An Application of Dense Sets in Analysis

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Abstract— **We give results on the pointwise convergence of pointwise equicontinuous sequence of maps between metric spaces and on the continuity of limit function, when the restriction of maps to a dense set, are assumed continuous. These results are useful in analysis as well as mathematical physics.**

Keywords— **Dense, Pointwise Convergence, Pointwise Equicontiuous, Restriction.**

I. INTRODUCTION

The role of dense sets in analysis is well documented. One of the striking results, for instance was obtained in [1], where the existence of a maximal dense sets containing all the points of continuity of a map was obtained under suitable conditions on the domain and the range of the map. The role of dense sets was also investigated in detail by the author in [2] with respect to the characterizations of continuous maps or a family of pointwise equicontinuity family of maps between metric spaces. Their role in uniform continuity and isometry between metric spaces was investigated in [3]. Also, the usefulness of dense sets for map gluing theorems is evident in [4]. Not only in Analysis but also in mathematical physics, dense sets play a simplifying role. For instance in the classic text on Mathematical Physics [5], dense sets were used to obtain pointwise convergence of pointwise equicontinuous family of maps [Theorem 1.26].

In this paper we give two results which are useful in mathematical physics. We also give an example to illustrate that the assumptions made in the result cannot be dropped. We shall make use of the following

Theorem 1.1 (Theorem 2.1 [2]):

For a map $f : X \rightarrow Y$, the following conditions are equivalent:

(a)   f is continuous ;

(b) $f|D$ is continuous and f is continuous at each point of D^C ;

(c)   for any sequence of points $\{d_n\}$ in D, $d_n \rightarrow x$ in X implies that $f(d_n) \rightarrow f(x)$.

We shall also refer to the following.

Theorem 1.2 (Theorem 1.25 [5]):

Let $f_n : X \rightarrow Y$ be a sequence of maps such that the family $\{f_n\}$ is pointwise equicontinuous and converges pointwise on X to the limit function f. Then f is continuous.

Theorem 1.3 (Theorem 1.26 [5]):

Let $f_n : X \rightarrow Y$ be a sequence of maps such that the family $\{f_n\}$ is pointwise equicontinuous on X and Y is complete. If $\{f_n\}$ converges pointwise on D, then $\{f_n\}$ converges pointwise on X.

Notation:

X, Y will denote arbitrary metric spaces and maps are not assumed to be continuous or surjective unless mentioned otherwise. D will denote arbitrary fixed dense subset of X and for a map $f : X \rightarrow Y$, $f|D$ will denote the restriction of f to D. For the complement of D in X, we use D^C.

II. RESULTS

We refer to the following well known:

Definition (definition 2.1 of [2]):

A family A of maps from X into Y is said to be :

(a)   equicontinuous at a point x of X if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon,x) > 0$ such that for every f in A and t in X, $\rho(f(x), f(t)) < \varepsilon$ whenever $\rho(x,t) < \delta$,

(b)   pointwise equicontinuous on a subset E of X if it is equicontinuous at each point of E.

We now generalize Theorem 1.3 above to Theorem 2.1 below which discusses the pointwise convergence of a pointwise equicontinuous sequence of maps between metric spaces.

Theorem 2.1:

Let $f_n : X \to Y$ be a sequence of maps such that the family $\{f_n\}$ is pointwise equicontinuous on D^C and Y is complete. If $\{f_n\}$ converges pointwise on D, then $\{f_n\}$ converges pointwise on X.

Proof: Since Y is complete, it is sufficient to show that $\{f_n(x)\}$ is a Cauchy sequence if x is in D^C. Since the family $\{f_n\}$ is equicontinuous at x, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that for all n, $\rho(f_n(x), f_n(d)) < \varepsilon/3$ **.... (i)** whenever $\rho(x, d) < \delta$ for $d \in D$. We fix one such d_0 in D. Then since $\{f_n(d_0)\}$ is convergent in Y, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that, for all m, $n \geq n_0$, $\rho(f_n(d_0), f_m(d_0)) < \varepsilon/3$ **.... (ii)**. Hence for all m, $n \geq n_0$ $\rho(f_n(x), f_m(x)) \leq \rho(f_n(x), f_n(d_0)) + \rho(f_n(d_0), f_m(d_0)) + \rho(f_m(d_0), f_m(x)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ by **(i) and (ii)** above. This proves that $(f_n(x))$ is a Cauchy sequence in Y and is, therefore, convergent.

We observe that in the above proof the point d_0 depends only on ε and since the integer n_0 depends only on d_0, it follows that n_0 also depends only on ε.

We now give an example to show that the assumption of completeness in Theorem 2.1 above cannot be dropped.

Example 2.1.

Let $f_n : \mathbf{R} \to \mathbf{R}^2 - \{(0,0)\}$ be defined by $f_n(x) = (x , 1/n)$. Then for the sequence $(f_n(x))$, $\lim_{n\to\infty} f_n(x)$ does not exist for x=0 in the space $\mathbf{R}^2 - \{(0, 0)\}$. If D is taken to be $\mathbf{R} - \{0\}$, then $\lim_{n\to\infty} f_n(x)$ exists for each point of D and is equal to (x, 0). Further, the family $\{f_n\}$ is equicontinuous at $\{0\}$. We also note that each f_n is, in fact, an isometry into $\mathbf{R}^2 - \{(0, 0)\}$.

We now discuss the continuity of pointwise limit of a sequence of functions, for which the following result is well known: The following Theorem 2.2 is an analog of Theorem 1.2 above.

Theorem 2.2:

Let $f_n : X \to Y$ be a sequence of maps converging pointwise on X to the limit function f. If the family $\{f_n | D\}$ is pointwise equicontinuous on D, then f is a continuous map from X into Y if and only if f is continuous at each point of D^C.

Proof: From Theorem 1.2, $f | D$ is continuous on D and the result follows from our Theorem 1.1.

In the above analog of Theorem 1.2, the maps f_n may not be even continuous.

The reader may find the following remark helpful in finding more results of the type mentioned in this paper.

Remark:

The proof of Theorem 2.1 above involves a variant of the well know Cantors diagonal process. Whereas the diagonal process involves writing a sequence of sequences in an array as below:

$$
\begin{array}{cccccc}
s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & \text{............} \\
s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & \text{............} \\
s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & \text{............} \\
s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & \text{............}
\end{array}
$$

and then taking the diagonal sequence (s_{nn}) as shown by the arrow in the array above; our technique involves a sequence of convergent sequences of points (usually taken from a dense subset of the domain of a map) as in the array below:

$$
\begin{array}{ccccccc}
d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & \text{........} & \to x_1 \\
d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & \text{........} & \to x_2 \\
.... & ..... & ..... & ..... & ..... & .... & \\
d_{n1} & d_{n2} & d_{n3} & d_{n4} & d_{n5} & \text{........} & \to x_n
\end{array}
$$

and choosing for each n, a suitable point $d_{nk(n)}$, above the main diagonal row, such that $\rho(d_{nk(n)}, x_n) < 1/n \to 0$ and so the sequence $(d_{nk(n)})$ converges if and only if the sequence of limits $\{x_n\}$ converges. This technique can be used both on the array shown above as well as on the array formed by the images of the convergent sequences under a map.

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