Numerical Solution of Second Order Fuzzy Differential Equations by Leapfrog Method

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***Abstract*— In this paper numerical solution of second order linear fuzzy differential equations [15] using Leapfrog method is considered. The obtained discrete solutions using Leapfrog method are compared with the exact solutions of the second order fuzzy differential equations and single-term Haar wavelet series (STHWS) method [15]. Tables and graphs are presented to show the efficiency of this method. This Leapfrog method can be easily implemented in a digital computer and the solution can be obtained for any length of time.**

***Keywords*— Haar wavelets, Single-term Haar wavelet series, Leapfrog method, Ordinary differential equations, Fuzzy differential equations.**

I. Introduction

Fuzziness is a basic type of uncertainty in real world. For a deterministic differential equation, the easiest way to introduce fuzziness is to assume that the initial value is a fuzzy variable. Recently a great deal of interest has been focused on the application of Leapfrog method to solve a wide variety of stochastic and deterministic problems [14,16].

In science and engineering, second order fuzzy differential equations often have to be solved [12, 15]. Although some cases can be solved analytically, the majority of second order fuzzy differential equations are too complicated to have analytical solutions. Even when analytical solutions can be found, they are not always useful in practice since the computational cost involved is very high. Many of the real world problems that arise in the studies of mechanical vibrations, electrical circuits, planetary motions, etc., can be formulated as second order fuzzy differential equations.

In this article we developed numerical methods for addressing second order fuzzy differential equations by an application of the Leapfrog method which was studied by S. Sekar and team of his researchers [10, 14, 16]. We refer [1-9, 11, 13] for the numerical treatment of fuzzy differential equations. Hence, we use this Leapfrog method in the present paper to study second order fuzzy differential equations with initial conditions. The organized paper is as follows: In Section 2, the Leapfrog method for solving second order fuzzy differential equations is introduced. In Section 3 presents general form of second order fuzzy differential equations. In Section 4, the Leapfrog and STHWS [15] method for solving second order fuzzy differential equations is solved.

II. Leapfrog Method

The most familiar and elementary method for approximating solutions of an initial value problem is Euler's Method. Euler's Method approximates the derivative in the form of $y' = f(t, y),\ y(t_0) = y_0,\ y \in R^d$ by a finite difference quotient $y'(t) \approx \left(y(t+h) - y(t)\right)/h$. We shall usually discretize the independent variable in equal increments:

$$t_{n+1} = t_n + h,\ n = 0,1,\ldots,t_0 .$$

Henceforth we focus on the scalar case, N = 1. Rearranging the difference quotient gives us the corresponding approximate values of the dependent variable:

$$y_{n+1} = y_n + hf(t_n, y_n),\ n = 0,1,\ldots,t_0$$

To obtain the leapfrog method, we discretize t_n as in $t_{n+1} = t_n + h,\ n = 0,1,\ldots,t_0$, but we double the time interval, h, and write the midpoint approximation $y(t+h) - y(t) \approx hy'\left(t + \frac{h}{2}\right)$ in the form

$$y'(t+h) \approx (y(t+2h) - y(t))/h$$

and then discretize it as follows:

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,\ldots,t_0$$

The leapfrog method is a linear *m* = 2-step method, with $a_0 = 0, a_1 = 1, b_{-1} = -1, b_0 = 2$ and $b_1 = 0$. It uses slopes evaluated at odd values of *n* to advance the values at points at even values of *n*, and vice versa, reminiscent of the children's game of the same name. For the same reason, there are multiple solutions of the leapfrog method with the same initial value $y = y_0$.

This situation suggests a potential instability present in multistep methods, which must be addressed when we analyze them—two values, y_0 and y_1 , are required to initialize solutions of $y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,\ldots,t_0$ uniquely, but the analytical problem $y' = f(t,y),\ y(t_0) = y_0,\ y \in R^d$ only provides one. Also for this reason, one-step methods are used to initialize multistep methods.

III. Second Order Fuzzy Differential Equations

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$ for all r ∈ [0, 1], which satisfy the following requirements [O. Kaleva (1990)]:

(i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1],

(ii) $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0, 1],

(iii) $\underline{u}(r) \le \overline{u}(r)$ ∀r ∈[0, 1],

Let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α-level intervals.

Lemma

Let $[\underline{v}(\alpha), \overline{v}(\alpha)]$, α ∈(0, 1] be a given family of non-empty intervals. If

(i) $[\underline{v}(\alpha), \overline{v}(\alpha)] \supset [\underline{v}(\beta), \overline{v}(\beta)]$ for 0 < α ≤ β,

and

(ii) $[\lim_{k\to\infty} \underline{v}(\alpha_k), \lim_{k\to\infty} \overline{v}(\alpha_k)] = [\underline{v}(\alpha), \overline{v}(\alpha)]$,

whenever (α_k) is a non-decreasing sequence converging to α ∈(0, 1], then the family $[\underline{v}(\alpha), \overline{v}(\alpha)]$, α ∈(0, 1], represent the α-level sets of a fuzzy number v in E.

Conversely if $[\underline{v}(\alpha), \overline{v}(\alpha)]$, α ∈(0, 1], are α-level sets of a fuzzy number v ∈E, then the conditions (i) and (ii) hold true.

Definition

Let I be a real interval. A mapping v : I ⟶ E is called a fuzzy process and we denoted the α-level set by $[v(t)]_\alpha = [\underline{v}(t,\alpha), \overline{v}(t,\alpha)]$. The Seikkala derivative $v'(t)$ of v is defined by $[v'(t)]_\alpha = \left[\underline{v}'(t,\alpha), \overline{v}'(t,\alpha)\right]$, provided that is a equation defines a fuzzy number $v'(t)$ ∈E.

Definition

Suppose u and v are fuzzy sets in E. Then their Hausdroff D : E × E ⟶ R_+ ∪ {0},

$$D(u,v) = \sup_{\alpha\in[0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\},$$

i.e.,D(u, v) is maximal distance between α-level sets of u and v.

In this section, we study the fuzzy initial value problem for a second-order linear fuzzy differential equation.

$$\left.\begin{array}{l} x''(t)+a(t)x'(t)+b(t)x(t)=\omega(t), \\ x(0)=c_1, \\ x'(0)=c_2, \end{array}\right\} \qquad (1)$$

where $c_1, c_2 \in R_f, a(t), b(t), \omega(t) \in R$. In this paper, we suppose $a(t), b(t) > 0$. Our strategy of solving (1) is based on the selection of derivative type in the fuzzy differential equation. We first give the following definition for the solutions of (1).

Definition

Let $x:[a,b] \to R_f$ be fuzzy-valued function and n,m = 1,2. One says x is an (n,m)-solution for problem (4.1). If $D_n^{(1)}x(t), D_{n,m}^{(2)}x(t)$ exist and $D_{n,m}^{(2)}x(t)+a(t)D_n^{(1)}x(t)+b(t)x(t)=\omega(t), x(0)=c_1, D_n^{(1)}x(0)=c_2$.

IV. Numerical examples

In this section, two examples are presented for second order linear fuzzy differential equations. Numerical solutions are obtained using two methods like Leapfrog method and single term Haar wavelet series method (STHWS) [S. Sekar and S. Senthilkumar (2014)].

The exact solutions and approximated solutions obtained by Leapfrog method and STHWS method. To show the efficiency of the Leapfrog method, we have considered the following problem taken from [15], with step size $t = 0.1$ along with the exact solutions.

The discrete solutions obtained by the two methods, Leapfrog method and the STHWS methods; the absolute errors between them are tabulated and are presented in Table 1 - 2. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of "*t*" and are presented in Fig. 1 to Fig. 4 for the following problem, using three dimensional effects.

Example 4.1

Consider the following second-order fuzzy differential equation with fuzzy initial value is given by S.Sekar and S. Senthilkumar (2014).

$$\left.\begin{array}{l} y''(t)=-y(t), \quad (t \ge 0) \\ y(0)=0 \\ y'(0)=[0.9+0.1r,\, 1.1-0.1r] \end{array}\right\}$$

The exact solution is as follows:

$$Y(t,r)=[(0.9+0.1r)\sin(t), (1.1-0.1r)\sin(t)]$$

Example 4.2

Consider the following second-order fuzzy differential equation with fuzzy initial value is given by S.Sekar and S. Senthilkumar (2014).

$$\left.\begin{array}{l} y''(t)=-y(t)+t, \quad (t \ge 0) \\ y(0)=[0.9+0.1r,\, 1.1-0.1r] \\ y'(0)=[1.8+0.2r,\, 2.2-0.2r] \end{array}\right\}$$

The exact solution under (1)-differentiability: $Y(t)=[Y_1(t,r), Y_2(t,r)]$ where

$$Y_1(t,r)=\left(\frac{4}{5}+\frac{1}{5}r\right)\sin(t)+\left(\frac{9}{10}+\frac{1}{10}r\right)\cos(t)+t$$

$$Y_2(t,r)=\left(\frac{6}{5}-\frac{1}{5}r\right)\sin(t)+\left(\frac{11}{10}-\frac{1}{10}r\right)\cos(t)+t$$

TABLE I

r	Example 4.1 Error Calculation				Example 4.2 Error Calculation			
	$y_1(t,r)$		$y_2(t,r)$		$y_1(t,r)$		$y_2(t,r)$	
	STHWS	**Leapfrog**	**STHWS**	**Leapfrog**	**STHWS**	**Leapfrog**	**STHWS**	**Leapfrog**
0	1.01E-12	1.12E-14	1.11E-12	1.11E-14	2.18E-10	2.18E-12	3.06E-10	3.06E-12
0.1	1.02E-12	1.52E-14	1.10E-12	1.10E-14	2.19E-10	2.19E-12	3.05E-10	3.05E-12
0.2	1.03E-12	1.83E-14	1.09E-12	1.09E-14	2.20E-10	2.20E-12	3.04E-10	3.04E-12
0.3	1.04E-12	1.14E-14	1.08E-12	1.08E-14	2.21E-10	2.21E-12	3.03E-10	3.03E-12
0.4	1.05E-12	1.55E-14	1.07E-12	1.07E-14	2.22E-10	2.22E-12	3.02E-10	3.02E-12
0.5	1.06E-12	1.78E-14	1.06E-12	1.06E-14	2.23E-10	2.23E-12	3.01E-10	3.01E-12
0.6	1.07E-12	1.96E-14	1.05E-12	1.05E-14	2.24E-10	2.24E-12	2.99E-10	2.99E-12
0.7	1.08E-12	1.44E-14	1.04E-12	1.04E-14	2.25E-10	2.25E-12	2.98E-10	2.98E-12
0.8	1.09E-12	1.78E-14	1.03E-12	1.03E-14	2.26E-10	2.26E-12	2.97E-10	2.97E-12
0.9	1.10E-12	1.10E-14	1.02E-12	1.02E-14	2.27E-10	2.27E-12	2.96E-10	2.96E-12
1	1.11E-12	1.13E-14	1.01E-12	1.01E-14	2.28E-10	2.28E-12	2.95E-10	2.95E-12

V. CONCLUSIONS

The obtained results (approximate solutions) of the second order linear fuzzy differential equation [15] show that the Leapfrog method works well for finding the state vector. The efficiency and the accuracy of the Leapfrog method have been illustrated by suitable examples. From the Table 1, it can be observed that for most of the time intervals, the absolute error is less in the Leapfrog method when compared to the single term Haar wavelet series method [15], which yields a small error, along with the exact solutions. From the Figures 1 – 4, it can be predicted that the Leapfrog method solution match well to the problem when compared to the single term Haar wavelet series method. Hence the Leapfrog method is more suitable for studying the second order linear fuzzy differential equation.

The researcher has successfully introduced Leapfrog method which has been exclusively developed for solving second order linear fuzzy differential equation. Hence, it can be said that the Leapfrog method is more suitable to study the second order linear fuzzy differential equation.

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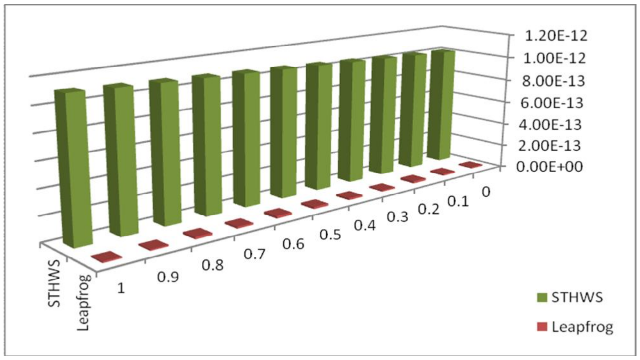

Fig. 1 Error estimation of Example 4.1 at y_1

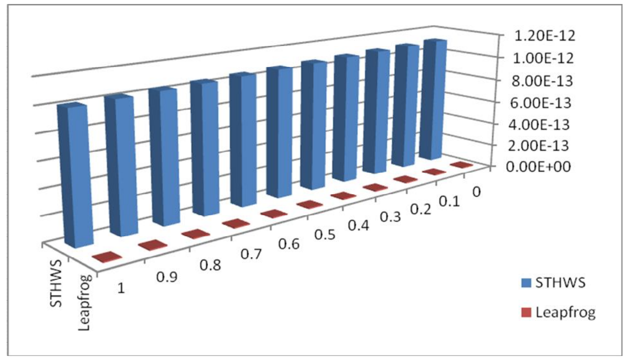

Fig. 2 Error estimation of Example 4.1 at y_2

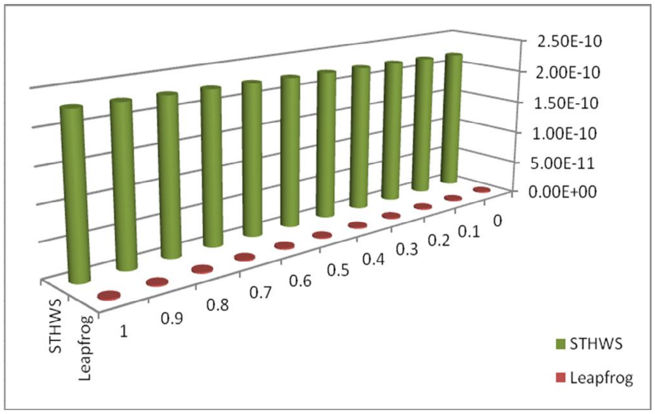

Fig. 3 Error estimation of Example 4.2 at y_1

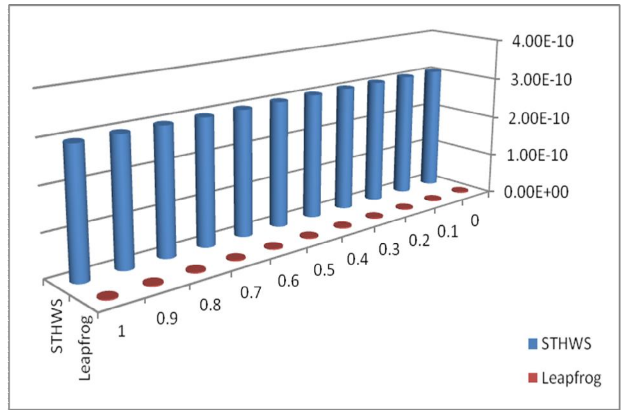

Fig. 4 Error estimation of Example 4.2 at y_2

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