(n , t)-Presented Modules and (n, t)-Coherent Rings

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Abstract: **Coherent rings were first introduced by Chase as a generalization of right Noetherian rings. His characterization has led to several generalizations of this concept. n-coherent rings have been studied by many authors like Glaz S. (1989), Costa (1994), Chen and Ding (1996),Weimin Xue (1999), Dexu Zhou (2004), Zhanmin Zhu(2011) . Some authors have also studied the notion of coherence with respect to a particular torsion theory. Wurfel (1973) characterizes rings which are coherent relative to a faithful hereditary torsion theory and M. F. Jones (1982) extended Wurfel's result to a setting of an arbitrary hereditary torsion theory and used her result to generalize Chase's theorem to the same setting.**

We introduce the notions of (n, t)-presented modules and (n, t)-coherent rings. Proposition 1.1 of Zhanmin Zhu (2011) and some results obtained by Chen and Ding (1996), are generalized to the setting of an hereditary torsion theory.

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I. INTRODUCTION

Our ring R will be associative with identity and modules unitary right R-modules unless otherwise stated.

An R-module M is saidto be finitely presented (f.p.) if there exists an exact sequence $F_1 \to F_0 \to \mathrm{M} \to 0$ with each F_i finitely generated and free. These modules have been extensively studied and various generalizations of these modules obtained. We will focus on two:

An R-module is said to be n-presented for some positive integer n if there exists an exact sequence $F_n \overset{d_n}{\to} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_0 \to M \to 0$ with each F_i finitely generated and free, see [7] and [16].

Let (**T, F**) be an hereditary torsion theory with associated radical t. An R-module M is said to be t-finitely generated (t-f.g.) if there exists a finitely generated sub module N of M such that $M/N \in \boldsymbol{T}$. An R-module M is t-finitely presented (t-f.p.) if there exists an exact sequence $0 \to K \to F \to M \to 0$ with F finitely generated and free and K t-finitely generated, see [12].

In section II, we define (n,t)-presented modules. An R-module is said to be (n,t)-presented if there exists an exact sequence of R-modules $F_n \overset{d_n}{\to} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_0 \to M \to 0$ with each F_i t-f.g. and free for some positive integer n and torsion radical t associated with some hereditary torsion theory. We use this definition to generalize proposition 1.1 of [16]. The main results of this section is Theorem 3, where we show that for an exact sequence of R-modules $0 \to A \to B \to C \to 0$, imposing conditions on any two of the modules in the exact sequence, we can obtain corresponding results on the third.

In section III, (n, t)-flat modules and (n, t)-FP-injective modules are introduced. A module M is said to be (n, t)-flat if Tor_n^R (M, A) = 0 for every left (n, t)-presented module A. A module M is said to be (n, t)-FP-injective if Ext_R^n (A, M) = 0 for every (n, t)-presented R-module A. These definitions are used to generalize some results obtained by J. Chen and N. Ding in [7].

(n, t)-coherent rings and some characterizations of these rings are studied in section IV. Also, we define the (n, t)-FP-injective dimensions of a module which in some sense "measures" how far a module is from being (n, t)-FP-injective.

II. (N, T)-PRESENTED MODULES

Definition 1: Let (**T, F**) be a hereditary torsion theory with corresponding torsion radical t. A right R-module M is said to be (n, t)- presented if there exists an exact sequence of right R-modules

$$F_n \overset{d_n}{\to} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \overset{d_0}{\to} \mathrm{M} \to 0$$

with each F_i t-finitely generated and free.

Remark 1:
(i) If M is n-presented then it is (n, t)-presented since any f.g. module is t-f.g.
(ii) If M is t-presented then it is (1, t) presented.

We begin the study of the properties of (n, t)-presented modules with the following lemmas:

Lemma 1: (Lemma 2.1, [12])
1. Any homomorphic image of a t-f.g. module is t-f.g.
2. A direct summand of a t-f.g. module is t-f.g.
3. A direct summand of t-f.p. module is t-f.p.

Lemma 2:(Proposition 2.2, [12])

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of R-modules.

1. If A and C are t-f.g., then B is t-f.g.
2. If A and C are t-f.p., then B is t-f.p.
3. If B is f.g. and C is t-f.p., then A is t-f.p.

Theorems 1 and 2 below generalizes Proposition 1.1 of Zhanmin Zhu (2011) with the assumption that every t-finitely generated module is a homomorphic image of a t-finitely generated free module.

Theorem 1: Let (T, F) be a hereditary torsion theory with corresponding torsion radical t. Assume that every t-f.g. R-module is a homomorphic image of a t-f.g. free R-module. Then the following are equivalent for a right R-module M:

1. M is (n, t)- presented.
2. There exists an exact sequence of right R-modules $0 \to K_n \to F_{n-1} \xrightarrow{d_{n-1}} \cdots F_0 \xrightarrow{d_0} M \to 0$ with each F_i t-f.g. and free and K_n t-f.g.
3. There exists an exact sequence of R-modules $0 \to K_n \to P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ with each P_i t-f.g. and projective and K_n t-f.g.
4. There exists an exact sequence of R-modules $P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ with each P_i t-f.g. and projective .

Proof 1: 1 $\Leftrightarrow$ 2 Suppose M is (n, t)-presented. This implies there exists an exact sequence $F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ with each F_i t-f.g. and free. Let $K_n = Im\, d_n = Ker\, d_{n-1}$. Then K_n is t-f.g. since a homomorphic image of a t-f.g. module is t-f.g. by Lemma 1. Hence we have the exact sequence $0 \to K_n \to F_{n-1} \xrightarrow{d_{n-1}} \cdots F_0 \xrightarrow{d_0} M \to 0$ with each F_i t-f.g. and free and K_n t-f.g.

Conversely, suppose there exists an exact sequence $0 \to K_n \xrightarrow{\alpha} F_{n-1} \xrightarrow{d_{n-1}} \cdots F_0 \xrightarrow{d_0} M \to 0$ with each F_i t-f.g. and free and K_n t-f.g. K_n t-f.g. implies there exists an exact sequence $F_n \xrightarrow{\beta} K_n \to 0$ with F_n t-f.g. and free by hypothesis. Combining these two exact sequences we have

$F_n \xrightarrow{\beta\alpha} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ with each F_i t-f.g. and free and thus M is (n, t)-presented.

2 $\Leftrightarrow$ 3 Suppose (2) holds. Since every free module is projective (3) follows.

Conversely, suppose (3). We use induction on n. When $n = 1$ we have the exact sequence $0 \to K_1 \xrightarrow{\pi} P_0 \to 0$ with P_0 t-f.g. and projective and K_1 t-f.g. By hypothesis, we have the exact sequence $F_0 \xrightarrow{\theta} P_0 \to 0$ with F_0 t-f.g. and free. Consequently we have the exact sequence $0 \to K \xrightarrow{\pi\theta} F_0 \to 0$ where $K = Ker\, \pi\theta$. By Schanuel's Lemma, $K \oplus P_0 \cong K_1 \oplus F_0$. F_0 and K_1 t-f.g. implies $K_1 \oplus F_0$ is t-f.g. by Lemma 2. Thus $K \oplus P_0$ is t-f.g. and therefore K is t-f.g. as a direct summand of a t-f.g. module. Theorem therefore holds when $n = 1$. Assume theorem is true for $n - 1$ and consider the exact sequence $0 \to K_n \to P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ with each P_i t-f.g. and projective and K_n t-f.g. Then we have the exact sequence $0 \to Im\, d_{n-1} \to P_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ with each P_i t-f.g. and projective and $Im\, d_{n-1}$ t-f.g. as a homomorphic image P_{n-1}. By induction hypothesis we have the exact sequence $0 \to K_{n-1} \xrightarrow{i} F_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ with each F_i t-f.g. and free and K_{n-1} t-f.g. K_{n-1} t-f.g. implies there exists an exact sequence $F_{n-1} \xrightarrow{\theta} K_{n-1} \to 0$ with F_{n-1} t-f.g. and free. Let $K = Ker\, i\theta$. Then we have the exact sequence $0 \to K \to F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$. By a generalization of Schanuel's Lemma, K is t-f.g. and (2) follows.

3 $\Leftrightarrow$ 4

Suppose there exists an exact sequence of modules

$0 \to K_n \xrightarrow{\alpha} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ with each P_i t-f.g. and projective and K_n t-f.g. Then we have the epimorphism $F_n \xrightarrow{\theta} K_n \to 0$ with F_n t-f.g. and free. F_n is projective since every free module is projective. Combining these two sequences we have $P_n \xrightarrow{\alpha\theta} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ where $P_n = F_n$ and each P_i t-f.g. and projective.

Conversely, suppose there exists an exact sequence

$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ with each P_i t-f.g. and projective . Then we have the exact sequence $0 \to Im\, d_n \to P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ with each P_i t-f.g. and projective and $Im\, d_n$ t-f.g. as a homomorphic image of the t-f.g. module F_n. ■

Theorem 2: Let (**T**, **F**) be a hereditary torsion theory with corresponding torsion radical t. Assume that every t-f.g. R-module is a homomorphic image of a t-f.g. free R-module. Then the following are equivalent for a right R-module M:

1. M is (n, t)-presented.
2. There exists an exact sequence of right R-modules

$0 \to K \to F \to M \to 0$ such that F is t-f.g. and free and K is (n-1, t)-presented.

3. M is t-f.g. and if the sequence of right R-modules $0 \to L \to F \to M \to 0$ is exact with F t-f.g. and free, then L is (n-1, t)-presented.

Proof 2: 1 ⟺ 2: Suppose M is (n,t)-presented. Then by Theorem 1, there exists an exact sequence $0 \to K_n \to F_{n-1} \xrightarrow{d_{n-1}} \cdots F_0 \xrightarrow{d_0} M \to 0$ with each F_i t-f.g. and free and K_n t-f.g. Let $K = \mathrm{Im}\, d_1$. Then we have the exact sequences $0 \to K_n \to F_{n-1} \xrightarrow{d_{n-1}} \cdots F_1 \xrightarrow{d_1} K \to 0$ and $0 \to K \to F_0 \to M \to 0$ and (2) holds.

Conversely, suppose (2) holds. Then we have the exact sequences

$0 \to K_{n-1} \to F_{n-2} \xrightarrow{d_{n-1}} \cdots F_0 \xrightarrow{d_0} K \to 0$ and

$0 \to K \xrightarrow{\beta} F \to M \to 0$. From these sequences we can construct the exact sequence

$0 \to K_{n-1} \to F_{n-2} \xrightarrow{d_{n-1}} \cdots F_0 \xrightarrow{\beta d_0} F \to M \to 0$ where $F, F_i, i = 0, 1, \cdots n-2$ are t-f.g. and free and K_{n-1} is t-f.g. Thus M is (n, t)-presented.

2 ⟺ 3: Suppose (2) holds. Then we have the exact sequence $0 \to K \to F \to M \to 0$ with F t-f.g. and free and K is (n-1, t)-presented. M is t-f.g. as a homomorphic image of F. Suppose there exists an exact sequence $0 \to L \to F' \to M \to 0$ with F' t-f.g. and free. Then by Schanuel's Lemma for projectives, $K \oplus F' \cong L \oplus F$. Hence L is (n-1, t)-presented.

Conversely, suppose (3) holds. Then M is t-f.g. By hypothesis, there exists an exact sequence $F \xrightarrow{\beta} M \to 0$ with F t-f.g. and free. Let $K = Ker\, \beta$. Then the sequence $0 \to K \to F \to M \to 0$ is exact with F t-f.g. and free. By (3) K is (n-1, t)-presented and hence (2) holds. ■

Theorem 3: Let (**T, F**) be an hereditary torsion theory with corresponding radical t. Let $n \geq 0$ and $0 \to A \to B \to C \to 0$ be an exact sequence of modules. Then

1. If both A and C are (n, t)-presented, then B is also (n, t)-presented.
2. If A is (n-1, t)-presented and B is (n, t)-presented, then C is (n, t)-presented.
3. If every t-f.g. module is a homomorphic image of a t-f.g. free module, B is (n, t)-presented and C is (n+1, t)-presented, then A is (n, t)-presented.
4. If $B = A \oplus C$ and every t-f.g. module is a homomorphic image of a t-f.g. free module, then B is (n, t)-presented if and only if A and C are (n, t)-presented.

Proof 3:

1. Suppose A and C are (n, t)-presented. Using the Horse Shoe Lemma for projectives, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \vdots & & \vdots & & \vdots & \\
0 \to & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & A & \to & B & \to & C & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

where $B_i = A_i \oplus C_i$ for each i. Also, B_i is free as a direct sum of two free modules. Moreover for each exact sequence $0 \to A_i \to B_i \to C_i \to 0$, B_i is t-finitely generated since A_i and C_i are t-finitely generated by Lemma 2. Hence B is (n, t)-presented.

2. Suppose A is (n-1, t)-presented and B is (n, t)-presented. We show that C is (n, t)-presented. B (n, t)-presented implies there exists an exact sequence $B_n \to B_{n-1} \to \cdots \to B_0 \to B \to 0$ with each B_i projective and t-f.g. From this sequence we obtain the exact sequence $0 \to K \to B_0 \to B \to 0$, where $K = Ker\,(B_0 \to B)$. Moreover, K is (n-1, t)-presented from the exact sequence $B_n \to B_{n-1} \to \cdots \to B_1 \to K \to 0$. We can then construct the following pull-back diagram.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & K & = & K & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & D & \to & B_0 & \to & C & \to 0 \\
 & \downarrow & & \downarrow & & \parallel & \\
0 \to & A & \to & B & \to & C & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}$$

A and K are (n-1, t)-presented and hence D is (n-1, t)-presented by part (1). Thus we have the exact sequence $D_{n-1} \to D_{n-2} \to \cdots \to D_0 \to D \to 0$ with each D_i projective and t-f.g. Combining this sequence with the exact sequence $0 \to D \to B_0 \to C \to 0$ we obtain the exact sequence $D_{n-1} \to D_{n-2} \to \cdots \to D_0 \to B_0 \to C \to 0$ with B_0 and each D_i projective and t-f.g. Thus C is (n, t)-presented.

3. Suppose B is (n, t)-presented and C is (n+1, t)-presented. We show that A is (n, t)-presented. C (n+1, t)-presented implies there exists an exact sequence $C_{n+1} \to C_n \to \cdots \to C_0 \to C \to 0$ where each C_i is projective and t-f.g.. We obtain the exact sequence $0 \to K \to C_0 \to C \to 0$ where $K = \mathrm{Ker}\,(C_0 \to C)$. More over, K is (n, t)-presented because of the exact sequence $C_{n+1} \to C_n \to \cdots \to C_1 \to K \to 0$. We thus have the following pull-back commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \to & D & \to & C_0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By part (1), B and K (n, t)-presented implies D is (n, t)-presented. Hence we have the exact sequence $D_n \to D_{n-1} \to \cdots \to D_0 \to D \to 0$ where each D_i is projective and t-f.g. From this sequence we obtain the exact sequence $0 \to L \to D_0 \to D \to 0$ where $L = \mathrm{Ker}\,(D_0 \to D)$ and L is (n-1, t)-presented from exact sequence $D_n \to D_{n-1} \to \cdots \to D_1 \to L \to 0$. From diagram, the sequence $0 \to A \to D \to C_0 \to 0$ is split exact since C_0 is projective. Thus $D = A \oplus C_0$ and the sequence $0 \to C_0 \to D \to A \to 0$ is exact. We can then form the commutative pull-back diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & E & \to & D_0 & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & C_0 & \to & D & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

L and C_0 are (n-1, t)-presented (C_0 is t-f.g. and hence (n-1, t)-presented by hypothesis) and hence from part (1), E is (n-1, t)-presented. This gives rise to the exact sequence $E_{n-1} \to E_{n-2} \to \cdots \to E_0 \to E \to 0$ with each E_i projective and t-f.g. Combining this exact sequence and the sequence $0 \to E \to D_0 \to A \to 0$ we obtain the exact sequence $E_{n-1} \to E_{n-2} \to \cdots \to E_0 \to D_0 \to A \to 0$ with D_0 and each E_i projective and t-f.g. Thus A is (n, t)-presented.

4. If A and C are (n, t)-presented, then B is (n, t)-presented by part (1).
Conversely, suppose B is (n, t)-presented. Then we have the exact Sequence $B_n \to B_{n-1} \to \cdots \to B_0 \to B \to 0$ with each B_i t-f.g. and free. By Lemma 1, B is t-f.g. as a homomorphic image of a t-f.g. module. Since $B = A \oplus C$, A and C are t-f.g. as a direct summand of a t-f.g. module by Lemma 1 and thus (n, t)-presented by hypothesis.

III. (N, T)-FLAT MODULES AND (N, T)-FP-INJECTIVE MODULES

The following definitions will be used in generalizing some of the results obtained by Chen and Ding in [7].

*Definition 2***:** Let (**T, F**) be an hereditary torsion theory with radical t, n a positive integer and M a right R-module.

1. M is said to be (n, t)-flat if $\mathrm{Tor}_n^R(M, N) = 0$ for all (n, t)-presented left R-modules N.
2. M is said to be (n, t)-FP-injective if $\mathrm{Ext}_R^n(N, M) = 0$ for all (n, t)-presented right R-modules N.

Proposition 1: Let $\{M_i\}_{i\in I}$ be a family of right R-modules and n a positive integer.

1. $\oplus_{i\in I}M_i$ is (n, t)-flat if and only if each M_i is (n, t)-flat.
2. $\prod_{i\in I}M_i$ is (n, t)-FP-injective if and only if each M_i is (n, t)-FP-injective.

Proof 4: (1) follows from the isomorphism

$$\mathrm{Tor}_n^R(\oplus_{i\in I}M_i\,,\ N)\ \cong\ \oplus_{i\in I}\mathrm{Tor}_n^R(M_i,\ N)\,.$$

(2) follows from the isomorphism $\mathrm{Ext}_R^n(N,\prod_{i\in I}M_i)\cong\prod_{i\in I}\mathrm{Ext}_R^n(N,M_i)$. ■

In what follows, for any module M, $M^+=\mathrm{Hom}\,(M,\ \mathbb{Q}/\mathbb{Z})$, the character module of M.

Proposition 2: Let M be a right R-module and n a positive integer. Then M is (n, t)-flat if and only if M^+ is (n, t)-injective.

Proof 5: Follows from the isomorphism $\mathrm{Ext}_R^n(N,M^+)\cong\ \mathrm{Tor}_n^R(N,M)^+$. ■

Proposition 3: Every pure submodule of an (n, t)-flat module is (n, t)-flat.

Proof 6: Let M be an (n, t)-flat module and M_1 a pure submodule of M. We have the exact sequence $0\to M_1\to M\to M/M_1\to 0$. This sequence induces the split exact sequence $0\to(M/M_1)^+\to M^+\to {M_1}^+\to 0$. By proposition 2, M^+ is (n, t)-injective. Since ${M_1}^+$ is a direct summand of M^+ , ${M_1}^+$ is (n,t)-injective by proposition 1 and hence M_1 is (n, t)-flat by proposition 2. ■

Lemma 3: An R-module M is (n,t)-FP-injective if and only if for every (n, t)-presentation $F_n\to F_{n-1}\to\cdots\to F_0\to A\to 0$ of an R-module A, every $f\colon K_n\to M$ can be extended to $g\colon F_{n-1}\to M$, where $K_n=\ker(F_n\to F_{n-1})$.

Proof 7: M is (n, t)-FP-presented if and only if $\mathrm{Ext}_R^n(A,M)=0$ for all (n, t)-presented modules A. By Dimension Shifting $\mathrm{Ext}_R^n(A,\ M)\cong\mathrm{Ext}_R^1(K_{n-1},M)$. Thus $\mathrm{Ext}_R^1(K_{n-1},M)=0$. From the exact sequence $0\to K_n\to F_{n-1}\to K_{n-1}\to 0$ we obtain the exact sequence $0\to\mathrm{Hom}(K_{n-1},M)\to\mathrm{Hom}(F_{n-1},M)\to\mathrm{Hom}(K_n,M)\to\mathrm{Ext}_R^1(K_{n-1},M)=0$ and the result follows. ■

Proposition 4: Every pure submodule of an (n, t)-FP-injective module is an (n, t)-FP-injective module.

Proof 8: Let M_1 be a pure submodule of an (n, t)-FP-injective module M. Let $F_n\to F_{n-1}\to\cdots\to F_0\to A\to 0$ be an (n, t)-presentation of a module A and $f\colon K_n\to M_1$ be a map. We obtain the following diagram with exact rows.

$$\begin{array}{ccccccc} 0\to & K_n & \xrightarrow{i} & F_{n-1} & \xrightarrow{p} & K_{n-1} & \to 0\\ & \downarrow f & \swarrow g & \downarrow k & & \downarrow h & \\ 0\to & M_1 & \xrightarrow{j} & M & \xrightarrow{q} & M/M_1 & \to 0\end{array}$$

where i and j are the inclusion maps, k exists by Lemma 3, since M is (n, t)-FP-injective by hypothesis and $k\circ i=j\circ f$. Hence $q\circ k\circ i=q\circ j\circ f=0$ i.e. $\mathrm{Im}(k\circ i)\subseteq\mathrm{Ker}\,q$ and we have the homomorphism $g\colon F_{n-1}\to M_1$ extending f. By Lemma 3, M_1 is (n, t)-FP-injective. ■

IV. (N, T)-COHERENT RINGS

Definition 3: A ring R is said to be (n, t)-coherent if every (n, t)-presented module is (n+1, t)-presented.

Remark 2: Every n-coherent ring is (n, t)-coherent since every n-presented module is (n, t)-presented but there exist (n, t)-presented modules that are not n-presented, see Jones, [12]. Thus an (n, t)-coherent ring need not be n-coherent.

Theorem 4: Let (**T, F**) be an hereditary torsion theory with corresponding torsion radical t. Assume that every t-f.g. R-module is a homomorphic image of a t-f.g. free R-module. Then the following are equivalent for a ring R:

1. R is right (n, t)-coherent.
2. If the Sequence $F_n\to F_{n-1}\to\cdots\to F_0\to M\to 0$ is exact where each F_i is t-f.g. free right R-module, then there exists an exact sequence of right R-modules $F_{n+1}\to F_n\to F_{n-1}\to\cdots\to F_0\to M\to 0$ where each F_i is t-f.g. free right R-module.
3. Every (n-1, t)-presented submodule of a t-f.g. projective right R-module is (n, t)-presented.

Proof 9: $1\Longrightarrow 2$

$$F_n\overset{d_n}{\to}F_{n-1}\overset{d_{n-1}}{\longrightarrow}\cdots\to F_0\overset{d_0}{\to}M\to 0$$

exact implies that

$$0\to K_n\overset{\alpha}{\to}F_n\overset{d_n}{\to}F_{n-1}\overset{d_{n-1}}{\longrightarrow}\cdots\to F_0\overset{d_0}{\to}M\to 0$$

is exact where $K_n=\mathrm{Ker}\,d_n$. By (1), M is (n+1, t)-presented and thus we have the exact sequence $0\to L_n\to P_n\overset{b_n}{\to}P_{n-1}\overset{b_{n-1}}{\longrightarrow}\cdots\to P_0\overset{b_0}{\to}M\to 0$

where each P_i is t-f.g. and free and L_n is t-f.g. by Theorem 1. By extended form of Schanuel's Lemma for projectives K_n is t-f.g. and by hypothesis there exists a t-f.g. free right R-module F_{n+1} such that $F_{n+1}\overset{\beta}{\to}K_n\to 0$. Hence the sequence

$$F_{n+1} \xrightarrow{\beta\alpha} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_0 \xrightarrow{d_0} M \to 0$$

is exact with each F_i t-f.g. and free.

2 ⟹ *1* is clear.

1 ⟹ *3* is clear.

1 ⟹ *3* Let M be (n, t)-presented. By Theorem 1, we have the exact sequence

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_0 \xrightarrow{d_0} M \to 0$$

where each P_i is t-f.g. and projective. Let $K_o = \text{Ker } d_0$. Then K_0 is an (n-1, t)-presented sub module of the t-f.g. and projective module P_0. By (3), K_0 is (n, t)-presented. Hence we have the exact sequences $0 \to K_0 \xrightarrow{\alpha} P_0 \xrightarrow{d_0} M \to 0$

and $P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{d_1} K_0 \to 0$ and combining them we have

$$P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{\alpha d_1} P_0 \to M \to 0 ..$$

Thus M is (n+1, t)-presented by Theorem 1. ■

Lemma 4: (Lemma 3.1, [12])

Let R and S be rings and **(T, F)** an hereditary torsion theory for right R-modules. Suppose Q is a left S-injective module and X is a left S right R bi-module that is torsion-free as a right R-module. Denote $\text{Hom}_S(.,Q)$ by $(.)^*$. If P is right R-module that is a t-f.g. (respectively, t-f.p.), then the canonical map $\varphi_P: P \otimes_R X^* \to \text{Hom}_R(P, X)^*$

given by $(g)(p \otimes f)\varphi_p = (gp)f$ for $g \in \text{Hom}_R(P,X)$, $p \in P$, $f \in X^*$ is an epimorphism (respectively, isomorphism).

Theorem 5: Let R and S be rings, n a fixed positive integer and (**T**, **F**) a hereditary torsion theory for right R-modules.. Let A be left S module which is (n, t)-presented, B a left S right R bimodule that is torsion free as a right R-module, C a right R module which is injective. Then there is an epimorphism

$$\varphi: Tor_n^S(Hom_R(B,C),A) \to \; Hom_R(Ext_S^n(A,B)\,,C).$$

Proof 10: A (n, t)-presented implies there exists an exact sequence

$$F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_0 \xrightarrow{d_0} A \to 0$$

with each F_i t-fg and free. Let

$$F_A := F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_0 \xrightarrow{d_0} 0$$

Define a map $: F_i \otimes B^* \to \text{Hom}(F_i,B)^*$ by $\varphi_{F_i}(x \otimes f)(g) = f[g(x)]$, where $x \in F_i$, $f \in B^*$, $g \in \text{Hom}(F_i,B)$ and $(.)^* = \text{Hom}(-,C)$\$.

Each φ_{F_i} is an epimorphism by Lemma 4 since each F_i is t-fg. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} F_n \otimes B^* & \to \cdots \to & F_1 \otimes B^* & \to & F_0 \otimes B^* & \to 0 \\ \downarrow \varphi_{F_n} & & \downarrow \varphi_{F_1} & & \downarrow \varphi_{F_0} & \\ \text{Hom}(F_n,B)^* & \to \cdots \to & \text{Hom}(F_1,B)^* & \to & \text{Hom}(F_0,B)^* & \to 0 \end{array}$$

Since each φ_{F_i} is an epimorphism, φ is an epimorphism. ■

Corollary 1: Let M be a torsion-free left R-module with respect to some hereditary torsion theory (**T, F**). If M^+ is (n, t)-flat right R-module, then M is (n, t)-f.p.-injective.

Proof 11: By Theorem 5, the sequence

$\text{Tor}_n^R(M^+, A) \to \text{Ext}_R^n(A,M)^+ \to 0$ is exact for every (n, t)-presented left R-module A and M^+ (n, t)-flat implies $\text{Tor}_n^R(M^+, A) = 0$ and hence $\text{Ext}_R^n(A,M)^+ = 0$ for all (n, t)-presented modules A. ■

Remark 4: We do not know if the map φ of Theorem 5 can be an isomorphism.

*Definition 4***:** Let M be right R-module. The (n, t)-FP-injective dimensions of M, denoted by (n, t)-FP-id(M), is the smallest integer $k \geq 0$ such that $Ext^{n+k}(N,M) = 0$, for all (n, t)-presented modules N.

Remark 4: The (n, t)-FP-id(M) "measures" how far M is from being (n, t)-FP-injective. In fact, when M is (n, t)-FP-injective, (n, t)-FP-id(M) = 0.

Theorem 6: Let R be an (n, t)-coherent ring and M a right R-module. Suppose every t-finitely generated module is a homomorphic image of a free t-finitely generated module and $n,k,r \in \mathbb{N}$. Then (n, t)-FP-id(M) = k if and only if $\text{Ext}^{n+r}(N,M) = 0$ for all $r \geq k$ and for all (n, t)-presented modules N but $\text{Ext}^{n+r}(N,M) \neq 0$ for some (n,t)-presented module N and for $r < k$.

Proof 12: Suppose (n, t)-FP-id(M) = k. We use induction on r. If r = k, then by hypothesis $\text{Ext}^{n+k}(N,M) = 0$ for all (n, t)-presented modules N and we are done. Assume theorem is true for r = s. N (n, t)-presented implies there exists an exact sequence

$0 \to K \to P \to N \to 0$, where K is (n-1, t)-presented and P is t-finitely generated free by Theorem 2. Since R is (n, t)-coherent, K is (n, t)-presented. Thus we have

$0 = \text{Ext}^{n+s}(K,M) \to \text{Ext}^{n+s+1}(N,M) \to \text{Ext}^{n+s+1}(P,M) = 0.$

by induction hypothesis and because P is projective. Hence $\mathrm{Ext}^{n+s+1}(N, M) = 0$ and theorem is true for r =s+1 and hence true for all r ≥ k. Also, by definition, k is smallest integer with this property and hence $\mathrm{Ext}^{n+r}(N, M) \neq 0$ for all r < k.

Conversely, suppose $\mathrm{Ext}^{n+r}(N, M) = 0$ for all r ≥ k and for all (n, t)-presented modules N. Then in particular when r = k we are done. ■

Theorem 7: Let R be an (n, t)-coherent ring and suppose every t-finitely generated module is a homomorphic image of a free t-finitely generated module. If $0 \to L \to M \to N \to 0$ is a short exact sequence of right R-modules and R-module homomorphisms, then

1. if (n, t)-FP-id(L) > (n, t)-FP-id(M), then

   (n, t)-FP-id(N) = 1+ (n, t)-FP-id(L).

2. if (n, t)-FP-id(L) = (n, t)-FP-id(M), then

   (n, t)- FP-id(N) ≤ 1 + (n, t)-FP-id(L).

3. if (n, t)-FP-id(L) < (n, t)-FP-id(M), then (n, t)-FP-id(N) = (n, t)-FP-id(M).

Proof 13:

1. Suppose (n, t)-FP-id(L) > (n, t)-FP-id(M) = k. Then for any (n, t)-presented right R-module X we have the exact sequence

$$0 = \mathrm{Ext}^{n+k}(X, M) \to \mathrm{Ext}^{n+k}(X, N) \to \mathrm{Ext}^{n+k+1}(X, L) \to \mathrm{Ext}^{n+k+1}(X, M) = 0.$$

Hence the map

$\varphi_{n+k} : \mathrm{Ext}^{n+k}(X, M) \to \mathrm{Ext}^{n+k+1}(X,\ L)$ is an isomorphism. Thus

(n, t)-FP-id(N) = 1 + (n, t)-FP-id(L).

2. Suppose (n, t)-FP-id(L) = (n, t)-FP-id(M) = k and X any (n, t)-presented right R-module. Therefore we have the exact sequence

$$0 = \mathrm{Ext}^{n+k}(X, M) \to \mathrm{Ext}^{n+k}(X, N) \to \mathrm{Ext}^{n+k+1}(X, L) = 0.$$

This implies that $\mathrm{Ext}^{n+k}(X, N) = 0$ and thus (n, t)-FP-id(N) ≤ (n, t)-FP-id(L).

4. k = (n, t)-FP-id(L) < (n, t)-FP-id(M) and X is any (n, t)-presented right R-module. Then for r ≥ k , we have the exact sequence

$$0 = \mathrm{Ext}^{n+r}(X, L) \to \mathrm{Ext}^{n+r}(X, M) \to \mathrm{Ext}^{n+r}(X, N) \to \mathrm{Ext}^{n+k}(X, L) = 0$$

Hence $\mathrm{Ext}^{n+r}(N, M) = \mathrm{Ext}^{n+r}(N, M)$ for all r ≥ k and thus (n, t)-FP-id(N) = (n, t)-FP-id(M). ■

IV. CONCLUSION

We defined a new class of modules and rings and gave some properties and characterizations of these rings and modules. Jones in [12] has observed that every t-f.g. module is not necessarily finitely generated. Thus to give characterizations of (n, t)-coherent rings similar to those given by Chase, Cheatham and Stone or Stentrom, see [1], we may have to find conditions under which the map φ of Theorem 5 can be an isomorphism. If this were to be possible, we may be able to generalize Theorem 3.1 of [7] in the setting of an hereditary torsion theory.

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