Pre-rough Connected Topologized Approximation Spaces

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Abstract

Rough thinking is one of the topological connections to uncertainty. The purpose of this paper is to introduce connectedness in approximation spaces using pre-open sets and rough set notions. The definition of pre-rough connected topologized approximation space is introduced.

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1. Preliminaries

This section presents a review of some fundamental notions of topological spaces and rough set theory.

A topological space [4] is a pair (X,τ) consisting of a set X and family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.
(T2) τ is closed under arbitrary union.
(T3) τ is closed under finite intersection.

Throughout this paper (X,τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space, and the family of all $\tau-$closed subsets of X is denoted by τ^*. The family τ of open subsets of X is also called a topology for X. A subset A of X in a topological space (X,τ) is said to be clopen if it is both open and closed in (X,τ).

A family $\mathrm{B} \subseteq \tau$ is called a base for (X,τ) iff every nonempty open subset of X can be represented as a union of subfamily of B. Clearly, a topological space can have many bases. A family $\mathrm{S} \subseteq \tau$ is called a subbase iff the family of all finite intersections of S is a base for (X,τ).

The τ-closure of a subset A of X is denoted by A^- and it is given by $A^- = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in \tau^*\}$. Evidently, A^- is the smallest closed subset of X which contains A. Note that A is closed iff $A = A^-$. The τ-interior of a subset A of X is denoted by A° and it is given by $A^\circ = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$. Evidently, A° is the largest open subset of X which contained in A. Note that A is open iff $A = A^\circ$.

A subset A of X in a topological space (X,τ) is called pre-open [7] (briefly p-open) if $A \subseteq A^{-\circ}$. The complement of a p-open set is called p-closed. The family of all p-open (resp. p-closed) sets is denoted by $PO(X)$ (resp. $PC(X)$). The p-closure of a subset A of X is denoted by A^{p-} and it is defined by $A^{p-} = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in PC(X)\}$.

Evidently, A^{p-} is the smallest p-closed subset of X which contains A. Note that A is p-closed iff $A = A^{p-}$. The p-interior of a subset A of X is denoted by $A^{p\circ}$ and it is defined by $A^{p\circ} = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in PO(X)\}$. Evidently, $A^{p\circ}$ is the largest p-open subset of X which contained in A. Note that A is p-open iff $A = A^{p\circ}$.

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X, R)$, where X is a set called the universe and R is an equivalence relation [6, 8]. The equivalence classes of R are also known as the granules, elementary sets or blocks. We shall use R_x to denote the equivalence class containing $x \in X$, and X/R to denote the set of all elementary sets of R. In the approximation space $K = (X, R)$, the upper (resp. lower) approximation of a subset A of X is given by

$$\overline{R}A = \{x \in X : R_x \cap A \neq \phi\} \ (\text{resp. } \underline{R}A = \{x \in X : R_x \subseteq A\}).$$

Pawlak noted [8] that the approximation space $K = (X, R)$ with equivalence relation R defines a uniquely topological space (X, τ) where τ is the family of all clopen sets in (X, τ) and X/R is a base of τ. Moreover, the upper (resp. lower) approximation of any subset A of X is exactly the closure (resp. interior) of A.

If R is a general binary relation, then the approximation space $K = (X, R)$ defines a uniquely topological space (X, τ_K) where τ_K is the topology associated to K (i.e. τ_K is the family of all open sets in (X, τ_K) and $S = \{xR : x \in X\}$ is a subbase of τ_K, where $xR = \{y \in X : x R y\}$) [2, 5].

Definition 1.1 [2]. Let $K = (X, R)$ be an approximation space with general relation R and τ_K is the topology associated to K. Then the triple $\kappa = (X, R, \tau_K)$ is called a topologized approximation space.

Definition 1.2 [2]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. The upper (resp. lower) approximation of A is denoted by $\overline{R}A$ (resp. $\underline{R}A$) and it is defined by

$$\overline{R}A = A^{-} \ (\text{resp. } \underline{R}A = A^{\circ}).$$

Proposition 1.1 [2]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If A and B are two subsets of X, then

i) $\underline{R}A \subseteq A \subseteq \overline{R}A$.

ii) $\underline{R}\phi = \overline{R}\phi = \phi$ and $\underline{R}X = \overline{R}X = X$.

iii) If $A \subseteq B$, then $\underline{R}A \subseteq \underline{R}B$.

iv) If $A \subseteq B$, then $\overline{R}A \subseteq \overline{R}B$.

v) $\underline{R}(X - A) = X - \overline{R}A$.

vi) $\overline{R}(X - A) = X - \underline{R}A$.

Definition 1.3 [2]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. The p-upper (resp. p-lower) approximation of A is denoted by $\overline{R}_p A$ (resp. $\underline{R}_p A$) and it is defined by

$$\overline{R}_p A = A^{p-} \text{ (resp. } \underline{R}_p A = A^{p\circ}\text{).}$$

Proposition 1.2 [2]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If A and B are two subsets of X, then

i) $\underline{R}_p A \subseteq A \subseteq \overline{R}_p A$

ii) $\underline{R}_p \phi = \overline{R}_p \phi = \phi$ and $\underline{R}_p X = \overline{R}_p X = X$.

iii) If $A \subseteq B$, then $\underline{R}_p A \subseteq \underline{R}_p B$.

iv) If $A \subseteq B$, then $\overline{R}_p A \subseteq \overline{R}_p B$.

v) $\underline{R}_p (X - A) = X - \overline{R}_p A$.

vi) $\overline{R}_p (X - A) = X - \underline{R}_p A$.

2. Pre-rough connected topologized approximation spaces

The present section is devoted to introduce the concept of pre-rough connectedness in approximation spaces with general binary relations. The following two definitions introduce concepts of definability for a subset A of X in a topologized approximation space $\kappa = (X, R, \tau_K)$.

Definition 2.1 [2]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. Then

i) A is called totally $R-$definable (exact) set if $\underline{R}A = A = \overline{R}A$,

ii) A is called internally $R-$definable set if $A = \underline{R}A$,

iii) A is called externally $R-$definable set if $A = \overline{R}A$,

iv) A is called $R-$indefinable (rough) set if $A \neq \underline{R}A$ and $A \neq \overline{R}A$.

Definition 2.2 [1]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. Then

i) A is called totally p-definable (p-exact) set if $\underline{R}_p A = A = \overline{R}_p A$,

ii) A is called internally p-definable set if $A = \underline{R}_p A$,

iii) A is called externally p-definable set if $A = \overline{R}_p A$,

iv) A is called p-indefinable (p-rough) set if $A \neq \underline{R}_p A$ and $A \neq \overline{R}_p A$.

Remark 2.1. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$.

- If A is exact set, then it is both internally $R-$definable and externally $R-$definable set.
- If A is p-exact set, then it is both internally p-definable and externally p-definable set.
- $\underline{R}A$ is the largest internally $R-$definable set contained in A.
- $\underline{R}_p A$ is the largest internally p-definable set contained in A.
- $\overline{R}A$ is the smallest externally $R-$definable set contains A.
- $\overline{R}_p A$ is the smallest externally p-definable set contains A.

Lemma 2.1. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. Then

i) A is exact set if and only if $X - A$ is exact.
ii) A is p-exact set if and only if $X - A$ is p-exact.
iii) A is internally $R-$definable (resp. externally $R-$definable) set if and only if $X - A$ is externally $R-$definable (resp. internally $R-$definable) set.
iv) A is internally p-definable (resp. externally p-definable) set if and only if $X - A$ is externally p-definable (resp. internally p-definable) set.

Proof. By using Proposition 1.1 and Proposition 1.2, the proof is obvious. □

The following definition introduces the concept of pre-rough disconnected topologized approximation space.

Definition 2.3. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. Then κ is said to be pre-rough (briefly p-rough) disconnected if there are two nonempty subsets A and B of X such that

$$A \cup B = X \text{ and } A \cap \overline{R}_p B = \overline{R}_p A \cap B = \phi .$$

The space $\kappa = (X, R, \tau_K)$ is said to be p-rough connected if it is not p-rough disconnected.

Proposition 2.1. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If X has a nonempty p-exact proper subset A, then $\kappa = (X, R, \tau_K)$ is p-rough disconnected.

Proof.

Suppose that A is a nonempty p-exact proper subset of X. Then by Lemma 2.1, we get $B = X - A$ is also a nonempty p-exact proper subset of X. Hence $A \cup B = X$ and $A \cap \overline{R}_p B = A \cap B = \overline{R}_p A \cap B = \phi$.
Thus $\kappa = (X, R, \tau_K)$ is p-rough disconnected. □

Example 2.1. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space such that $X = \{a, b, c, d\}$ and $R = \{(a,a),(b,b),(d,d)(a,b),(b,a)\}$. Then $aR = \{a,b\} = bR$, $cR = \phi$ and $dR = \{d\}$. Hence

$$\mathrm{S} = \{\phi, \{d\}, \{a,b\}\},\ \mathrm{B} = \{X, \phi, \{d\}, \{a,b\}\},\ \tau_K = \{X, \phi, \{d\}, \{a,b\}, \{a,b,d\}\},$$

$$PO(X) = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\},$$

and

$$PC(X) = \{\phi, X, \{b,c,d\}, \{a,c,d\}, \{a,b,c\}, \{c,d\}, \{b,c\}, \{a,c\}, \{c\}, \{b\}, \{a\}\}.$$

Since $A = \{a\}$ is a nonempty p-exact proper subset of X, then the space $\kappa = (X, R, \tau_K)$ is p-rough disconnected.

Proposition 2.2. Let $\kappa = (X, R, \tau_K)$ be a p-rough disconnected topologized approximation space, then there is a nonempty p-exact proper subset of X.

Proof.

Let $\kappa = (X, R, \tau_K)$ be a p-rough disconnected topologized approximation space. Then there exist two nonempty subsets A and B of X such that
$A \cup B = X$ and $A \cap \overline{R}_p B = \overline{R}_p A \cap B = \phi$. But $A \subseteq \overline{R}_p A$, hence $A \cap B = \phi$. Thus $A = X - B$. Also $A = X - \overline{R}_p B$, since $A \cap \overline{R}_p B = \phi$ and $A \cup \overline{R}_p B \supseteq A \cup B = X$.

Hence $A = \underline{R}_p A$ and $B = \overline{R}_p B$. Similarly $B = \underline{R}_p B$ and $A = \overline{R}_p A$. Therefore there exists a nonempty p-exact proper subset A of X. □

Theorem 2.1. A topologized approximation space $\kappa = (X, R, \tau_K)$ is p-rough disconnected if and only if there exists a nonempty p-exact proper subset of X.

Proof.

By using Proposition 2.1 and Proposition 2.2, the proof is obvious. □

Definition 2.4 [3]. Let $\kappa = (X, R_1, \tau_K)$, $Q = (Y, R_2, \tau_Q)$ be two topologized approximation spaces. Then a mapping $f : \kappa \to Q$ is called p-rough continuous if $f^{-1}(\underline{R_2} V) \subseteq \underline{R_1}_p f^{-1}(V)$ for every subset V of Y in Q.

In Definition 2.4, f^{-1} does not mean the inverse function, but it means the inverse image.

Theorem 2.2. Let $f : \kappa \to Q$ be a mapping from a topologized approximation space $\kappa = (X, R_1, \tau_K)$ to a topologized approximation space $Q = (Y, R_2, \tau_Q)$. Then the following statements are equivalent.

i) f is p-rough continuous.

ii) The inverse image of each internally R_2 $-$definable set in Q is internally p-definable set in κ.

iii) The inverse image of each externally R_2 $-$definable set in Q is externally p-definable set in κ.

iv) $f(\overline{R_1}_p A) \subseteq \overline{R_2} f(A)$ for every subset A of X in κ.

v) $\overline{R_1}_p f^{-1}(B) \subseteq f^{-1}(\overline{R_2} B)$ for every subset B of Y in Q.

Proof.

(i)$\Rightarrow$(ii) Let f be p-rough continuous and let V be an internally R_2 $-$definable set in Q. Then $\underline{R_2} V = V$ and $f^{-1}(V)$ is a subset of X in κ. By (i), we get $f^{-1}(V) = f^{-1}(\underline{R_2} V) \subseteq \underline{R_1}_p f^{-1}(V)$. Then $f^{-1}(V) \subseteq \underline{R_1}_p f^{-1}(V)$. But $\underline{R_1}_p f^{-1}(V) \subseteq f^{-1}(V)$. Hence $f^{-1}(V) = \underline{R_1}_p f^{-1}(V)$. Therefore $f^{-1}(V)$ is internally p-definable set in κ.

(ii)$\Rightarrow$(i) Let A be a subset of Y in Q. Since $\underline{R_2} A \subseteq A$, then $f^{-1}(\underline{R_2} A) \subseteq f^{-1}(A)$. But $\underline{R_2} A$ is internally R_2 $-$definable set in Q, then by (ii), we get $f^{-1}(\underline{R_2} A)$ is internally p-definable set in κ contained in $f^{-1}(A)$. Hence $f^{-1}(\underline{R_2} A) \subseteq \underline{R_1}_p f^{-1}(A) \subseteq f^{-1}(A)$, since $\underline{R_1}_p f^{-1}(A)$ is the largest internally p-definable set contained in $f^{-1}(A)$. Thus $f^{-1}(\underline{R_2} A) \subseteq \underline{R_1}_p f^{-1}(A)$ for every subset A of Y in Q. Therefore f is p-rough continuous.

(ii)$\Rightarrow$(iii) Let F be an externally R_2-definable set in Q, then by Lemma 2.1, we get $Y-F$ is internally R_2-definable. Thus by (ii), we have $f^{-1}(Y-F)$ is internally p-definable set in κ.

Since $f^{-1}(Y-F)=X-f^{-1}(F)$, then $X-f^{-1}(F)$ is internally p-definable set in κ. Hence $f^{-1}(F)$ is externally p-definable set in κ.

Similarly we can prove (iii)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iv) Let A be a subset of X in κ, then $\overline{R_2}\, f(A)$ is an externally R_2-definable set in Q. Hence $Y-\overline{R_2}\, f(A)$ is internally R_2-definable set in Q. Thus by (ii), we get $f^{-1}\left(Y-\overline{R_2}\, f(A)\right)=X-f^{-1}\left(\overline{R_2}\, f(A)\right)$ is internally p-definable set in κ, and so $f^{-1}\left(\overline{R_2}\, f(A)\right)$ is externally p-definable set containing A in κ. Thus $A\subseteq \overline{R}_{1p} A\subseteq f^{-1}\left(\overline{R_2}\, f(A)\right)$, since $\overline{R}_{1p} A$ is the smallest externally p-definable set containing A in κ. Hence

$$f\left(\overline{R}_{1p} A\right)\subseteq f\left[f^{-1}\left(\overline{R_2}\, f(A)\right)\right]\subseteq \overline{R_2}\, f(A).$$

Therefore $f\left(\overline{R}_{1p} A\right)\subseteq \overline{R_2}\, f(A)$ for every subset A in κ.

(iv)$\Rightarrow$(v) Let B be a subset of Y in Q. Let $A=f^{-1}(B)$, then A is a subset of X in κ. By (iv), we get

$$f\left(\overline{R}_{1p} A\right)\subseteq \overline{R_2}\, f(A)=\overline{R_2}\, f\left(f^{-1}(B)\right)\subseteq \overline{R_2}\, B.$$

Hence $\overline{R}_{1p} A\subseteq f^{-1}\left(\overline{R_2}\, B\right)$. Thus $\overline{R}_{1p} A=\overline{R}_{1p}\, f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}\, B\right)$.

Therefore $\overline{R}_{1p}\, f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}\, B\right)$ for every subset B of Y in Q.

(v)$\Rightarrow$(ii) Let G be an internally R_2-definable set in Q, then $B=Y-G$ is externally R_2-definable set in Q. Thus by (v), we get

$$\overline{R}_{1p}\, f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}\, B\right).$$

Since B is externally R_2-definable set, then $f^{-1}\left(\overline{R_2}\, B\right)=f^{-1}(B)$. Thus $\overline{R}_{1p}\, f^{-1}(B)\subseteq f^{-1}(B)$. But $f^{-1}(B)\subseteq \overline{R}_{1p}\, f^{-1}(B)$, then $\overline{R}_{1p}\, f^{-1}(B)=f^{-1}(B)$.

Hence $f^{-1}(B)$ is externally p-definable set in κ.

Since $f^{-1}(B)=f^{-1}(Y-G)=X-f^{-1}(G)$, then $X-f^{-1}(G)$ is externally p-definable set in κ. Therefore $f^{-1}(G)$ is internally p-definable set in κ. □

Example 2.2. Let $\kappa=(X,R_1,\tau_K)$, $Q=(Y,R_2,\tau_Q)$ be two topologized approximation spaces such that $X=\{a,b,c,d\}$, $Y=\{y_1,y_2,y_3,y_4\}$,

$R_1=\{(a,a),(b,b),(d,d)(a,b),(b,a)\}$ and $R_2=\{(y_1,y_1),(y_4,y_4),(y_1,y_2)\}$. Then $\tau_K=\{X,\phi,\{d\},\{a,b\},\{a,b,d\}\}$ and $\tau_Q=\{Y,\phi,\{y_4\},\{y_1,y_2\},\{y_1,y_2,y_4\}\}$. Hence

$$PO(X)=\{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\},$$

Define a mapping $f:K\to Q$ such that

$$f(a) = y_4,\ f(b) = y_3,\ f(c) = y_1 \text{ and } f(d) = y_2.$$

Then f is not a p-rough continuous mapping, since $V = \{y_1, y_2\}$ is an internally R_2 − definable set in Q , but $f^{-1}(V) = \{c, d\}$ is not an internally p-definable set in κ.

Proposition 2.3. Let $\kappa = (X, R_1, \tau_K)$ and $Q = (Y, R_2, \tau_Q)$ be two topologized approximation spaces. If $f : \kappa \to Q$ is a p-rough continuous mapping, then the inverse image of each exact set in Q is p-exact set in κ.

Proof.

Let A be an exact set in Q , then A is both internally and externally R_2 − definable set in Q . Hence by Theorem 2.2, we get $f^{-1}(A)$ is both internally and externally p-definable set in κ. Therefore $f^{-1}(A)$ is a p-exact set in κ. □

3. Conclusions

In this paper, we used p-open sets to introduce the definition of p-rough connected to pologized approximation space.

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