An Inequality with the Sequence of Prime Numbers

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***Abstract*— In this paper we will present an inequality with the sequence of prime numbers $2, 3, 5, 7, \ldots$. We prove that there exists a positive constant real number κ, such that for every real number $\lambda > \kappa$, there exists a natural number n_λ, such that for every natural number $n > n_\lambda$, it is true the inequality $\frac{2}{2-1}\cdot\frac{3}{3-1}\cdot\ldots\cdot\frac{p_n}{p_n-1} < \frac{\lambda p_n}{n}$, where p_n is the n-th prime number. The constant number κ is equal with $e^{M+\sigma}$, where M is the Merten's constant and σ is the sum of the convergent series $\sum\frac{1}{p(p-1)}$. The constant κ has an approximate value $\kappa \approx 2.812$.**

***Keywords*— prime number, inequality, series of prime number, prime number theorem.**

I. Introduction

Let's note $p_1, p_2, p_3, \ldots$ the sequence of prime numbers. We will show that there exists a positive constant real number κ, such that for every real number $\lambda > \kappa$, there exists a natural number n_λ, such that for every natural number $n > n_\lambda$, it is true the inequality

$$\frac{p_1}{p_1-1}\cdot\ldots\cdot\frac{p_n}{p_n-1} < \frac{\lambda p_n}{n}.$$

We will show that $\sum\frac{1}{p(p-1)}$ is a convergent series and if we note $\sigma = \sum\frac{1}{p(p-1)}$, then $\kappa = e^{M+\sigma}$, where M is Merten's constant.

We will show that $\sigma \approx 0.7723$, and since $M \approx 0.2615$ then the approximate value of the constant real number κ is 2.812.

II. Prove the Inequality

For every positive real number x we have $1 + x < e^x$. Indeed, considering the function $f(x) = e^x - x - 1$ with domain $[0, +\infty[$, we have $f(0) = 0$ and $f'(x) = e^x - 1 > 0$ for every $x > 0$. Consequently, $f(x) > 0$ for every $x > 0$, namely $e^x > 1 + x$ for every $x > 0$.

Since for every prime number p we have

$$\frac{p}{p-1} = 1 + \frac{1}{p-1},$$

and $\frac{1}{p-1} > 0$, then for every index n it is true the inequality

$$\prod_{i=1}^{n}\frac{p_i}{p_i-1} = \prod_{i=1}^{n}\left(1 + \frac{1}{p_i-1}\right) < e^{\sum_{i=1}^{n}\frac{1}{p_i-1}}.$$

But $\frac{1}{p-1} = \frac{1}{p} + \frac{1}{p(p-1)}$, then

$$\sum_{i=1}^{n}\frac{1}{p_i-1}=\sum_{i=1}^{n}\frac{1}{p_i}+\sum_{i=1}^{n}\frac{1}{p_i(p_i-1)},$$

consequently, for every natural number n it is true the inequality:

$$\prod_{i=1}^{n}\frac{p_i}{p_i-1}<e^{\sum_{i=1}^{n}\frac{1}{p_i}}\cdot e^{\sum_{i=1}^{n}\frac{1}{p_i(p_i-1)}}. \quad (1)$$

Let's prove now that the series $\sum\frac{1}{p(p-1)}$ is convergent. Indeed, since the terms of the series are positive and since for every natural number n we have

$$\sum_{i=1}^{n}\frac{1}{p_i(p_i-1)}=\frac{1}{p_1(p_1-1)}+\frac{1}{p_2(p_2-1)}+\cdots+\frac{1}{p_n(p_n-1)}=$$

$$=\left(\frac{1}{p_1-1}-\frac{1}{p_1}\right)+\left(\frac{1}{p_2-1}-\frac{1}{p_2}\right)+\cdots+\left(\frac{1}{p_n-1}-\frac{1}{p_n}\right)=$$

$$=\frac{1}{p_1-1}-\left(\frac{1}{p_1}-\frac{1}{p_2-1}\right)-\cdots-\left(\frac{1}{p_{n-1}}-\frac{1}{p_n-1}\right)-\frac{1}{p_n}<\frac{1}{p_1-1}=1,$$

then its partial sums are upper bounded, consequently this series is convergent. Let's note

$$\sum\frac{1}{p(p-1)}=\sigma .$$

From Dirichlet theorem [1] we have

$$\sum_{i=1}^{n}\frac{1}{p_i}=\ln(\ln p_n)+M+O\left(\frac{1}{\ln p_n}\right),$$

where M is Merten's constant [2]. Then, based on (1) for every natural number n it is true the inequality

$$\prod_{i=1}^{n}\frac{p_i}{p_i-1}<e^{\ln(\ln p_k)+M+O\left(\frac{1}{\ln p_k}\right)}\cdot e^{\sigma}=e^{M+\sigma}\cdot e^{O\left(\frac{1}{\ln p_n}\right)}\cdot\ln p_n,$$

otherwise, for every natural number n it is true the inequality

$$\prod_{i=1}^{n}\frac{p_i}{p_i-1}<\kappa\cdot e^{O\left(\frac{1}{\ln p_n}\right)}\cdot\ln p_n, \quad (2)$$

where $\kappa=e^{M+\sigma}$.

Let's now consider a real number λ, such that $\kappa<\lambda$.

Since $\lim_{n\to\infty}O\left(\frac{1}{\ln p_n}\right)=0$, then

$$\lim_{n\to\infty}e^{O\left(\frac{1}{\ln p_n}\right)}=1. \quad (3)$$

From prime number theorem [3] we have

$$\lim_{n\to\infty}\frac{n\ln p_n}{p_n}=1. \quad (4)$$

Consequently, from (3) and (4) we have

$$\lim_{n\to\infty}\frac{e^{O\left(\frac{1}{\ln p_n}\right)}\cdot n\ln p_n}{p_n}=1.$$

Moreover, since $\kappa<\lambda$ we can write

$$\lim_{n\to\infty}\frac{\kappa\cdot e^{O\left(\frac{1}{\ln p_n}\right)}\cdot n\ln p_n}{\lambda p_n}=\frac{\kappa}{\lambda}=\theta<1.$$

Then, for $\varepsilon=1-\theta$ there exists a natural number n_λ, such that for every natural number $n>n_\lambda$, it is true the inequality

$$\frac{\kappa\cdot e^{O\left(\frac{1}{\ln p_n}\right)}\cdot n\ln p_n}{\lambda p_n}-\theta<\varepsilon.$$

Thus, for every natural number $n>n_\lambda$ it is true the inequality

$$\frac{\kappa\cdot e^{O\left(\frac{1}{\ln p_n}\right)}\cdot n\ln p_n}{\lambda p_n}<\theta+\varepsilon=1,$$

or otherwise

$$\kappa\cdot e^{O\left(\frac{1}{\ln p_n}\right)}\cdot\ln p_n<\lambda\frac{p_n}{n}. \qquad (5)$$

Finally, by (2) and (5) we can say that: there exists a natural number n_λ, such that for every natural number $n>n_\lambda$, is true the inequality

$$\prod_{i=1}^{n}\frac{p_i}{p_i-1}<\kappa\cdot e^{O\left(\frac{1}{\ln p_n}\right)}\cdot\ln p_n<\lambda\frac{p_n}{n},$$

namely

$$\frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\cdot\ldots\cdot\frac{p_n}{p_n-1}<\frac{\lambda p_n}{n}. \qquad (6)$$

Let's find now the value of constant k with precision to two digits after the decimal point. We have

$$\sum\frac{1}{p(p-1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{96\cdot 97}+\left(\frac{1}{100\cdot 101}+\frac{1}{102\cdot 103}+\frac{1}{106\cdot 107}+\cdots\right),$$

but, since

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{96\cdot 97}=0.7713\ldots,$$

and

$$\frac{1}{100\cdot 101}+\frac{1}{102\cdot 103}+\frac{1}{106\cdot 107}\cdots=\left(\frac{1}{100}-\frac{1}{101}\right)+\left(\frac{1}{102}-\frac{1}{103}\right)+\left(\frac{1}{106}-\frac{1}{107}\right)+\cdots$$

$$<\frac{1}{100}-\left(\frac{1}{101}-\frac{1}{102}\right)-\left(\frac{1}{103}-\frac{1}{106}\right)-\cdots<\frac{1}{100},$$

then

$$\sigma=\sum\frac{1}{p(p-1)}\approx 0.7713+0.01=0.7723.$$

Finally, since $M \approx 0.2615$, then

$$\kappa \approx e^{M+\sigma} = e^{0.2615+0.7723} = e^{1.0338} \approx 2.812 .$$

CONCLUSIONS

Depending on the values of the real number λ we can find different natural numbers n_λ, that verify the inequality (6).

If the real number λ is very close to the constant number κ, then the natural number n_λ increases indefinitely, while if the real number λ is further distant from the real number κ, then the natural number n_λ decreases continuously.

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