

# On Homomorphism and Algebra of Functions on BE-algebras

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## Abstract

Since the introduction of the concepts of BCK and BCI algebras by K. Iseki in 1966, some more systems of similar type have been introduced and studied by a number of authors in the last two decades. K. H. Kim and Y. H. Yon studied dual BCK algebra and M.V.algebra in 2007. H. S. Kim and Y. H. Kim in 2006 have introduced the concept of BE-algebra as a generalization of dual BCK- algebra. Here we want to present the different properties of homomorphism and multiplier maps on BE-algebra.

**Key words:** BCK-algebra, BCI-algebra, BE-algebra, homomorphism, multiplier maps.

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## I. Introduction:

Since the introduction of the concepts of BCK and BCI algebras ([6,7]) by K. Iseki in 1966, some more systems of similar type have been introduced and studied by a number of authors in the last two decades. K. H. Kim and Y.H. Yon studied dual BCK algebra and M.V. algebra in 2007 ([8]). It is known that BCK-algebras is a proper subclass of BCI-algebras. There are so many generalizations of BCK/BCI-algebras, such as BCH-algebras ([10]), dual BCK-algebras ([8]),  $d$ -algebras ([5]), etc. In ([3]), H. S. Kim and Y. H. Kim introduced the concept of BE-algebra as a generalization dual BCK-algebra. S. S. Ahn and Y. H. Kim ([11]), S. S. Ahn and K. S. So ([12]) introduced the notion of ideals and upper sets in BE-algebra discussed several properties of ideals. A. Walendziak ([1]) introduced the notion of commutative BE-algebras and discussed some of its properties. H. S. Kim and K. J. Lee in ([4]) generalized the notions of upper sets and generalized upper sets and introduced the concept of extended upper sets and with the help of this concept they gave several descriptions of filters in BE-algebras. The concept of CI- algebra has been introduced by B. L. Meng in ([2]) as a generalization of BE-algebra and studied some of its important properties and relations with BE-algebras.

## II. Preliminaries:

**Definition 2.1.** Let  $(X; *, 1)$  be a system of type  $(2, 0)$  consisting of a non-empty set  $X$ , a binary operation “ $*$ ” and a fixed element  $1$ . The system  $(X; *, 1)$  is called a BE- algebra ([3]) if the following conditions are satisfied:

$$(BE 1) \quad x * x = 1$$

$$(BE 2) \quad x * 1 = 1$$

$$(BE 3) \quad 1 * x = x$$

$$(BE 4) \quad x * (y * z) = y * (x * z), \quad \forall x, y, z \in X.$$

**Note 2.1.** In any BE-algebra one can define a binary relation “ $\leq$ ” as  $x \leq y$  if and only if

$$x * y = 1, \quad \forall x, y, \in X.$$

**Example 2.1.** First of all we present a simplest example of a BE-algebra which is of much importance. Let  $X = \{0, 1\}$  and the binary operation  $*$  is defined on  $X$  by the following Cayley table

*	0	1
0	1	1
1	0	1

Then  $(X; *, 1)$  is a BE-algebra.

**Example 2.2.** Let  $\mathbb{N}$  be the set of natural numbers and let  $*$  be the binary operation defined on  $\mathbb{N}$  as follows:

$$x * y = y \text{ if } x = 1 \text{ and } 1 \text{ if } x \neq 1.$$

Then  $(\mathbb{N}; *, 1)$  is a BE-algebra.

**Lemma 2.1.** In a BE-algebra the following identities are true ([3]):

- (1)  $x * (y * x) = 1$ ;
- (2)  $x * ((x * y) * y) = 1$ .

**Definition 2.2.** Let  $(X; *, 1)$  be a BE-algebra. An element  $x \in X$  is said to commute with  $y \in X$  if  $(x * y) * y = (y * x) * x$ . If this condition is true for all  $x, y \in X$ , then  $(X; *, 1)$  is called a commutative BE-algebra ([1]).

**Definition 2.3.** A BE-algebra  $(X; *, 1)$  is said to be self distributive ([3]) if  $x * (y * z) = (x * y) * (x * z), \forall x, y, z \in X$ .

**Definition 2.4.** A mapping  $f$  from a BE– algebra  $(X; *, 1)$  to another BE– algebra

$(Y; \odot, e)$  is said to be a homomorphism  $f(x * y) = f(x) \odot f(y)$  for all  $x, y \in X$ .

### III. Main results:

**Definition 3.1.** Let  $(X; *, 1)$  be a BE-algebra and let  $F(X)$  be the set of all functions from  $X$  onto itself. Let  $e$  denote the identity function defined as  $e(x) = 1$  for all  $x \in X$ . Two functions  $f$  and  $g \in F(X)$  are said to be equal iff  $f(x) = g(x)$  for all  $x \in X$ .

Now we prove the following result:

**Theorem 3.1.** The system  $(F(X); \odot, e)$  is a BE-algebra under the operation  $\odot$  defined as follows:

$$\text{For } f \text{ and } g \in F(X), f \odot g \text{ is defined as } (f \odot g)(x) = f(x) * g(x) \text{ for all } x \in X.$$

**Proof:** (1) For every  $f \in F(X)$ , we have

$$(f \odot f)(x) = f(x) * f(x) = e(x) \text{ for all } x \in X. \text{ So } f \odot f = e$$

(2) If  $f \in F(X)$  then  $(f \odot e)(x) = f(x) * e(x) = f(x) * 1 = 1 = e(x)$  for all  $x \in X$ .

$$\text{So } f \odot e = e$$

(3) If  $f \in F(X)$  then  $(e \odot f)(x) = e(x) * f(x) = 1 * f(x) = f(x)$  for all  $x \in X$ .

$$\text{So } e \odot f = f.$$

(4) Let  $f, g, h \in F(X)$ . We have

$$\begin{aligned} (f \odot (g \odot h))(x) &= f(x) * (g(x) * h(x)) \\ &= g(x) * (f(x) * h(x)) \end{aligned}$$

$$= (g \odot (f \odot h))(x) \text{ for all } x \in X.$$

$$\text{So } f \odot (g \odot h) = g \odot (f \odot h)$$

Hence  $F(X)$  is a BE-algebra.

In view of the above result we can obtain the following similar result:

**Theorem 3.2.** If  $(X; *, 1)$  and  $(Y; \odot, e)$  be two BE-algebras and if  $\text{Hom}(X, Y)$  be the set of all homomorphisms from  $X$  to  $Y$  then  $\text{Hom}(X, Y)$  is a BE-algebra with the operations defined in the previous theorem.

Next we study examples and properties of multiplier maps.

First of all we prove the following result:

**Lemma 3.1.** Let  $(X; *, 1)$  be a BE-algebra and let  $f: X \rightarrow X$  be such that  $f(x * y) = f(x) * y$  for all  $x, y \in X$ . Then  $f$  is the identity mapping on  $X$ .

**Proof:** We have  $x * 1 = 1$ , for all  $x \in X$ .

$$\text{This gives } f(1) = f(x * 1) = f(x) * 1 = 1$$

$$\text{Again } 1 * x = x. \text{ This gives } f(1 * x) = f(x)$$

$$\Rightarrow f(1) * x = f(x)$$

$$\Rightarrow 1 * x = f(x)$$

$$\Rightarrow x = f(x) \text{ for all } x \in X.$$

Hence  $f$  is the identity function.

In view of the above result a multiplier map on a BE-algebra is defined as follows:

**Definition 3.2.** A map  $f: X \rightarrow X$  is called a multiplier if  $f(x * y) = x * f(y)$  for all  $x, y \in X$ .

**Example 3.1.** Let  $(X; *, 1)$  be a BE-algebra. For a fixed element  $a \in X$ , let  $f_a(x) = a * x$  for all  $x \in X$ . Then  $f_a(x * y) = a * (x * y) = x * (a * y) = x * f_a(y)$

So  $f_a$  is a multiplier.

$$\text{Also } f_a(1) = a * 1 = 1$$

**Example 3.2.** Let  $X = \{1, a, b, c\}$  and the operation  $*$  is defined as follows:

	1	a	b	c
*				
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then  $(X; *, 1)$  be a BE-algebra. Let us define a map  $f: X \rightarrow X$  by

$$f(x) = 1 \text{ if } x = 1, a, c \text{ and } f(x) = a \text{ if } x = b$$

Then  $f$  is a multiplier of  $X$ .

**Example 3.3.** The identity mapping  $e(x) = x$  for all  $x \in X$  and the constant unit mapping  $f(x) = 1$  for all  $x \in X$  are multiplier maps.

**Lemma 3.2.** If  $f$  is a multiplier on a BE-algebra  $X$ , then  $f(1) = 1$ .

**Proof:** Since  $f$  is a multiplier map, we have

$$f(x * y) = x * f(y) \text{ for all } x, y \in X.$$

Putting  $f(1)$  for  $x$  and  $1$  for  $y$ , we get

$$f(f(1) * 1) = f(1) * f(1) = 1 \text{ i.e. } f(1) = 1$$

**Lemma 3.3.** Let  $(X; *, 1)$  be a BE-algebra and let  $f: X \rightarrow X$  be a multiplier. Then

- (1)  $x \leq f(x)$ , for all  $x \in X$ .
- (2)  $x \leq y \Rightarrow x \leq f(y)$ , for all  $x, y \in X$ .

**Proof:** (1) We have  $x * x = 1$ , for all  $x \in X$ . So  $f(x * x) = f(1)$

$$\Rightarrow x * f(x) = 1$$

$$\Rightarrow x \leq f(x)$$

(2) We have  $x \leq y \Rightarrow x * y = 1$

$$\Rightarrow f(x * y) = f(1)$$

$$\Rightarrow x * f(y) = 1$$

$$\Rightarrow x \leq f(y)$$

**Definition 3.3.** Let  $X$  be a BE-algebra. Then the composition of two maps  $f, g: X \rightarrow X$  denoted by  $f \circ g: X \rightarrow X$  is defined as  $(f \circ g)(x) = f(g(x))$ , for all  $x \in X$ .

**Proposition 3.1.** Composite of two multiplier maps is a multiplier.

**Proof:** Let  $f$  and  $g$  be two multiplier maps on a BE-algebra  $(X; *, 1)$ . Then for  $x, y \in X$ ,

$$\text{we have } (f \circ g)(x * y) = f(g(x * y))$$

$$= f(x * g(y))$$

$$= x * f(g(y))$$

$$= x * (f \circ g)(y)$$

So fog is a multiplier map.

**Definition 3.4.** If  $f$  and  $g$  be two self maps on a BE-algebra  $(X; *, 1)$ , then  $f + g$  is defined as

$$(f + g)(x) = f(x) + g(x) = (g(x) * f(x)) * f(x)$$

**Definition 3.5.** A non-empty subset  $S$  of a BE-algebra  $X$  is said to be a sub algebra ([3,2]) if  $x * y \in S, \forall x, y, \in X$

Kim[9] has established the following results:

**Theorem 3.3.** Let  $(X; *, 1)$  be a BE-algebra.

- (a) If  $f_1$  and  $f_2$  are two multiplier maps then so is  $f_1 + f_2$ .
- (b) Let  $f$  be a multiplier on  $X$  and  $F = \{x \in X \mid f(x) = x\}$ . then  $F$  is a sub algebra of  $X$ .
- (c) If  $f$  be a multiplier on  $X$  and  $x \in F$ , then  $x + y \in F$ , for every  $y \in X$ .

Next we prove the following result:

**Theorem 3.4.** Let  $X$  be a commutative BE-algebra and  $f$  be a multiplier map. If  $x \in F$  and

$x \leq y$ , then  $y \in F$  and  $f(y) \in F$ .

**Proof:** Let  $X$  be a commutative BE-algebra and  $x \in F, x \leq y$ . We have

$$\begin{aligned} f(y) &= f(1 * y) = f((x * y) * y) \\ &= f((y * x) * x) \\ &= (y * x) * f(x) \\ &= (y * x) * x \\ &= (x * y) * y \\ &= 1 * y \\ &= y \end{aligned}$$

So  $y \in F$ . Again from Lemma 3.3. , we have  $x \leq f(y)$ . So  $f(y) \in F$ .

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