Divisor Cordial Labelling of Some Disconnected Graphs

P. Lawrence Rozario Raj[#1] and R. Lawrence Joseph Manoharan[*2]

[#] *P.G. and Research Department of Mathematics, St. Joseph's College, Tiruchirappalli – 620 002, Tamil Nadu, India.*

[*] *Department of Mathematics, Hindustan College of Arts and Science, Chennai – 603 103, Tamil Nadu, India.*

Abstract - In this paper, the divisor cordial labeling of disconnected graphs $P_n \cup P_m$, $C_n \cup C_m$, $P_n \cup C_m$, $P_n \cup K_{1,m}$, $P_n \cup K_{1,m,m}$, $P_n \cup W_m$, $P_n \cup S_m$, $C_n \cup K_{1,m}$, $C_n \cup K_{1,m,m}$, $C_n \cup W_m$, $C_n \cup S_m$, $W_n \cup S_m$, $W_n \cup W_m$ and $S_n \cup S_m$ are presented.

AMS subject classifications : 05C78

Keywords - Disconnected graph, divisor cordial labeling, divisor cordial graph.

I. Introduction

By a graph, we mean a finite, disconnected, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [4]. For standard terminology and notations related to number theory we refer to Burton [2] and graph labeling, we refer to Gallian [3]. In [1], Cahit introduce the concept of cordial labeling of graph. In [12], Varatharajan et al. introduce the concept of divisor cordial labeling of graph. The divisor cordial labeling of various types of graph are presented in [5-11,13]. The brief summaries of definition which are necessary for the present investigation are provided below.

Definition :1.1

A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph.

Definition :1.2

A mapping $f : V(G) \to \{0,1\}$ is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f. If for an edge e = uv, the induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e) = | f(u) - f(v) |$. Then $v_f(i)$ = number of vertices of having label i under f and $e_f(i)$ = number of edges of having label i under f*.

Definition :1.3

A binary vertex labeling f of a graph G is called a cordial labeling if $| v_f(0) - v_f(1) | \le 1$ and $| e_f(0) - e_f(1) | \le 1$. A graph G is cordial if it admits cordial labeling.

Definition :1.4

Let a and b be two integers. If a divides b means that there is a positive integer k such that b = ka. It is denoted by a | b. If a does not divide b, then we denote $a \nmid b$.

Definition :1.5

Let G = (V(G), E(G)) be a simple graph and $f : \to \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if f(u) | f(v) or f(v) | f(u) and the label 0 otherwise. The function f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \le 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.

Definition :1.6

The shell S_n is the graph obtained by taking $n - 3$ concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex.

Definition :1.7

A wheel W_n is a graph with n+1 vertices, formed by connecting a single vertex to all the vertices of cycle C_n. It is denoted by $W_n = C_n + K_1$.

Definition :1.8

A complete biparitite graph $K_{1,n}$ is called a star and it has n+1 vertices and n edges. $K_{1,n,n}$ is the graph obtained by the subdivision of the edges of the star $K_{1,n}$.

II. MAIN THEOREMS

Theorem : 2.1

The disconnected graph $P_n \cup P_m$ is divisor cordial graph, where $n,m \geq 2$.

Proof.

Let G be the disconnected graph $P_n \cup P_m$.
Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_m$ be the vertices of P_n and P_m respectively.
Then $|V(G)| = n+m$ and $|E(G)| = n + m - 2$.
Define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., n+m\}$ as follows
Label the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_m$ in the following order.

$$\begin{array}{lllll} 1, & 2, & 2^2, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \leq n+m$ and $s \geq 1$, $k_s \geq 0$.

Case (i) : n+m is odd and $f(v_1)$ is even.

Then, $e_f(0) = e_f(1) + 1 = \frac{n+m-1}{2}$.

Case (ii) : n+m is odd and $f(v_1)$ is odd.

Then, $e_f(1) = e_f(0)+1 = \frac{n+m-1}{2}$.

Case (iii) : n+m is even and $f(v_1)$ is even.

Then, $e_f(0) = e_f(1) = \frac{n+m-2}{2}$.

Case (iv) : n+m is even and $f(v_1)$ is odd.

Subcase (a) : n+m = 6 and $f(v_1)$ is odd.

Interchange the labels of u_1 and v_1.
Then, $e_f(0) = e_f(1) = 2$

Subcase (b) : $n+m \neq 6$ and $f(v_1)$ is odd.

Interchange the labels of u_2 and v_m.
Then, $e_f(0) = e_f(1) = \frac{n+m-2}{2}$.

Therefore, $|e_f(0) - e_f(1)| \leq 1$.
Hence G is divisor cordial graph.

Example : 2.1

(i) The graph $P_6 \cup P_5$ and its divisor cordial labeling is given in Figure 2.1(a).

Figure 2.1(a)

(ii) The graph $P_7 \cup P_5$ and its divisor cordial labeling is given in Figure 2.1(b).

Figure 2.1(b)

Theorem : 2.2

The disconnected graph $C_n \cup C_m$ is divisor cordial graph, where $n,m \geq 3$.

Proof.

Let G be the disconnected graph $C_n \cup C_m$.

Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_m$ be the vertices of C_n and C_m respectively.
Then $|V(G)| = n+m$ and $|E(G)| = n + m$.
Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m \}$ as follows

$f(u_m) = p$, where p is the largest prime number and $p \leq n+m$.

Label the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_{m-1}$ in the following order other than p.

$$
\begin{array}{lllll}
1, & 2, & 2^2, & ..., & 2^{k_1}, \\
3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\
5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\
... & ... & ... & ... & ..., \\
... & ... & ... & ... & ...
\end{array}
$$

where $(2s-1)2^{k_s} \leq n+m$ and $s \geq 1$, $k_s \geq 0$.

Case (i) : n+m is odd and $f(v_1)$ is even.

Then, $e_f(0) = e_f(1) + 1 = \frac{n+m+1}{2}$.

Case (ii) : n+m is odd and $f(v_1)$ is odd.

Then, $e_f(1) = e_f(0)+1 = \frac{n+m+1}{2}$.

Case (iii) : n+m is even and $f(v_1)$ is even.

Then, $e_f(0) = e_f(1) = \frac{n+m}{2}$.

Case (iv) : n+m is even and $f(v_1)$ is odd.

Subcase (a) : n+m = 6 and $f(v_1)$ is odd.

Interchange the labels of u_1 and v_1.

Then, $e_f(0) = e_f(1) = 3$

Subcase (b) : $n+m \neq 6$ and $f(v_1)$ is odd.

Interchange the labels of u_2 and v_m.

Then, $e_f(0) = e_f(1) = \frac{n+m}{2}$.

Therefore, $|e_f(0) - e_f(1)| \leq 1$.
Hence G is divisor cordial graph.

Example : 2.2

The graph $C_8 \cup C_5$ and its divisor cordial labeling is given in Figure 2.2.

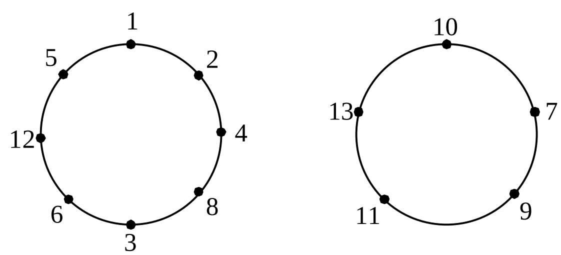

Figure 2.2

Theorem : 2.3

The disconnected graph $P_n \cup C_m$ is divisor cordial graph, where $n \geq 2$ and $m \geq 3$.

Proof.

Let G be the disconnected graph $P_n \cup C_m$.
Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_m$ be the vertices of P_n and C_m respectively.
Then $|V(G)| = n+m$ and $|E(G)| = n + m - 1$.
Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m\}$ as follows

$f(v_m) = p$, where p is the largest prime number and $p \leq n+m$.

Label the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_{m-1}$ in the following order other than p.

$1, \quad 2, \quad 2^2, \quad ..., \quad 2^{k_1},$

$3, \quad 3\times 2 \quad 3\times 2^2 \quad ..., \quad 3\times 2^{k_2},$

$5, \quad 5\times 2 \quad 5\times 2^2 \quad ..., \quad 5\times 2^{k_3},$

$... \quad ... \quad ... \quad ... \quad ...,$

$... \quad ... \quad ... \quad ... \quad ...$

where $(2s-1)2^{k_s} \le n+m$ and $s \ge 1$, $k_s \ge 0$.

Case (i) : n+m is even and $f(v_1)$ is odd.

Then, $e_f(1) = e_f(0) + 1 = \frac{n+m}{2}$.

Case (ii) : n+m is even and $f(v_1)$ is even.

Then, $e_f(0) = e_f(1) + 1 = \frac{n+m}{2}$.

Case (iii) : n+m is odd and $f(v_1)$ is odd.

Then, $e_f(0) = e_f(1) = \frac{n+m-1}{2}$.

Case (iv) : n+m is odd and $f(v_1)$ is even.

Interchange the labels of u_1 and v_m.

Then, $e_f(0) = e_f(1) = \frac{n+m-1}{2}$.

Therefore, $|e_f(0) - e_f(1)| \le 1$.

Hence G is divisor cordial graph.

Example : 2.3

The graph $P_5 \cup C_6$ and its divisor cordial labeling is given in Figure 2.3.

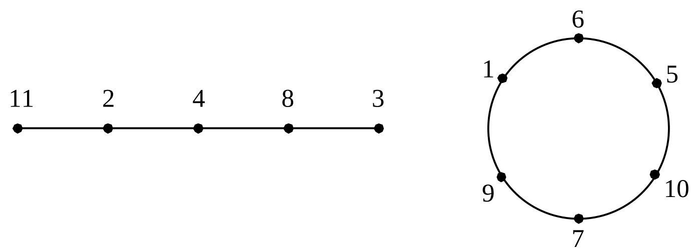

Figure 2.3

Theorem : 2.4

The disconnected graph $P_n \cup K_{1,m}$ is divisor cordial graph, where $n \ge 2$ and $m \ge 1$.

Proof.

Let G be the disconnected graph $P_n \cup K_{1,m}$.

Let $u_1, u_2, ..., u_n$ and $v, v_1, v_2, ..., v_m$ be the vertices of P_n and $K_{1,m}$ respectively.

Then $|V(G)| = n+m+1$ and $|E(G)| = n + m - 1$.

Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m+1\}$ as follows

$f(v) = 2$

Label the vertices $u_1, u_2, ..., u_n$ in the following order.

$1, \quad 2^2, \quad 2^3, \quad ..., \quad 2^{k_1},$

$3, \quad 3\times 2 \quad 3\times 2^2 \quad ..., \quad 3\times 2^{k_2},$

$5, \quad 5\times 2 \quad 5\times 2^2 \quad ..., \quad 5\times 2^{k_3},$

$... \quad ... \quad ... \quad ... \quad ...,$

$... \quad ... \quad ... \quad ... \quad ...$

where $(2s-1)2^{k_s} \le n+1$ and $s \ge 1$, $k_s \ge 0$ and label the remaining vertices $v_1, v_2, ..., v_m$ from n+2 to n+m+1.

Then,

$e_f(0) = e_f(1) + 1 = \frac{n+m}{2}$, when either n and m are odd or n and m are even.

$e_f(1) = e_f(0) = \frac{n+m-1}{2}$, when either n is even and m is odd or n is odd and m is even.

Therefore, $|e_f(0) - e_f(1)| \le 1$.
Hence G is divisor cordial graph.

Example : 2.4

The graph $P_6 \cup K_{1,8}$ and its divisor cordial labeling is given in Figure 2.4.

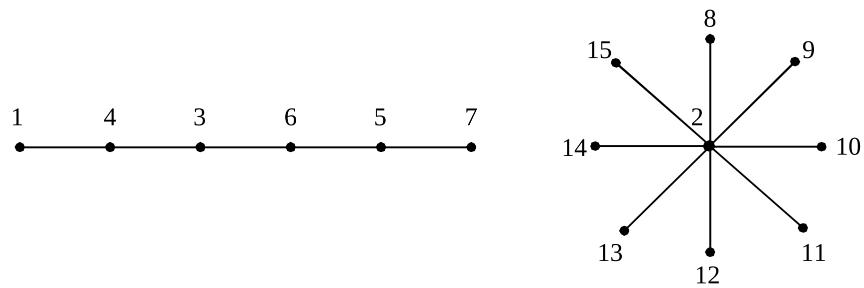

Figure 2.4

Theorem : 2.5

The disconnected graph $P_n \cup K_{1,m,m}$ is divisor cordial graph, where $n \ge 2$ and $m \ge 1$.

Proof.

Let G be the disconnected graph $P_n \cup K_{1,m,m}$
Let $u_1, u_2, ..., u_n$ and $v, v_1, v_2, ..., v_m, v_{m+1}, v_{m+2}, ..., v_{2m}$ be the vertices of P_n and $K_{1,m,m}$ respectively.
Then $|V(G)| = n+2m+1$ and $|E(G)| = n + 2m - 1$.
Define vertex labeling $f : V(G) \to \{1, 2, ..., n+2m+1\}$ as follows

$f(v) = 2$

Label the vertices $u_1, u_2, ..., u_n$ in the following order.

$1,\ 2^2,\ 2^3,\ ...,\ 2^{k_1},$

$3,\ 3\times 2\ \ 3\times 2^2\ \ ...,\ 3\times 2^{k_2},$

$5,\ 5\times 2\ \ 5\times 2^2\ \ ...,\ 5\times 2^{k_3},$

... ...,

... ...

where $(2s-1)2^{k_s} \le n+1$ and $s \ge 1$, $k_s \ge 0$.

Case (i) : n is odd

$f(u_i) = n+1+2i$, for $1 \le i \le m$

$f(u_{m+i}) = n+2i$, for $1 \le i \le m$

Case (ii) : n is even

$f(u_i) = n+2i$, for $1 \le i \le m$

$f(u_{m+i}) = n+1+2i$, for $1 \le i \le m$

Then,

$e_f(1) = e_f(0) = \frac{n+2m-1}{2}$, when n is odd.

$e_f(0) = e_f(1) + 1 = \frac{n+2m}{2}$, when n is even.

Therefore, $|e_f(0) - e_f(1)| \le 1$.
Hence G is divisor cordial graph.

Example : 2.5

The graph $P_7 \cup K_{1,5,5}$ and its divisor cordial labeling is given in Figure 2.5.

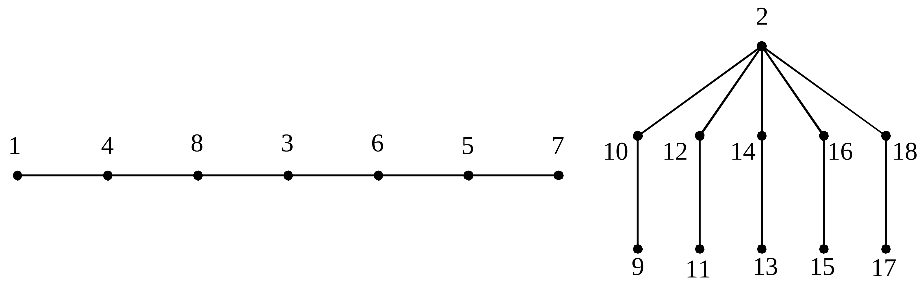

Figure 2.5

Theorem : 2.6

The disconnected graph $P_n \cup W_m$ is divisor cordial graph, where $n \geq 2$ and $m \geq 3$.

Proof.

Let G be the disconnected graph $P_n \cup W_m$.
Let $u_1, u_2, ..., u_n$ and $v, v_1, v_2, ..., v_m$ be the vertices of P_n and W_m respectively.
Then $|V(G)| = n+m+1$ and $|E(G)| = n + 2m - 1$.
Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m+1\}$ as follows

$f(v) = 1$

Label the vertices $u_1, u_2, ..., u_n$ in the following order.

$$\begin{array}{lllll} 2, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \leq n+1$ and $s \geq 1$, $k_s \geq 0$ and label the remaining vertices $v_1, v_2, ..., v_m$ from n+2 to n+m+1.
If (n+2) divides (m–1), then interchange the labels of v_{m-1} and v_m.
Then,

$$e_f(1) = e_f(0) = \frac{n+2m-1}{2}, \text{ when n is odd.}$$

$$e_f(0) = e_f(1) + 1 = \frac{n+2m}{2}, \text{ when n is even.}$$

Therefore, $|e_f(0) - e_f(1)| \leq 1$.
Hence G is divisor cordial graph.

Example : 2.6

The graph $P_5 \cup W_7$ and its divisor cordial labeling is given in Figure 2.6.

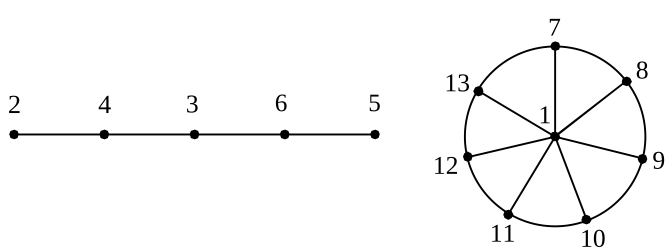

Figure 2.6

Theorem : 2.7

The disconnected graph $P_n \cup S_m$ is divisor cordial graph, where $n \geq 2$ and $m \geq 4$.

Proof.

Let G be the disconnected graph $P_n \cup S_m$.
Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_m$ be the vertices of P_n and S_m respectively.
Then $|V(G)| = n+m$ and $|E(G)| = n + 2m - 4$.
Define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., n+m\}$ as follows

$f(v_1) = 1$

Label the vertices $u_1, u_2, ..., u_n$ in the following order.

$$\begin{array}{lllll} 2, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \leq n+1$ and $s \geq 1$, $k_s \geq 0$ and label the remaining vertices $v_2, v_3, ..., v_m$ from n+2 to n+m.

Then,

$$e_f(1) = e_f(0) + 1 = \frac{n+2m-3}{2}, \text{ when n is odd.}$$

$$e_f(0) = e_f(1) = \frac{n+2m-4}{2}, \text{ when n is even.}$$

Therefore, $|e_f(0) - e_f(1)| \leq 1$.
Hence G is divisor cordial graph.

Example : 2.7

The graph $P_5 \cup S_6$ and its divisor cordial labeling is given in Figure 2.7.

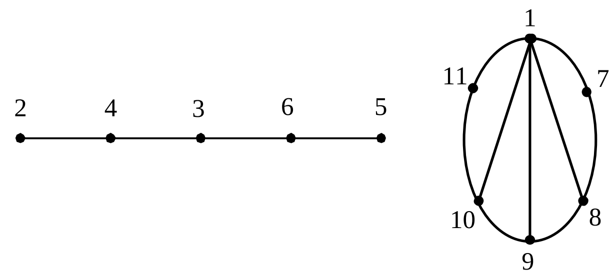

Figure 2.7

Theorem : 2.8

The disconnected graph $C_n \cup K_{1,m}$ is divisor cordial graph, where $n \geq 3$ and $m \geq 1$.

Proof.

Let G be the disconnected graph $C_n \cup K_{1,m}$.
Let $u_1, u_2, ..., u_n$ and $v, v_1, v_2, ..., v_m$ be the vertices of C_n and $K_{1,m}$ respectively.
Then $|V(G)| = n+m+1$ and $|E(G)| = n + m$.
Define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., n+m+1\}$ as follows

$f(v) = 2$

Label the vertices $u_1, u_2, ..., u_n$ in the following order.

$$\begin{array}{lllll} 1, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \leq n+1$ and $s \geq 1$, $k_s \geq 0$ and label the remaining vertices $v_1, v_2, ..., v_m$ from n+2 to n+m+1.

Then,

$e_f(1) = e_f(0) = \frac{n+m}{2}$, when either n and m are odd or n and m are even.

$e_f(1) = e_f(0) + 1 = \frac{n+m+1}{2}$, when either n is even and m is odd or n is odd and m is even.

Therefore, $|e_f(0) - e_f(1)| \le 1$.
Hence G is divisor cordial graph.

Example : 2.8

The graph $C_7 \cup K_{1,6}$ and its divisor cordial labeling is given in Figure 2.8.

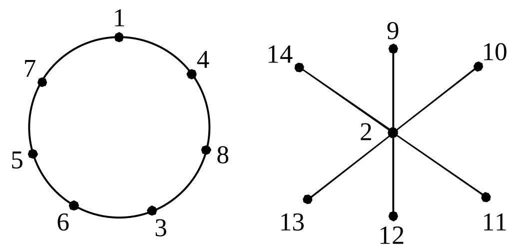

Figure 2.8

Theorem : 2.9

The disconnected graph $C_n \cup K_{1,m,m}$ is divisor cordial graph, where $n \ge 3$ and $m \ge 1$.

Proof.

Let G be the disconnected graph $C_n \cup K_{1,m,m}$
Let $u_1, u_2, ..., u_n$ and $v, v_1, v_2, ..., v_m, v_{m+1}, v_{m+2}, ..., v_{2m}$ be the vertices of C_n and $K_{1,m,m}$ respectively.
Then $|V(G)| = n+2m+1$ and $|E(G)| = n + 2m$.
Define vertex labeling $f : V(G) \to \{1, 2, ..., n+2m+1\}$ as follows

$f(v) = 2$

Label the vertices $u_1, u_2, ..., u_n$ in the following order.

$$\begin{array}{lllll} 1, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \le n+1$ and $s \ge 1$, $k_s \ge 0$.

Case (i) : n is odd

$f(u_i) = n+1+2i$, for $1 \le i \le m$

$f(u_{m+i}) = n+2i$, for $1 \le i \le m$

Case (ii) : n is even

$f(u_i) = n+2i$, for $1 \le i \le m$

$f(u_{m+i}) = n+1+2i$, for $1 \le i \le m$

From above cases,

$e_f(1) = e_f(0) + 1 = \frac{n+2m+1}{2}$, when n is odd.

$e_f(0) = e_f(1) = \frac{n+2m}{2}$, when n is even.

Therefore, $|e_f(0) - e_f(1)| \le 1$.
Hence G is divisor cordial graph.

Example : 2.9

The graph $C_5 \cup K_{1,6,6}$ and its divisor cordial labeling is given in Figure 2.9.

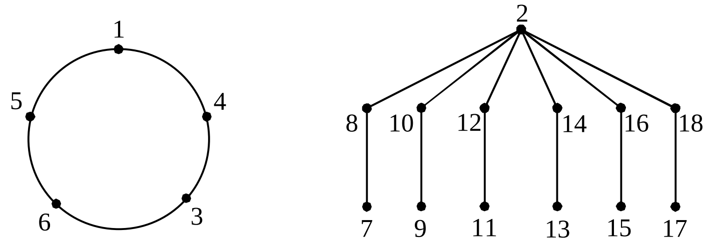

Figure 2.9

Theorem : 2.10

The disconnected graph $C_n \cup W_m$ is divisor cordial graph, where $n, m \geq 3$.

Proof.

Let G be the disconnected graph $C_n \cup W_m$.

Let $u_1, u_2, ..., u_n$ and $v, v_1, v_2, ..., v_m$ be the vertices of C_n and W_m respectively.

Then $|V(G)| = n+m+1$ and $|E(G)| = n + 2m$.

Case (i) : n is odd

Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m+1\}$ as follows

$f(v) = 1$

Label the vertices $u_1, u_2, ..., u_n$ in the following order.

$$\begin{array}{lllll} 2, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \leq n+1$ and $s \geq 1$, $k_s \geq 0$ and label the remaining vertices $v_1, v_2, ..., v_m$ from n+2 to n+m+1.

If n+2 divides m–1, then interchange the labels of v_{m-1} and v_m.

Case (ii) : n is even

Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m+1\}$ as follows

$f(u_n) = n+2$,

$f(v) = 1$,

Label the vertices $u_1, u_2, ..., u_{n-1}$ in the following order.

$$\begin{array}{lllll} 2, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \leq n$ and $s \geq 1$, $k_s \geq 0$ and label the remaining vertices $v_1, v_2, ..., v_m$ from n+1, n+3 to n+m+1.

If (n+1) divides m, then interchange the labels of v_{m-1} and v_m.

From the above cases,

$$e_f(0) = e_f(1) + 1 = \frac{n+2m+1}{2}, \text{ when n is odd.}$$

$$e_f(0) = e_f(1) = \frac{n+2m}{2}, \text{ when n is even.}$$

Therefore, $|e_f(0) - e_f(1)| \leq 1$.

Hence G is divisor cordial graph.

Example : 2.10

The graph $C_8 \cup W_6$ and its divisor cordial labeling is given in Figure 2.10.

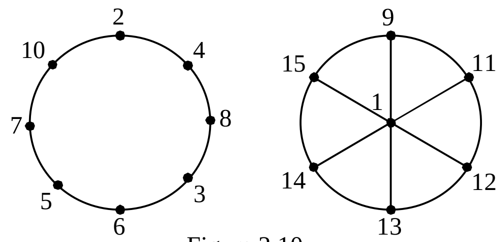

Figure 2.10

Theorem : 2.11

The disconnected graph $C_n \cup S_m$ is divisor cordial graph, where $n \geq 3$ and $m \geq 4$.

Proof.

Let G be the disconnected graph $C_n \cup S_m$.
Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_m$ be the vertices of C_n and S_m respectively.
Then $|V(G)| = n+m$ and $|E(G)| = n + 2m - 3$.

Case (i) : n is odd

Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m\}$ as follows

$f(v_1) = 1$

Label the vertices $u_1, u_2, ..., u_n$ in the following order.

$$\begin{array}{lllll} 2, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \leq n+1$ and $s \geq 1$, $k_s \geq 0$ and label the remaining vertices $v_2, ..., v_m$ from n+2 to n+m.

Case (ii) : n is even

Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m\}$ as follows

$f(u_n) = n+2$,

$f(v_1) = 1$,

Label the vertices $u_1, u_2, ..., u_{n-1}$ in the following order.

$$\begin{array}{lllll} 2, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2s-1)2^{k_s} \leq n$ and $s \geq 1$, $k_s \geq 0$ and label the remaining vertices $v_2, v_3, ..., v_m$ from n+1, n+3 to n+m.

From the above cases,

$$e_f(1) = e_f(0) = \frac{n+2m-3}{2}, \text{ when n is odd.}$$

$$e_f(1) = e_f(0) + 1 = \frac{n+2m-2}{2}, \text{ when n is even.}$$

Therefore, $|e_f(0) - e_f(1)| \leq 1$.

Hence G is divisor cordial graph.

Example : 2.11

The graph $C_6 \cup S_7$ and its divisor cordial labeling is given in Figure 2.11.

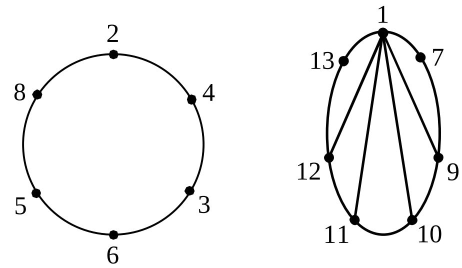

Figure 2.11

Theorem : 2.12

The disconnected graph $W_n \cup S_m$ is divisor cordial graph, where $n \geq 3$ and $m \geq 4$.

Proof.

Let G be the disconnected graph $W_n \cup S_m$.
Let $u, u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_m$ be the vertices of W_n and S_m respectively.
Then $|V(G)| = n+m+1$ and $|E(G)| = 2n+2m-3$.
Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m+1\}$ as follows

Case (i) : $n < m$

Subcase (i) : $n = 3$

$f(u) = 2$, $f(u_1) = 4$, $f(u_2) = 6$ and $f(u_3) = 7$.
$f(v_1) = 1$, $f(v_2) = 3$, $f(v_3) = 5$ and $f(v_4) = 8$.
Label the remaining vertices $v_5, v_6, ..., v_m$ from 9, 10 to m+4.

Subcase (ii) : $n \geq 4$

$f(v_i) = 2i - 1$, for $1 \leq i \leq n+1$
$f(v_{n+1+i}) = 2n+1+i$, for $1 \leq i \leq m - n - 1$
$f(u) = 2$,

For n is even

$f(u_i) = 2i+2$, for $1 \leq i \leq n$

For n is odd

$f(u_i) = 2i+2$, for $1 \leq i \leq n-2$
$f(u_{n-1}) = 2n+2$,
$f(u_n) = 2n$

Case (ii) : $n = m$

Subcase (i) : $2n+1 \equiv 0 \pmod 3$

$f(v_i) = 2i$, for $1 \leq i \leq n$
$f(u) = 1$,
$f(u_i) = 2i + 1$, for $1 \leq i \leq n-2$
$f(u_{n-1}) = 2n+1$,
$f(u_n) = 2n - 1$.

Subcase (ii) : $2n+1 \equiv 1,2 \pmod 3$

$f(v_i) = 2i$, for $1 \leq i \leq n$
$f(u) = 1$,
$f(u_i) = 2i + 1$, for $1 \leq i \leq n$

Case (iii) : $n > m$

Subcase (i) : $n+m+1 \equiv 0 \pmod 3$

$f(v_i) = 2i$, for $1 \leq i \leq m$
$f(u) = 1$,
$f(u_i) = 2i + 1$, for $1 \leq i \leq m$
$f(u_{m+i}) = 2m+1+i$, for $1 \leq i \leq n - m - 2$
$f(u_{n-1}) = n+m+1$,
$f(u_n) = n+m$,

Subcase (ii) : $n+m+1 \equiv 1,2 \pmod 3$

$f(v_i) = 2i$, for $1 \leq i \leq m$
$f(u) = 1$,
$f(u_i) = 2i + 1$, for $1 \leq i \leq m$

$f(u_{m+i}) = 2m+1+i$, for $1 \le i \le n - m$

From the above cases,

$e_f(1) = e_f(0) + 1 = n+m-2$.

Therefore, $|e_f(0) - e_f(1)| \le 1$.

Hence G is divisor cordial graph.

Example : 2.12

The graph $W_8 \cup S_5$ and its divisor cordial labeling is given in Figure 2.12.

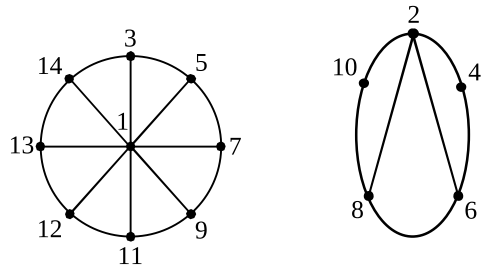

Figure 2.12

Theorem : 2.13

The disconnected graph $W_n \cup W_m$ is divisor cordial graph, where $n, m \ge 3$.

Proof.

Let G be the disconnected graph $W_n \cup W_m$.

Let $u, u_1, u_2, ..., u_n$ and $v, v_1, v_2, ..., v_m$ be the vertices of W_n and W_m respectively.

Then $|V(G)| = n+m+2$ and $|E(G)| = 2n+2m$.

Define vertex labeling $f : V(G) \to \{1, 2, ..., n+m+2\}$ as follows

Case (1) : $n = 3$ and $m = 4$.

$f(u) = 2$, $f(u_1) = 4$, $f(u_2) = 6$ and $f(u_3) = 7$.

$f(v) = 1$, $f(v_1) = 3$, $f(v_2) = 5$, $f(v_3) = 9$ and $f(v_3) = 8$.

Case (2) : $n = 3$ and $m > 4$.

$f(u) = 2$, $f(u_1) = 4$, $f(u_2) = 6$ and $f(u_3) = 7$.

$f(v) = 1$, $f(v_1) = 3$, $f(v_2) = 5$ and $f(v_3) = 8$.

Subcase (i) : $m+5 \equiv 1,2 \pmod 3$

$f(v_{n+i}) = 8+i$, for $1 \le i \le m - 3$

Subcase (ii) : $m+5 \equiv 0 \pmod 3$

$f(v_{n+i}) = 8+i$, for $1 \le i \le m - 5$

$f(v_{m-1}) = n+m+2$,

$f(v_m) = n+m+1$.

Case (3) : $n > 3$ and $m > 5$ and $n < m$.

$f(u) = 2$,

$f(v) = 1$,

Subcase (i) : n is even and $n+m+2 \equiv 1,2 \pmod 3$

$f(u_i) = 2i+2$, for $1 \le i \le n$

$f(v_i) = 2i+1$, for $1 \le i \le n$

$f(v_{n+i}) = 2n+2+i$, for $1 \le i \le m - n$

Subcase (ii) : n is odd and $n+m+2 \equiv 1,2 \pmod 3$

$f(u_i) = 2i+2$, for $1 \le i \le n - 2$

$f(u_{n-1}) = 2n+2$,

$f(u_n) = 2n$,

$f(v_i) = 2i+1$, for $1 \le i \le n$

$f(v_{n+i}) = 2n+2+i$, for $1 \le i \le m - n$

Subcase (iii) : n is even and $n+m+2 \equiv 0 \pmod 3$

$f(u_i) = 2i+2$, for $1 \le i \le n$

$f(v_i) = 2i+1$, for $1 \le i \le n$

$f(v_{n+i}) = 2n+2+i$, for $1 \le i \le m - n - 2$

$f(v_{m-1}) = n+m+2$,
$f(v_m) = n+m+1$.

Subcase (iv) : n is odd and $n+m+2 \equiv 0 \pmod 3$

$f(u_i) = 2i+2$, for $1 \le i \le n-2$
$f(u_{n-1}) = 2n+2$,
$f(u_n) = 2n$,
$f(v_i) = 2i+1$, for $1 \le i \le n-2$
$f(v_{m-1}) = 2n+1$,
$f(v_m) = 2n-1$.

Case (4) : $n = m = 3$

$f(u) = 1$, $f(u_1) = 5$, $f(u_2) = 6$ and $f(u_3) = 7$.
$f(u) = 3$, $f(v_1) = 2$, $f(v_2) = 4$ and $f(v_3) = 8$.

Case (5) : $n > 3$ and $n = m$.

$f(u) = 2$,
$f(v) = 1$,

Subcase (i) : n is even and $n \equiv 0,2 \pmod 3$

$f(u_i) = 2i+2$, for $1 \le i \le n$
$f(v_i) = 2i+1$, for $1 \le i \le n$

Subcase (ii) : n is odd and $n \equiv 0,2 \pmod 3$

$f(u_i) = 2i+2$, for $1 \le i \le n-2$
$f(u_{n-1}) = 2n+2$,
$f(u_n) = 2n$,
$f(v_i) = 2i+1$, for $1 \le i \le n$

Subcase (iii) : n is even and $n \equiv 1 \pmod 3$

$f(u_i) = 2i+2$, for $1 \le i \le n$
$f(v_i) = 2i+1$, for $1 \le i \le n-2$
$f(v_{m-1}) = 2n+1$,
$f(v_m) = 2n-1$.

Subcase (iv) : n is odd and $n \equiv 1 \pmod 3$

$f(u_i) = 2i+2$, for $1 \le i \le n-2$
$f(u_{n-1}) = 2n+2$,
$f(u_n) = 2n$,
$f(v_i) = 2i+1$, for $1 \le i \le n$
$f(v_{n+i}) = 2n+2+i$, for $1 \le i \le m-n-2$
$f(v_{m-1}) = n+m+2$,
$f(v_m) = n+m+1$.

Case (6) : $n = 4$ and $m = 3$.

$f(u) = 1$, $f(u_1) = 3$, $f(u_2) = 5$, $f(u_3) = 9$ and $f(u_3) = 8$.
$f(v) = 2$, $f(v_1) = 4$, $f(v_2) = 6$ and $f(v_3) = 7$.

Case (7) : $n > 4$ and $m = 3$.

$f(u) = 1$, $f(u_1) = 3$, $f(u_2) = 5$ and $f(u_3) = 8$.
$f(v) = 2$, $f(v_1) = 4$, $f(v_2) = 6$ and $f(v_3) = 7$.

Subcase (i) : $n+5 \equiv 1,2 \pmod 3$

$f(u_{3+i}) = 8+i$, for $1 \le i \le n-3$

Subcase (ii) : $n+5 \equiv 0 \pmod 3$

$f(u_{3+i}) = 8+i$, for $1 \le i \le n-5$
$f(u_{n-1}) = n+5$,
$f(u_n) = n+4$.

Case (8) : $n > 5$ and $m > 3$ and $m < n$.

$f(v) = 2$,
$f(u) = 1$,

Subcase (i) : m is even and $n+m+2 \equiv 1,2 \pmod 3$

$f(v_i) = 2i+2$, for $1 \le i \le m$
$f(u_i) = 2i+1$, for $1 \le i \le m$
$f(u_{m+i}) = 2m+2+i$, for $1 \le i \le n-m$

Subcase (ii) : m is odd and $n+m+2 \equiv 1,2 \pmod 3$

$f(v_i) = 2i+2$, for $1 \le i \le m-2$
$f(v_{m-1}) = 2m+2$,
$f(v_m) = 2m$,
$f(u_i) = 2i+1$, for $1 \le i \le m$
$f(u_{m+i}) = 2m+2+i$, for $1 \le i \le m-n$

Subcase (iii) : m is even and $n+m+2 \equiv 0 \pmod 3$
$f(v_i) = 2i+2$, for $1 \le i \le m$
$f(u_i) = 2i+1$, for $1 \le i \le m$
$f(u_{m+i}) = 2m+2+i$, for $1 \le i \le n-m-2$
$f(u_{n-1}) = n+m+2$,
$f(u_n) = n+m+1$.

Subcase (iv) : m is odd and $n+m+2 \equiv 0 \pmod 3$
$f(v_i) = 2i+2$, for $1 \le i \le m-2$
$f(v_{m-1}) = 2m+2$,
$f(v_m) = 2m$,
$f(u_i) = 2i+1$, for $1 \le i \le m$
$f(u_{m+i}) = 2m+2+i$, for $1 \le i \le n-m-2$
$f(u_{n-1}) = n+m+2$,
$f(u_n) = n+m+1$.

From the above cases,
$e_f(1) = e_f(0) = n+m$.
Therefore, $|e_f(0) - e_f(1)| \le 1$.
Hence G is divisor cordial graph.

Example : 2.13

The graph $W_5 \cup W_8$ and its divisor cordial labeling is given in Figure 2.13.

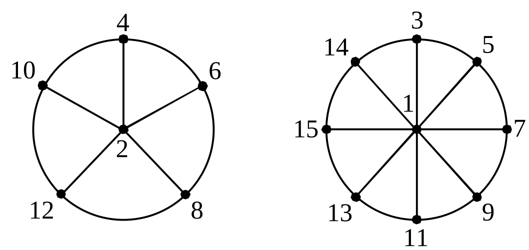

Figure 2.13

Theorem : 2.14

The disconnected graph $S_n \cup S_m$ is divisor cordial graph, where $n, m \ge 4$.

Proof.

Let G be the disconnected graph $S_n \cup S_m$.
Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_m$ be the vertices of S_n and S_m respectively.
Then $|V(G)| = n+m$ and $|E(G)| = 2n+2m-6$.
Define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., n+m\}$ as follows

Case (i) : $n < m$
$f(u_i) = 2i$, for $1 \le i \le n-1$
$f(u_n) = 2n-1$.
$f(v_i) = 2i-1$, for $1 \le i \le n-1$
$f(v_n) = 2n$.
$f(v_{n+i}) = 2n+i$, for $1 \le i \le m-n$

Case (ii) : $n = m$
$f(u_i) = 2i$, for $1 \le i \le n-1$
$f(u_n) = 2n-1$.
$f(v_i) = 2i-1$, for $1 \le i \le n-1$
$f(v_n) = 2n$.

Case (iii) : $n > m$

$f(u_i) = 2i-1$, for $1 \le i \le m-1$

$f(u_m) = 2n$.

$f(v_i) = 2i$, for $1 \le i \le m-1$

$f(v_m) = 2n-1$.

$f(u_{m+i}) = 2n+i$, for $1 \le i \le n-m$

From the above cases,

$e_f(1) = e_f(0) = n + m - 3$.

Therefore, $|e_f(0) - e_f(1)| \le 1$.

Hence G is divisor cordial graph.

Example : 2.14

The graph $S_8 \cup S_6$ and its divisor cordial labeling is given in Figure 2.14.

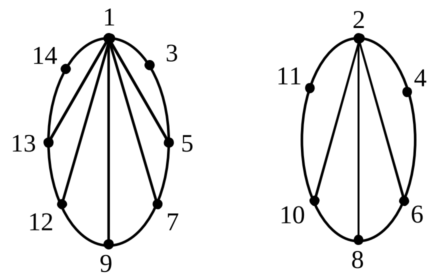

Figure 2.14

III. CONCLUSIONS

In this paper, we prove the divisor cordial labeling of disconnected graphs $P_n \cup P_m$, $C_n \cup C_m$, $P_n \cup C_m$, $P_n \cup K_{1,m}$, $P_n \cup K_{1,m,m}$, $P_n \cup W_m$, $P_n \cup S_m$, $C_n \cup K_{1,m}$, $C_n \cup K_{1,m,m}$, $C_n \cup W_m$, $C_n \cup S_m$, $W_n \cup S_m$, $W_n \cup W_m$ and $S_n \cup S_m$.

REFERENCES

[1] I. Cahit, "Cordial graphs: A weaker version of graceful and harmonious graphs", Ars Combinatoria, Vol 23, pp. 201-207, 1987.

[2] David M. Burton, Elementary Number Theory, Second Edition, Wm. C. Brown Company Publishers, 1980.

[3] J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 16, # DS6, 2013.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, 1972.

[5] P. Lawrence Rozario Raj and R. Valli, Some new families of divisor cordial graphs, International Journal of Mathematics Trends and Technology, Vol 7, No. 2, 2014.

[6] P.Maya and T.Nicholas, Some New Families of Divisor Cordial Graph, Annals of Pure and Applied Mathematics Vol. 5, No.2, pp. 125-134, 2014.

[7] A. Muthaiyan and P. Pugalenthi, "Some new divisor cordial graphs", International Journal of Mathematics Trends and Technology, Vol 12, No. 2, 2014.

[8] A. Nellai Murugan and M. Taj Nisha, "A study on divisor cordial labelling of star attached paths and cycles", Indian Journal of Research, Vol.3, Issue 3, pp. 12-17, 2014.

[9] A. Nellai Murugan and V.Brinda Devi, "A study on path related divisor cordial graphs", International Journal of Scientific Research, Vol.3, Issue 4, pp. 286 – 291, 2014.

[10] S. K. Vaidya and N. H. Shah, "Some Star and Bistar Related Divisor Cordial Graphs", Annals of Pure and Applied Mathematics, Vol 3, No.1, pp. 67-77, 2013.

[11] S. K. Vaidya and N. H. Shah, "Further Results on Divisor Cordial Labeling", Annals of Pure and Applied Mathematics, Vol 4, No.2, pp. 150-159, 2013.

[12] R. Varatharajan, S. Navanaeethakrishnan and K. Nagarajan, "Divisor cordial graphs", International J. Math. Combin., Vol 4, pp. 15-25, 2011.

[13] R. Varatharajan, S. Navanaeethakrishnan and K. Nagarajan, "Special classes of divisor cordial graphs", International Mathematical Forum, Vol 7, No. 35, pp. 1737-1749, 2012.