Dual Transformations and Instantaneous Screw Axes

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***Abstract*— In this study, we investigate instantaneous screw axes with the help of a dual transformation which is defined in [1]. This transformation makes clear the relations between Euclidean space and Lorentzian space. The aim of this paper is to observe instantaneous screw axes under this dual transformation.**

***Keywords*— Dual transformation, space motion, instantaneous screw axis.**

I. Introduction

In this study, we investigate instantaneous screw axes with the help of a dual transformation which is defined between SO(n+1)\{a_{00}=0} and SO(n,1) in [1]. The dual transformation points out the relations between Euclidean space and Lorentzian space. In [2], one-parameter motions are explained and it is shown that the axes of instantaneous motion matrices is not invariant under the dual transformation. The aim of this paper is to carry these calculations into instantaneous kinematics and to show that this dual transformation does not leave invariant the instantaneous screw axis of motion matrices.

II. Preliminaries

Definition 1. If a and a^* are real numbers and $\varepsilon^2 = 0$, the combination $A = a + a^*$ is called a dual number, where ε is the dual unit.

The set of all dual numbers forms a commutative ring over the real number field and is denoted by D. Then the set

$$D^3 = \{\widehat{a} = (A_1, A_2, A_3) | A_i \in D, 1 \leq i \leq 3\}$$

is a modüle over the ring D which is called D- module or dual space. The elements of D^3 are called dual vectors. Thus a dual vector $\hat{a}$ can be written as

$$\widehat{a} = a + \varepsilon a^*$$

where a and a^* are real vectors at R^3 (See [8]).

Definition 2. A, 3x3 skew-symmetric matrix [B] has only 3 independent elements, that is

$$[B] = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \tag{1}$$

These elements can be assembled into the vector $b = (b_1, b_2, b_3)$ (See [5]).

Definition 3. Lorentz – Minkowski space is the metric space $E_1^3 = (R^3, <,>)$ where the metric <,> is given by

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3, \qquad u = (u_1, u_2, u_3), v = (v_1, v_2, v_3),$$

The metric <,> is called the Lorentzian metric.

We remark that <,> is a non-degenerate metric of index 1. We also call E_1^3 as Minkowski space and <,> as the Minkowski metric.

We can also write

$$\langle u, v\rangle = u^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} v = u^t G v$$

(See [3]).

Definition 4. In E_1^3, every 3x3 semi skew-symmetric matrix determines a vector under the isomorphism ϕ that is

$$\phi : \lozenge_1(3) \to E_1^3$$

$$W = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ w_2 & -w_1 & 0 \end{bmatrix} \to \phi(W) = W^v = (w_1, w_2, w_3) \tag{2}$$

where $\lozenge_1(3)$ is the lie algebra of $O_1(3, R)$ which is the group of 3x3 semi-orthogonal matrices.

Remark 1. If we use D dual numbers ring instead of R - real numbers, elements of matrices can be paired as a vector as in Eq. 1 and Eq. 2.

Theorem 1. Let A be an orthogonal matrix as below.

$$A = \left(\begin{array}{c|c} B & C \\ \hline D & a_{33} \end{array}\right)$$

where a_{33} is non zero. Let X and Y be sets as below

$$X = \{A \in SO(3) | a_{33} \neq 0\},$$
$$Y = \{A \in SO(2,1) | a_{33} \neq 0\}.$$

There is a dual transformation between X and Y.

$$\begin{aligned} f \quad : \quad & X \to Y \\ A \mapsto f(A) \quad & = \tfrac{1}{a_{33}} \left(\begin{array}{c|c} a_{33}(B^{-1})^T & C \\ \hline -D & 1 \end{array}\right) \end{aligned}$$

where T is transpose.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto \quad f(A) = \tfrac{1}{a_{33}} \begin{bmatrix} a_{22} & -a_{21} & a_{13} \\ -a_{12} & a_{11} & a_{23} \\ -a_{31} & -a_{32} & 1 \end{bmatrix}$$
$$(3)$$

(See [1],[2]).

Remark 2. According to our other studies on dual transformation, if use dual numbers as elements of matrices instead of real numbers, matrices can be calculated as in Eq. 3.

III. ON THE INSTANTANEOUS SCREW AXIS

In [2], it is shown that the axes of instantaneous motion matrices is not invariant under the dual transformation. Fig. 1 indicates the instantaneous rotation motion in Euclidean space (See [2]). Now we carry these calculations into instantaneous kinematics. Fig. 2 represents the instantaneous screw axis in Euclidean space.

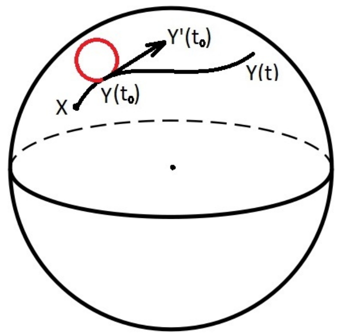

Fig. 1 Instantaneous Rotation Motion

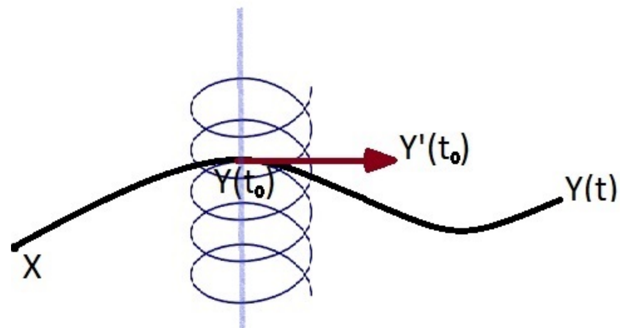

Fig. 2 Instantaneous Screw Axis

Theorem 2. Dual transformation does not leave invariant the instantaneous screw axis of motion matrices $\hat{A}$ and *f(Â)*.

Proof. Let $\hat{A} \in S\hat{O}(3)$ be a dual orthogonal matrix as below.

$$\widehat{A} = \begin{bmatrix} \widehat{a}_{11} & \widehat{a}_{12} & \widehat{a}_{13} \\ \widehat{a}_{21} & \widehat{a}_{22} & \widehat{a}_{23} \\ \widehat{a}_{31} & \widehat{a}_{32} & \widehat{a}_{33} \end{bmatrix}$$

We find the instantaneous screw axis with the help of $\hat{A}'.\hat{A}^T$ (See [6]). We should obtain a dual skew-symmetric matrix as below.

$$\begin{aligned}\widehat{A}'\widehat{A}^T &= [\widehat{B}] = \begin{bmatrix} 0 & -\widehat{b_3} & \widehat{b_2} \\ \widehat{b_3} & 0 & -\widehat{b_1} \\ -\widehat{b_2} & \widehat{b_1} & 0 \end{bmatrix} \\ &\cong \widehat{b} = (\widehat{b_1}, \widehat{b_2}, \widehat{b_3}) \cong (\overrightarrow{b} + \varepsilon \overrightarrow{b^*})\end{aligned}$$

$$\begin{aligned}\widehat{A}'\widehat{A}^T &= \begin{bmatrix} \widehat{a}'_{11} & \widehat{a}'_{12} & \widehat{a}'_{13} \\ \widehat{a}'_{21} & \widehat{a}'_{22} & \widehat{a}'_{23} \\ \widehat{a}'_{31} & \widehat{a}'_{32} & \widehat{a}'_{33} \end{bmatrix} \begin{bmatrix} \widehat{a}_{11} & \widehat{a}_{21} & \widehat{a}a_{31} \\ \widehat{a}_{12} & \widehat{a}_{22} & \widehat{a}_{32} \\ \widehat{a}_{13} & \widehat{a}_{23} & \widehat{a}_{33} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \widehat{a}'_{11}\widehat{a}_{21} + \widehat{a}'_{12}\widehat{a}_{22} + \widehat{a}'_{13}\widehat{a}_{23} & \widehat{a}'_{11}\widehat{a}_{31} + \widehat{a}'_{12}\widehat{a}_{32} + \widehat{a}'_{13}\widehat{a}_{33} \\ \widehat{a}'_{21}\widehat{a}_{11} + \widehat{a}'_{22}\widehat{a}_{12} + \widehat{a}'_{23}\widehat{a}_{13} & 0 & \widehat{a}'_{21}\widehat{a}_{31} + \widehat{a}'_{22}\widehat{a}_{32} + \widehat{a}'_{23}\widehat{a}_{33} \\ \widehat{a}'_{31}\widehat{a}_{11} + \widehat{a}'_{32}\widehat{a}_{12} + \widehat{a}'_{33}\widehat{a}_{13} & \widehat{a}'_{31}\widehat{a}_{21} + \widehat{a}'_{32}\widehat{a}_{22} + \widehat{a}'_{33}\widehat{a}_{23} & 0 \end{bmatrix}\end{aligned}$$

From Eq. 1, we obtain the direction vector of instantaneous screw axis as below.

$$\overrightarrow{x} = (\widehat{a}'_{31}\widehat{a}_{21} + \widehat{a}'_{32}\widehat{a}_{22} + \widehat{a}'_{33}\widehat{a}_{23}, \widehat{a}'_{11}\widehat{a}_{31} + \widehat{a}'_{12}\widehat{a}_{32} + \widehat{a}'_{13}\widehat{a}_{33}, \widehat{a}'_{21}\widehat{a}_{11} + \widehat{a}'_{22}\widehat{a}_{12} + \widehat{a}'_{23}\widehat{a}_{13})$$

The instantaneous screw axis will be the unit form of the vector $\hat{\vec{x}}$.

$$\widehat{\overrightarrow{x_0}} = \frac{\widehat{\overrightarrow{x}}}{\left\|\widehat{\overrightarrow{x}}\right\|}$$

According to E. Study map, $\hat{\vec{x}}_0$ is unit dual vector that corresponds a line in E^3. Now, we calculate the instantaneous screw axis of *f(Â)* as below. We should obtain a dual skew-symmetric matrix as below.

$$\begin{aligned}\widehat{W} &= \begin{bmatrix} 0 & -\widehat{w_3} & \widehat{w_2} \\ \widehat{w_3} & 0 & -\widehat{w_1} \\ \widehat{w_2} & -\widehat{w_1} & 0 \end{bmatrix} \\ &\cong \widehat{w} = (\widehat{w_1}, \widehat{w_2}, \widehat{w_3}) \cong (\overrightarrow{w} + \varepsilon \overrightarrow{w^*})\end{aligned}$$

Then, we calculate the matrices below.

$$\widehat{f(A)}^{-1} = \varepsilon \widehat{f(A)}^T \varepsilon = \frac{1}{\widehat{a}_{33}} \begin{bmatrix} \widehat{a}_{22} & -\widehat{a}_{12} & \widehat{a}_{31} \\ -\widehat{a}_{21} & \widehat{a}_{11} & \widehat{a}_{32} \\ -\widehat{a}_{13} & -\widehat{a}_{23} & 1 \end{bmatrix}$$

$$\widehat{f(A)}'\widehat{f(A)}^{-1} = \frac{1}{\widehat{a}_{33^2}} \begin{bmatrix} 0 & -\widehat{a}_{12}\widehat{a}'_{22} - \widehat{a}_{11}\widehat{a}'_{21} - \widehat{a}_{23}\widehat{a}'_{13} & \widehat{a}'_{22}\widehat{a}_{31} - \widehat{a}'_{21}\widehat{a}_{32} + \widehat{a}'_{13} \\ -\widehat{a}'_{12}\widehat{a}_{22} - \widehat{a}'_{11}\widehat{a}_{21} - \widehat{a}'_{23}\widehat{a}_{13} & 0 & -\widehat{a}'_{12}\widehat{a}_{31} - \widehat{a}'_{11}\widehat{a}_{32} - \widehat{a}'_{23} \\ -\widehat{a}'_{31}\widehat{a}_{22} + \widehat{a}'_{32}\widehat{a}_{21} & \widehat{a}'_{31}\widehat{a}_{12} - \widehat{a}'_{32}\widehat{a}_{11} & 0 \end{bmatrix}$$

From Eq. 2, we obtain the direction vector of instantaneous screw axis as below.

$$\overrightarrow{y} = \frac{1}{\widehat{a}_{33^2}}(-\widehat{a}'_{31}\widehat{a}_{12} + \widehat{a}'_{32}\widehat{a}_{11}, \widehat{a}'_{22}\widehat{a}_{31} - \widehat{a}'_{21}\widehat{a}_{32} + \widehat{a}'_{13}, -\widehat{a}'_{12}\widehat{a}_{22} - \widehat{a}'_{11}\widehat{a}_{21} - \widehat{a}'_{23}\widehat{a}_{13})$$

The instantaneous screw axis will be the unit form of the vector $\hat{\vec{y}}$.

$$\widehat{\overrightarrow{y_0}} = \frac{\widehat{\overrightarrow{y}}}{\left\|\widehat{\overrightarrow{y}}\right\|}$$

Thus, the direction vectors of unit dual vectors are not the same.

Example 1. Let $\hat{A}$ be a dual orthogonal matrix is given below.

$$\widehat{A}=\begin{bmatrix} \frac{1}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \frac{\sin\theta+\varepsilon\cos\theta}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & 0 \\ \frac{-\sin\theta\cos\theta-\varepsilon(\cos 2\theta)}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \frac{\cos\theta-\varepsilon\sin\theta}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \sin\theta+\varepsilon\cos\theta \\ \frac{\sin^2\theta+\varepsilon(\sin 2\theta)}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \frac{-\sin\theta-\varepsilon\cos\theta}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \cos\theta-\varepsilon\sin\theta \end{bmatrix}$$

Under the dual transformation we obtain *f(A)* as below.

$$\widehat{f(A)}=\frac{1}{\cos\theta-\varepsilon\sin\theta}\begin{bmatrix} \frac{\cos\theta-\varepsilon\sin\theta}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \frac{\sin\theta\cos\theta+\varepsilon(\cos 2\theta)}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & 0 \\ \frac{-\sin\theta-\varepsilon\cos\theta}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \frac{1}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \sin\theta+\varepsilon\cos\theta \\ \frac{-\sin^2\theta-\varepsilon(\sin 2\theta)}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & \frac{\sin\theta+\varepsilon\cos\theta}{\sqrt{\sin^2\theta+1+\varepsilon(\sin 2\theta)}} & 1 \end{bmatrix}$$

We can calculate the instantaneous screw axes $\vec{x}$ and $\vec{y}$ by the help of Theorem 2. Firstly we obtain $\vec{x}$ from matrix $\hat{A}'.\hat{A}^T$. Then, we get $\vec{y}$ from $f(\hat{A})'.f(\hat{A})^{-1}$. It can be seen from these calculations that $\vec{x}$ is not equal to $\vec{y}$.

IV. CONCLUSIONS

We can conclude that the dual transformation acts like a bridge between Euclidean space and Lorentzian space. Because of this role, we examine the invariants or variants under this dual transformation. In this study, we show that dual transformation does not leave invariant the instantaneous screw axis of motion matrices.

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