - 25Gno_a^2$$

$$(v) \quad 5[x(a,b,k) + y(a,b,k) + z(a,b,k)]$$

can be written as the difference of two squares.

$$\begin{aligned} (vi) \quad &(2-k)^3y(a,1,k) - (2-k)k[Star_a - 4SO_a + 5Gno_a + 3] \\ &+ (2-k)[8PP_a + 16T_a - 24Pen_a + 12Oct_a - 2] \end{aligned}$$

is a quartic integer.

#### Pattern 2:

By taking  $5 = (1+2i)(1-2i)$  in (3), we get

$$(u + iv)(u - iv) = (1+2i)(1-2i)(a^2 + b^2)^4 \quad \dots\dots\dots(8)$$

Now we define

$$u + iv = (1+2i)(a + ib)^4 \quad \dots\dots\dots(9)$$

On equating real and imaginary parts we get,

$$\left. \begin{array}{l} u = a^4 - 8a^3b - 6a^2b^2 + 8ab^3 + b^4 \\ v = 2a^4 + 4a^3b - 12a^2b^2 - 4ab^3 + 2b^4 \end{array} \right\} \dots\dots(10)$$

Then the integral solutions to (1) are given by

$$\left. \begin{array}{l} x(a,b,k) = (1+2k)a^4 - 4(2-k)a^3b - 6(1+2k)a^2b^2 \\ \quad + 4(2-k)ab^3 + (1+2k)b^4 \\ y(a,b,k) = (1-2k)a^4 - 4(2+k)a^3b - 6(1-2k)a^2b^2 \\ \quad + 4(2+k)ab^3 + (1-2k)b^4 \\ z(a,b) = 2a^4 + 4a^3b - 12a^2b^2 - 4ab^3 + 2b^4 \\ w(a,b) = a^2 + b^2 \end{array} \right\} \dots\dots(11)$$

*Observations:*

1.  $x(a,1,k) + y(a,1,k) = z(a,1,k) - 120Tetra_a - 1$
2.  $z(a,1,k) - w^2(a,1) + 16T_a + 5Gno_{a^2} + 4$  is a quartic integer.
3.  $x(a,b,k) - y(a,b,k) = 2kz(a,b,k)$
4.  $10[x(a,b,k) + y(a,b,k)] + 40z(a,b)$  can be written as the difference of two squares.
5.  $(2+k)x(a,1,k) - (2-k)y(a,1,k) = 5k[3Hex_{a^2} - Dec_{a^2}] + 10k[6(OH_a) - 2Pro_a - 8PP_a + 1]$
6.  $(1-2k)x(a,1,k) - (1+2k)y(a,1,k) = 10k[3(OH_a) + SO_a - 2Gno_a - 2]$
7.  $6(1+2k)x(a,1,k) + 48(1+2k)^2 - 2(k-2)(1+2k)[9(OH_a) + 3SO_a + Star_a - 12T_a - 1]$   
is a Nasty number.

*Pattern 3:*

Let us choose 5 as  $5 = \frac{(11+2i)(11-2i)}{25}$

Then (3) can be written as

$$(u+iv)(u-iv) = \frac{(11+2i)(11-2i)}{25}(a^2+b^2)^4 \dots\dots(12)$$

Now we define

$$u+iv = \frac{(11+2i)}{5}(a+ib)^4 \dots\dots(13)$$

By equating real and imaginary parts on both sides, we obtain the values of u and v as

$$\left. \begin{array}{l} u = \frac{1}{5}[11a^4 - 8a^3b - 66a^2b^2 + 8ab^3 + 11b^4] \\ v = \frac{1}{5}[2a^4 + 44a^3b - 12a^2b^2 - 44ab^3 + 2b^4] \end{array} \right\} \dots\dots(14)$$

With these values of u and v, the solutions x,y,z and w of (1) are

$$\left. \begin{array}{l} x(a,b,k) = \frac{1}{5}[(11+2k)a^4 + 4(11k-2)a^3b \\ \quad - 6(11+2k)a^2b^2 - 4(11k-2)ab^3 + (11+2k)b^4] \\ y(a,b,k) = \frac{1}{5}[(11-2k)a^4 - 4(11k+2)a^3b \\ \quad - 6(11-2k)a^2b^2 + 4(11k+2)ab^3 + (11-2k)b^4] \\ z(a,b) = \frac{1}{5}[2a^4 + 44a^3b - 12a^2b^2 - 44ab^3 + 2b^4] \\ w(a,b) = a^2 + b^2 \end{array} \right\} \dots\dots(15)$$

Since our aim is to find integral solutions,

Let us take  $a = 5A$  and  $b = 5B$

Then the solutions are given by,

$$\left. \begin{array}{l} x(A,B,k) = 125(11+2k)A^4 + 500(11k-2)A^3B - 750(11+2k)A^2B^2 \\ \quad - 500(11k-2)AB^3 + 125(11+2k)B^4 \end{array} \right\}$$

$$y(A,B,k) = 125(11-2k)A^4 - 500(11k+2)A^3B - 750(11-2k)A^2B^2 + 500(11k+2)AB^3 + 125(11-2k)B^4$$

$$\left. \begin{array}{l} z(A,B,k) = 250A^4 + 5500A^3B - 1500A^2B^2 - 5500AB^3 + 250B^4 \\ w(A,B) = 25A^2 + 25B^2 \end{array} \right\} \dots\dots(16)$$

**Observations:**

1.  $x(A,B,k) - y(A,B,k) = 2kz(A,B)$
2.  $2w(A,1) - 25Gno_A^2 - 75 = 0$
3.  $11z(A,1,k) - x(A,1,k) - y(A,1,k) = 31250[SO_A + 2Oct_A - 3Hex_A]$
4.  $x(A,1,k) + y(A,1,k) - 250[ICT_A^2 - 11TED_A - 4SO_A + 20TA + 11] \equiv 0 \pmod{15250}$
5.  $20[x(A,1,k) - y(A,1,k)]k^3 - 5000[22SO_A - 12Pro_A - 5Gno_A - 3]k^4$   
is a quartic integer.
6.  $(11-2k)x(A,1,k) - (11+2k)y(A,1,k) = 62500k[SO_A + Dec_A - 2Hex_A]$

**CONCLUSION :**

One may search for other patterns of solutions and the corresponding observations.

**REFERENCES :**

- [1]. R.D.Carmichael, The Theory of numbers and Diophantine Analysis, Dover Publications, New York(1959).
2. Mordell L.J., “Diophantine Equations”, Academic Press, New York, 1970.
3. Dickson. L. E., History of Theory of Numbers, Vol.2, Chelsia Publishing Co., New York (1952).
4. Telang S.G., Number Theory, Tata Mcgraw Hill Publishing Company, New Delhi (1996).
5. Nigel, P.Smart, The Algorithmic Resolutions of Diophantine Equations, Cambridge University Press, London (1999).
6. M.A.Gopalan and V.Mythili, On the Quartic equation  $x^4 - (2y+c)x^2 - 6x + y^2 - y = 0$ , Acta Ciencia Indica, Vol.XXIX M, No. 4, 755 (2003).
7. M.A.Gopalan and R.Anbuselvi, Integral solutions of  $x^2 \pm xy + y^2 = (x - My)^4$ , Acta Ciencia Indica, Vol.XXXII M, No. 4, 1743 (2006).
8. M.A.Gopalan, Manju Somnath and N.Vanitha, Parametric Integral solutions of  $x^2 + y^3 = z^4$ , Acta Ciencia Indica, Vol.XXXIII M, No. 4, 1261 (2007).
9. M.A.Gopalan and R.Anbuselvi, Integral solutions of  $x^2 - y^2 = (x - My)^4$ , Pure and Applied Matematika Sciences, Vol. LXVIII. No.1-2, Sep. 2008.
10. M.A.Gopalan and S.Devibala, On the Binary Non-homogeneous Quartic Equation  $x^4 - (2y+c)x^2 - 6x + y^2 - ay - d = 0$ , Reflections des ERA-JMS, Vol.3 Issue 1, 2008 Pages 9 of pages 9-14.
11. M.A.Gopalan and R.Anbuselvi, Integral solutions of Ternary quartic equation  $x^2 + y^2 = z^4$ , Acta Ciencia Indica, Vol.XXIV M, No. 1, 267 (2008).
12. M.A.Gopalan and V.Pandichelvi, On Ternary Quartic Diophantine Equation  $x^2 + ky^3 = z^4$ , Pasific Asian Journal of Mathematics, Vol.2, No.1-2, Jan-Dec 2008.
13. M.A.Gopalan and G.Janaki, Integral solutions of Ternary quartic equation  $x^2 - y^2 + xy = z^4$ , Impact Journal of Science and Technology, Vol.2(2), No.71-76, 2008.
14. M.A.Gopalan and G.Janaki, Observation on  $2(x^2 - y^2) + 4xy = z^4$ , Acta Ciencia Indica, Vol.XXXV M, No. 2, 445 (2009).
15. M.A.Gopalan and R.Anbuselvi, Integral solutions of binary quartic equation  $x^3 + y^3 = (x - y)^4$ , Reflections des ERA-JMS, Vol.4 Issue 3, 2009, Pages 271 of pages 271-280.
16. M.A.Gopalan, S.Vidhyalakshmi and S.Devibala, Ternary Quartic Diophantine Equation  $2^{4n+3}(x^3 - y^3) = z^4$ , Impact Journal of Science and Technology, Vol.4, No. 57-60, 2010.
17. M.A.Gopalan, A.Vijayasanchar and Manju Somnath, Integral solutions of  $x^2 - y^2 = z^4$ , Impact Journal of Science and Technology, Vol.2(4), No. 149-157, 2010.
18. M.A.Gopalan and P.Shamuganandham, On the Biquadratic equation,  $x^4 + y^4 + z^4 = 2w^4$  Impact Journal of Science and Technology, Vol.4, No.4, 111-115, 2010.
19. M.A.Gopalan, Manju Somnath and G.Sangeetha, Integral solutions of the Non-homogeneous quartic equation  $x^4 - y^4 = (k^2 + 1)(z^2 - w^2)$ , Archimedes Journal of Mathematics, Volume 1, Number 1, March 2011, 1(1), 51-57.
20. M.A.Gopalan, P.Shamuganandham, On the Biquadratic equation  $x^4 + y^4 + (x + y)z^3 = 2(k^2 + 3)^2w^4$ , Bessel Journal of Mathematics, 2(2), 2012, 87-91.
21. M.A.Gopalan, S.Vidhyalakshmi and K.Lakshmi, On the Biquadratic equation with four unknowns  $x^2 + xy + y^2 = (z^2 + zw + w^2)^2$ , International Journal of Pure and Applied Mathematical Sciences, ISSN 0972-9828, Volume 5, Number 1 (2012), pp. 73-77.
22. Manju Somnath, G.Sangeetha and M.A.Gopalan, Integral solutions of a Biquadratic equation with four unknowns  $xy + (k^2 + 1)z^2 = 5w^4$ , Pacific Asian Journal of Mathematics, Vol.6, No.2, July-December 2012, pp. 185-190.
23. M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, Integral solutions of the Biquadratic equation with four unknowns  $(x + y + z + w)^2 = xyzw + 1$ , IOSR Journals of Mathematics, e-ISSN : 2278-5728, p-ISSN : 2319-765X, Volume 7, Issue 4, (Jul-Aug 2013), pp.11-13.
24. K.Meena, S.Vidhyalakshmi, M.A.Gopalan, S.Aarthy Thangam, On the Biquadratic equation with four unknowns  $x^3 + y^3 = 39zw^3$ , International Journal of Engineering and Research- Online, Vol 2, Issue 1, 2014.