Tetrabonacci Subgroup of the Symmetric Group over the Loubéré Magic Squares Semigroup

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ABSTRACT

By the Loubéré Magic Squares, we understand the set of magic squares constructed with the De La Loubéré Procedure. It is seemingly very close to triviality that this set equipped with the matrix binary operation of addition forms a semigroup if the underlining set or multi set so considered in the square is of the natural numbers. In this paper, we introduce the permutation and its composition over the Loubéré Magic Squares Semigroup logically to form a subgroup of the Symmetric Group which by analogy to the Fibonacci Group is termed the Tetrabonacci Group. We also present a new definition, a new procedure and a new generalization of the Loubéré –.

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1. INTRODUCTION

The works of [1, 2, 3] showcased that an idea of algebraic structures over the magic squares is conceived. The concepts of Fibonacci Semigroups [4] as well as the concepts of Symmetric Groups [5] are widespread in the Modern Algebra Literatures.

The underlining sequence–we consider for this work– in the aforementioned magic squares is the arithmetic sequence with unity common difference. This does not sidetrack generalization, and the generalization of the general concepts explicated in this work as we finally set a conjecture.

In the permutation of the Symmetric Group, only the final effect is recognized; but for the construction of the aforementioned subgroup, both the motion which may be considered as clockwise + NE-W-S or anticlockwise + NW-E-S Loubéré Procedures of construction – where NE stands for North-East, W stands for due West, S stands for due South, and NW and E stand for analogous – and the effect which has to do with the final result of disjoint cycles elements of the permutation. This is of double interest for magic squares by themselves are energizing and important. See the work of [6] for some of such stimulations.

With the composition of the maps on the 4 elements per the general 8 miscellany effects [7] of rotations and/or reflections of 1 Loubéré Magic Square defined on the set of their permutations forms the Tetrabonacci Group – as we call for brevity – an idea analogous to the Fibonacci Group. The reason for such a brevity is not too far.

2. PRELIMINARIES

We now present new definitions (2.3, 2.5), a new procedure (2.4) and a new generalization (2.6) for the Loubéré –.

Definition 2.1. A basic magic square of order n can be defined as an arrangement of arithmetic sequence of common difference of 1 from 1 to n^2 in an $n \times n$ square grid of cells such that every row, column and diagonal add up to the same number, called the magic sum M(S) expressed as $M(S) = \frac{n^3+n}{2}$ and a centre piece C as $C = \frac{M(S)}{n}$.

Definition 2.2. Main Row or Column is the column or row of the Loubéré Magic Squares containing the first term and the last term of the arithmetic sequence in the square.

Definition 2.3. A Loubéré Magic Square of type I is a magic square of arithmetic sequence entries such that the entries along the main column or row have a common difference and the main column or row is its central column or central row.

2.4 Loubéré Procedure (NE-W-S or NW-E-S, the cardinal points). Consider an empty $n \times n$ square of grids (or cells). Start, from the central column or row at a position $\left\lceil \frac{n}{2} \right\rceil$ where $\lceil \ \ \rceil$ is the greater integer number less than or equal to, with the number 1.

The fundamental movement for filling the square is diagonally up, right (clock wise or NE or SE) or up left (anti clock wise or NW or SW) and one step at a time. If a filled cell (grid) is encountered, then the next consecutive number moves vertically down ward one square instead. Continue in this fashion until when a move would leave the square, it moves due N or E or W or S (depending on the position of the first term of the sequence) to the last row or first row or first column or last column. Thus, the square grid of cells $[a_{ij}]_{n\times n}$ is said to be Loubéré Magic Square if the following conditions are satisfied.

i. $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = k$;

ii. $\mathrm{trace}\,[a_{ij}]_{n\times n} = \mathrm{trace}\,[a_{ij}]_{n\times n}^{T} = k$; and

iii. $a_{1,\lceil\frac{n}{2}\rceil}$, $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}$, $a_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n}$, $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}$, $a_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row.

$\lceil\ \rceil$ is the greater integer less or equal to, T is the transpose (of the square), k is the magic sum (magic product is defined analogously) usually expressed as $k = \frac{n}{2}[2a + (n-1)j]$ – from the sum of arithmetic sequence, where j is the common difference along the main column or row and a is the first term of the sequence– and $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil} = \frac{k}{n}$.

Definition 2.5. Loubéré Magic Squares of type II are magic squares constructed with Loubéré Procedure with repeating-pattern- sequence.

2.6. The Generalized 3×3 Loubéré Magic Square

Let $\mathbb{Z}$ denotes the set of integer numbers, $\overline{\vee}$ denotes the exclusive 'or' and $\vee$ denotes the inclusive 'or'. Then the general 3×3 Loubéré Magic Square is given by

$$G_{3\times 3\ L} := \left\{ \left(\begin{bmatrix} f+7d & f & f+5d \\ f+2d & f+4d & f+6d \\ f+3d & f+8d & f+d \end{bmatrix} \vee \begin{bmatrix} c+b & c-(b+d) & c+d \\ c-b+d & c & c+b-d \\ c-d & c+(b+d) & c-b \end{bmatrix} \right)^{1\ or\ C} \overline{\vee} \left(\begin{bmatrix} d & f & c \\ c & d & f \\ f & c & d \end{bmatrix} \vee \begin{bmatrix} c & f & d \\ f & d & c \\ d & c & f \end{bmatrix} \right)^{1\ or\ M\ or\ C} : c, d, f \in \mathbb{Z} \right\}$$

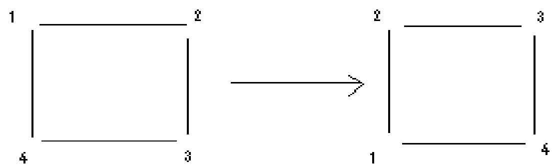
where S^M denotes the miscellany effects of rotations and\or reflections of S and S^C denotes the composition of S.

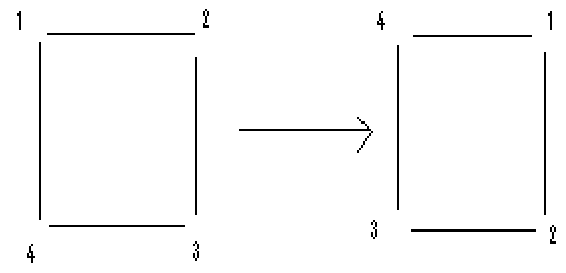
The advantage of this generalization is that it covered both miscellany effects and 9×9 Composite Loubéré –. With rotations and/or reflections, a single Loubéré Magic Square will give 7 miscellany effects, and are covered in the generalization. For the effects, see [7].

Definition 2.7. A permutation is a bijective map from a set to itself.

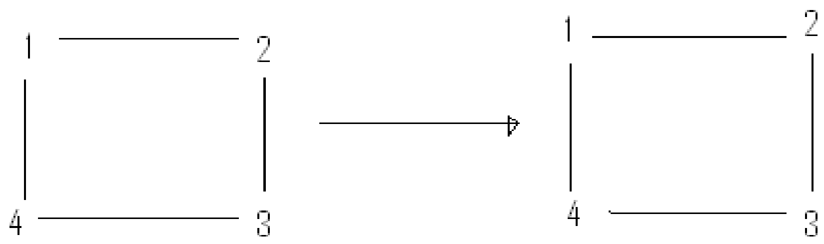
The collection of all such maps enclosed with composition of maps forms the symmetric group, always of order n factorial ($n!$) , and is called the Symmetric Group of length n. The n! order is obvious from the definition of a permutation map and a factorial function.

Example I. Consider the geometric construction of 2 per 24 elements of the Symmetric Group of length 4 and their composition as follows: Map the numbered (usually) vertices of the square to vertices of the same square as follows:





i.e. in the mapping notation as $\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$, and in the cyclic notation as $(1\ \ 2\ \ 3\ \ 4)$ and $(1\ \ 4\ \ 3\ \ 2)$ respectively.
The composition * of the two elements is
$(1\ 2\ 3\ 4)*(1\ 4\ 3\ 2)=(1\ 4\ 3\ 2)*(1\ 2\ 3\ 4)=(1)$, the identity.
This confirmed that the 2 elements are inverses of each other with the identity, (1) standing for $\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$ and for the map:



This map and its composition is imbibed and applied over the squares with sandwich of clockwise+ NE-W-S/ NW-E-S and anticlockwise+ NE-W-S/ NW-E-S Procedures.

Definition 2.8. The Fibonacci groups are defined by the presentation
$F(r,n)=\langle a_1,a_2,\dots,a_n: a_1a_2\dots a_r=a_{r+1}, a_2a_3\dots a_{r+1}=a_{r+2},\dots a_na_1\dots a_{r-1}=a_r\rangle$ where $r,n>0$ and all subscripts are reduced modulo n.

The generalization of the Fibonacci Groups (and their associated semigroups) are denoted and defined by the presentation
$F(r,n,k)=\langle a_1,a_2,\dots,a_n: a_ia_{i+1}\dots a_{i+r-1}=a_{i+k+r-1}, i=1,2,\dots,n\rangle$
where $r,n>0$ and all subscripts are assumed to be reduced modulo n. [8]

3. MAIN RESULT

We construct the Tetrabonacci Subgroup of the Symmetric Groups of length 9 in a 3×3 Loubéré Magic Square Semigroups and of length 25 in a 5×5 Loubéré Magic Square Semigroups, and we finally set a conjecture that these results can be generalized.

The Tetrabonacci Subgroups under discuss are always of order 4, and are always having the identity element, which is a basic Loubéré Magic Square with the clockwise and NE-W-S Procedures; the next element is the rotation by 90 degrees followed by a reflection along the main column, the 3rd element is a rotation of the first by 180 degrees and the last element is a rotation of the second one by 180 degrees. This process will establish the Tetrabonacci Subgroups of the Symmetric Groups of any finite length as we propose.

Examples II

1. The Tetrabonacci Subgroup of the Symmetric Group of Length 9 over the 3× 3 Loubéré Magic Squares

$$a := \begin{array}{|c|c|c|}\hline 8 & 1 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 9 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 8 & 1 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 9 & 2 \\ \hline \end{array}$$

$$b := \begin{array}{|c|c|c|}\hline 8 & 1 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 9 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 2 & 7 & 6 \\ \hline 9 & 5 & 1 \\ \hline 4 & 3 & 8 \\ \hline \end{array}$$

$$c := \begin{array}{|c|c|c|}\hline 8 & 1 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 9 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 2 & 9 & 4 \\ \hline 7 & 5 & 3 \\ \hline 6 & 1 & 8 \\ \hline \end{array}$$

$$d := \begin{array}{|c|c|c|}\hline 8 & 1 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 9 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 8 & 3 & 4 \\ \hline 1 & 5 & 9 \\ \hline 6 & 7 & 2 \\ \hline \end{array}$$

This maps 4 – where 1 is a constant (an identity map) – out of the 8 miscellany effects of rotations and/or reflections of a Loubéré Magic Square via sandwich.
$a = (1)$, $b = (1\ 7)(2\ 8)(3\ 9)$, $c = (1\ 9)(2\ 8)(3\ 7)(4\ 6)$ and $d = (1\ 3)(4\ 6)(7\ 9)$.
This is the usual cyclic notations of the permutation notations widespread in introductory abstract algebra, see also [9].

It can be laconically visualized that $d = b * c, b = c * d$ and $c = d * b$ where $*$ is a composition of map. Thus, $(\{(1), (1\ 7)(2\ 8)(3\ 9), (1\ 9)(2\ 8)(3\ 7)(4\ 6), (1\ 3)(4\ 6)(7\ 9)\}, *)$ is a Tetrabonacci Subgroup of the Symmetric Group over the Loubéré Magic Squares Semigroup.

2. The Tetrabonacci Subgroup of the Symmetric Group of Length 25 over the 5×5 Loubéré Magic Squares

$$a:=\begin{bmatrix}17&24&1&8&15\\23&5&7&14&16\\4&6&13&20&22\\10&12&19&21&3\\11&18&25&2&9\end{bmatrix}\longrightarrow\begin{bmatrix}17&24&1&8&15\\23&5&7&14&16\\4&6&13&20&22\\10&12&19&21&3\\11&18&25&2&9\end{bmatrix}$$

$$b:=\begin{bmatrix}17&24&1&8&15\\23&5&7&14&16\\4&6&13&20&22\\10&12&19&21&3\\11&18&25&2&9\end{bmatrix}\longrightarrow\begin{bmatrix}9&3&22&16&15\\2&21&20&14&8\\25&19&13&7&1\\18&12&6&5&24\\11&10&4&23&17\end{bmatrix}$$

$$c:=\begin{bmatrix}17&24&1&8&15\\23&5&7&14&16\\4&6&13&20&22\\10&12&19&21&3\\11&18&25&2&9\end{bmatrix}\longrightarrow\begin{bmatrix}9&2&25&18&11\\3&21&19&12&10\\22&20&13&6&4\\16&14&7&5&23\\15&8&1&24&17\end{bmatrix}$$

$$d:=\begin{bmatrix}17&24&1&8&15\\23&5&7&14&16\\4&6&13&20&22\\10&12&19&21&3\\11&18&25&2&9\end{bmatrix}\longrightarrow\begin{bmatrix}17&23&4&10&11\\24&5&6&12&18\\1&7&13&19&25\\8&14&20&21&2\\15&16&22&3&9\end{bmatrix}$$

$a = (1)$, the identity element,

$b = (1\ 22)(2\ 23)(3\ 24)(4\ 25)(5\ 21)(6\ 19)(7\ 20)(8\ 16)(9\ 17)(10\ 18)$,

$c = (1\ 25)(2\ 24)(3\ 23)(4\ 22)(5\ 21)(6\ 20)(7\ 19)(8\ 18)(9\ 17)(10\ 16)(11\ 15)(12\ 14)$, and

$d = (1\ 4)(2\ 3)(6\ 7)(8\ 10)(11\ 15)(12\ 14)(16\ 18)(19\ 20)(22\ 25)(23\ 24)$.

The set $\{a, b, c, d\}$ equipped with the composition of map forms the aforementioned group.

4. CONCLUSION

Conjecture 4.1. There exists a Tetrabonacci Subgroup of the Symmetric Group of any finite odd squared length l over the $\sqrt{l} \times \sqrt{l}$ Loubéré Magic Square.

The Loubéré Magic Squares equipped with the matrix binary operation of addition form a semigroup (for semigroup definition, see [10]) if the underlining set considered is the set of natural numbers. 1×1 Loubéré Magic Square is considered trivial for it is isomorphic to the underlined set or multi set.

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