Concrete Examples of the Minimal Bull Eye Composite Loubéré Magic Square Infinite Additive Abelian Groups

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ABSTRACT

In this paper, concrete examples of the Minimal Bull Eye Composite Loubéré Magic Square Infinite Additive Abelian Groups are constructed. The minimum Loubéré Magic Square is the 3×3 and so the minimal Composite Loubéré Magic Square is the $3^2 \times 3^2$. We construct its Bull Eye and proved that the Bull Eye also forms an Infinite Additive Abelian Group.

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1. Introduction

By the Loubéré Magic Square, we understand the magic square constructed with the De La Loubéré Procedure. In this pioneering work, Bull Eye Loubéré Magic Squares are constructed as well as Bull Eye Composite −.

We showcased that the set of the Minimal Bull Eye Composite forms an infinite additive abelian group, an idea that does not require a deep mathematics to conclude its generalization.

The whole concept is based on intimate survey of the structure of the centre piece formula, the mighty formula of magic squares realm. We will rightly so introduce a new result on it.

The minimum Loubéré Magic Square is the 3×3 for 1×1 Square is trivial for it is isomorphic to the underlined set of entries of the square so considered and 2×2 Loubéré − does not exist. Since the minimum is considered to be the 3×3, the minimum composite−is $3^2 \times 3^2$ and its minimal Bull Eye is 15×15 because an infix adjoinment of 17 times the face-centre magic squares is more than necessary.

2. PRELIMINARIES

Definition 2.1. Main Row or Column is the column or row of the Loubéré Magic Squares containing the first term and the last term of the arithmetic sequence in the square.

2.2.Loubéré Procedure (NE-W-S or NW-E-S, the cardinal points). Consider an empty $n \times n$ square grid of cells. Start, from the central column or row at a position $\left\lceil \frac{n}{2} \right\rceil$ where $\lceil \ \rceil$ is the greater integer number less than or equal to, with the number 1. The fundamental movement for filling the square is diagonally up, right (clock wise or NE or SE) or up left (anti clock wise or NW or SW) and one step at a time. If a filled cell (grid) is encountered, then the next consecutive number moves vertically down ward one square instead. Continue in this fashion until when a move would leave the square, it moves due N or E or W or S (depending on the position of the first term of the sequence) to the last row or first row or first column or last column.

The square grid of cells $[a_{ij}]_{n \times n}$ is said to be Loubéré Magic Square if the following conditions are satisfied.

i. $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = k$;

ii. $\mathrm{trace}[a_{ij}]_{n \times n} = \mathrm{trace}[a_{ij}]_{n \times n}^{T} = k$; and

iii. $a_{1,\left\lceil \frac{n}{2} \right\rceil}$, $a_{\left\lceil \frac{n}{2} \right\rceil,\left\lceil \frac{n}{2} \right\rceil}$, $a_{n,\left\lceil \frac{n}{2} \right\rceil}$ are on the same main column or row and $a_{\left\lceil \frac{n}{2} \right\rceil,n}$, $a_{\left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil}$, $a_{\left\lceil \frac{n}{2} \right\rceil,1}$ are on the same main column or row,

where $\lceil \ \rceil$ is the greater integer less or equal to, T is the transpose (of the square), k is the magic sum (magic product is defined analogously) usually expressed as $k = \frac{n}{2}[2a + (n-1)j]$ − from the sum of arithmetic sequence, where j is the common difference along the main column or row and a is the first term of the sequence− and $a_{\left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil} = \frac{k}{n}$.

Definition 2.3. A Composite Loubéŕe Magic Square is a magic square such that each of its cell (grid) is a Loubéŕe Magic Square. See also [1].

Definition 2.4. A Bull Eye Loubéré Magic Square is a magic square constructed by an infix adjoinment (in a special way) of the centre piece entry or its Loubéré Magic Squares into the cells(grids) of a Loubéré Magic Square or its composite.

Remarks 2.5. This study is based on close study of the centre piece formula C and magic sum M(S) of an $n \times n$ Loubéré Magic Square relationship $C = \frac{M(S)}{n}$, an existing mighty formula of this realm.

Theorem 2.6. The centre piece C of an $n \times n$ Loubéré Magic Square with first entry (first term of its underlined sequence) and common difference j along the main column

$$c = a + \frac{(n-1)}{2} j$$

.

Proof. The magic sum of the square is $S = \frac{n}{2}[2a + (n-1)j]$ from High School Gaussian Formula of sum of arithmetic sequence. And, from Remark 2.5,

$$C = \frac{M(S)}{n} = \frac{\frac{n}{2}[2a + (n-1)j]}{n} = a + \frac{(n-1)}{2} j$$

.

Theorem 2.7. The centre piece of the minimal composite Loub éré Magic Squares are expressed as $C_m = \frac{(a_{f_m} + a_{l_m})}{2} + \frac{(m-1)}{4}\left(j_{f_m} + j_{l_m}\right)$ where m =1, 2, 3, ...

Proof. This follows from $C = \frac{M(S)}{n}$ and $C = a + \frac{(n-1)}{2} j$ where C is the centre piece magic square of the composite−.

Theorem 2.8. The Set of the centre pieces of the Minimal Composite Loubéré Magic Squares equipped with the integer numbers binary operation of addition forms an infinite additive abelian group.

Proof. Let C denotes the set of the centre pieces of Minimal Composite − and $\oplus$ denotes the matrix binary operation of addition. Then

$C = \left\{C_m = \frac{(a_{f_m} + a_{l_m})}{2} + \frac{(m-1)}{4}\left(j_{f_m} + j_{l_m}\right) : m \in \mathbb{Z}_+\right\}$, where $\mathbb{Z}_+$ is the set of positive integer numbers and $(C, \oplus)$ satisfies the **Properties of Infinite Additive Abelian Group** as follows:

i. **Closure Property.** Clearly,

$C_1 + C_2 = \frac{(a_{f_1} + a_{l_1})}{2} + \frac{(a_{f_2} + a_{l_2})}{2} + \frac{(m-1)}{4}\left((j_{f_1} + j_{l_1}) + (j_{f_2} + j_{l_2})\right)$ is the centre piece of $n^2 \times n^2$ Composite Loubéré Magic Square with first term $\frac{(a_{f_1} + a_{l_1})}{2} + \frac{(a_{f_2} + a_{l_2})}{2}$ and common difference along the main column $\frac{(j_{f_1} + j_{l_1}) + (j_{f_2} + j_{l_2})}{2}$. Hence, C is closed.

ii. **Associativity Property.** This is an inherited property of the set of integer numbers for:

$$C_1 + (C_2 + C_3) = \frac{(a_{f_1} + a_{l_1})}{2} + \left(\frac{(a_{f_2} + a_{l_2})}{2} + \frac{(a_{f_3} + a_{l_3})}{2}\right) + \frac{(m-1)}{4}\left((j_{f_1} + j_{l_1}) + \left((j_{f_2} + j_{l_2}) + (j_{f_3} + j_{l_3})\right)\right)$$
$$= \frac{(a_{f_1} + a_{l_1})}{2} + \frac{(a_{f_2} + a_{l_2})}{2} + \frac{(a_{f_3} + a_{l_3})}{2} + \frac{(m-1)}{4}\left((j_{f_1} + j_{l_1}) + (j_{f_2} + j_{l_2}) + (j_{f_3} + j_{l_3})\right)$$
$$= \left(\frac{(a_{f_1} + a_{l_1})}{2} + \frac{(a_{f_2} + a_{l_2})}{2}\right) + \frac{(a_{f_3} + a_{l_3})}{2} + \frac{(m-1)}{4}\left((j_{f_1} + j_{l_1}) + (j_{f_2} + j_{l_2})\right) + (j_{f_3} + j_{l_3})$$
$$= (C_1 + C_2) + C_3$$

iii. The **Identity element** 0 is the zero face-centre magic square.

iv. Given an arbitrary centre piece $C_p = \frac{(a_{f_p} + a_{l_p})}{2} + \frac{(m-1)}{4}\left(j_{f_p} + j_{l_p}\right)$ having first term $\frac{(a_{f_p} + a_{l_p})}{2}$ and common difference along the main column or row $\frac{j_{f_p} + j_{l_p}}{2}$ of an $n^2 \times n^2$ Composite−, there exists another centre piece (call the

inverse of C_p) C_q of another $n^2 \times n^2$ Composite– having first term $-\frac{(a_{f_p}+a_{l_p})}{2}$ and common difference along the main column or row $\frac{-j_{f_p}-j_{l_p}}{2}$, thus having the formulae $C_q = -\frac{(a_{f_p}+a_{l_p})}{2} + \frac{(m-1)}{4}\left(-j_{f_p} - j_{l_p}\right)$ such that $C_p + C_q = \frac{(a_{f_p}+a_{l_p})}{2} + -\frac{(a_{f_p}+a_{l_p})}{2} + \frac{(m-1)}{4}\left(j_{f_p} + j_{l_p} - j_{f_p} - j_{l_p}\right) = 0 = C_q + C_p$

v. Obviously,

$$C_1 + C_2 = \frac{(a_{f_1} + a_{l_1})}{2} + \frac{(a_{f_2} + a_{l_2})}{2} + \frac{(m-1)}{4}\left((j_{f_1} + j_{l_1}) + (j_{f_2} + j_{l_2})\right)$$
$$= \frac{(a_{f_2}+a_{l_2})}{2} + \frac{(a_{f_1}+a_{l_1})}{2} + \frac{(m-1)}{4}\left((j_{f_2} + j_{l_2}) + (j_{f_1} + j_{l_1})\right) = C_2 + C_1$$

whence, we are dealing with integer numbers entries.

And, $(C, \oplus)$ is an infinite additive abelian group.

3. CONCRETE EXAMPLES OF THE MINIMAL BULL EYE COMPOSITE LOUBéré MAGIC SQUARES INFINITE ADDITIVE ABELIAN GROUPS

Example 1. The following are examples of 3×3 Loubéré Magic Square

$$\begin{bmatrix} 4 & -3 & 2 \\ -1 & 1 & 3 \\ 0 & 4 & -2 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Example 2. The following is an example of a $3^2 \times 3^2$ Composite Loubéré Magic Square

$$\begin{bmatrix} 15 & 08 & 13 & 08 & 01 & 06 & 13 & 06 & 11 \\ 10 & 12 & 14 & 03 & 05 & 07 & 08 & 10 & 12 \\ 11 & 16 & 09 & 04 & 09 & 02 & 09 & 14 & 07 \\ 10 & 03 & 08 & 12 & 05 & 10 & 14 & 07 & 12 \\ 05 & 07 & 09 & 07 & 09 & 11 & 09 & 11 & 13 \\ 06 & 11 & 04 & 08 & 13 & 06 & 10 & 15 & 08 \\ 11 & 04 & 09 & 16 & 09 & 14 & 09 & 02 & 07 \\ 06 & 08 & 10 & 11 & 13 & 15 & 04 & 06 & 08 \\ 07 & 12 & 05 & 12 & 17 & 10 & 05 & 10 & 03 \end{bmatrix}$$

Example 3. The following is an example of a Bull Eye Loubéré Magic Square of the 3×3 Loubéré Magic Square

$$\begin{bmatrix} 04 & 01 & -3 & 01 & 02 \\ 01 & 01 & 01 & 01 & 01 \\ -1 & 01 & 01 & 01 & 03 \\ 01 & 01 & 01 & 01 & 01 \\ 00 & 01 & 04 & 01 & -2 \end{bmatrix}$$

We take another example of its 5×5 Loubéré Magic Square as follows:

$$\begin{bmatrix} 17 & 24 & 13 & 01 & 13 & 08 & 15 \\ 23 & 05 & 13 & 07 & 13 & 14 & 16 \\ 13 & 13 & 13 & 13 & 13 & 13 & 13 \\ 04 & 06 & 13 & 13 & 13 & 20 & 22 \\ 13 & 13 & 13 & 13 & 13 & 13 & 13 \\ 10 & 12 & 13 & 19 & 13 & 21 & 03 \\ 11 & 18 & 13 & 25 & 13 & 02 & 09 \end{bmatrix}$$

And, for 7×7 as:

$$\begin{bmatrix} 30 & 39 & 48 & 25 & 01 & 25 & 10 & 19 & 28 \\ 38 & 47 & 07 & 25 & 09 & 25 & 18 & 27 & 29 \\ 46 & 06 & 08 & 25 & 17 & 25 & 26 & 35 & 37 \\ 25 & 25 & 25 & 25 & 25 & 25 & 25 & 25 & 25 \\ 05 & 14 & 16 & 25 & 25 & 25 & 34 & 36 & 45 \\ 25 & 25 & 25 & 25 & 25 & 25 & 25 & 25 & 25 \\ 13 & 15 & 24 & 25 & 33 & 25 & 42 & 44 & 04 \\ 21 & 23 & 32 & 25 & 41 & 25 & 43 & 03 & 12 \\ 22 & 31 & 40 & 25 & 49 & 25 & 02 & 11 & 20 \end{bmatrix}$$

Remark 3.1. By citing 3 examples here, we can see that we need $8n + 9$ where $n \in \mathbb{Z}_+$ number of centre pieces infix adjoinment in general.

Example 4. The following is an example of a minimal Bull Composite Loubéré Magic Square.

$$\begin{bmatrix} 15 & 08 & 13 & 12 & 05 & 10 & 08 & 01 & 06 & 12 & 05 & 10 & 13 & 06 & 11 \\ 10 & 12 & 14 & 07 & 09 & 10 & 03 & 05 & 07 & 07 & 09 & 11 & 08 & 10 & 12 \\ 11 & 16 & 09 & 08 & 13 & 06 & 04 & 09 & 02 & 08 & 13 & 06 & 09 & 14 & 07 \\ 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 \\ 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 \\ 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 \\ 10 & 03 & 08 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 14 & 07 & 12 \\ 05 & 07 & 09 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 09 & 11 & 13 \\ 06 & 11 & 04 & 08 & 13 & 06 & 08 & 13 & 06 & 12 & 05 & 10 & 12 & 05 & 10 \\ 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 \\ 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 \\ 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 \\ 11 & 04 & 09 & 12 & 05 & 10 & 16 & 09 & 14 & 12 & 05 & 10 & 09 & 02 & 07 \\ 06 & 08 & 10 & 07 & 09 & 11 & 11 & 13 & 15 & 07 & 09 & 11 & 04 & 06 & 08 \\ 07 & 12 & 05 & 08 & 13 & 06 & 12 & 17 & 10 & 08 & 13 & 06 & 05 & 10 & 03 \end{bmatrix}$$

Theorem 3.2. $3^2 \times 5^2$ Composite Loubéré Magic Square is not Isomorphic to $5^2 \times 3^2$ Composite−.

Proof.

$$\begin{bmatrix} 24 & 31 & 08 & 15 & 22 & 17 & 24 & 01 & 08 & 15 & 22 & 29 & 06 & 13 & 20 \\ 30 & 12 & 14 & 21 & 23 & 23 & 05 & 07 & 14 & 16 & 28 & 10 & 12 & 19 & 21 \\ 11 & 13 & 20 & 27 & 29 & 04 & 06 & 13 & 20 & 22 & 09 & 11 & 18 & 25 & 27 \\ 17 & 19 & 26 & 28 & 10 & 10 & 12 & 19 & 21 & 3 & 15 & 17 & 24 & 26 & 08 \\ 18 & 25 & 32 & 09 & 16 & 11 & 18 & 25 & 02 & 09 & 16 & 23 & 30 & 07 & 14 \\ 19 & 26 & 03 & 10 & 17 & 21 & 28 & 05 & 12 & 19 & 23 & 30 & 07 & 14 & 21 \\ 25 & 07 & 09 & 16 & 18 & 27 & 09 & 11 & 18 & 20 & 29 & 11 & 13 & 20 & 22 \\ 06 & 08 & 15 & 22 & 24 & 08 & 10 & 17 & 24 & 26 & 10 & 12 & 19 & 26 & 28 \\ 12 & 14 & 21 & 23 & 05 & 14 & 16 & 23 & 25 & 07 & 16 & 18 & 25 & 27 & 09 \\ 13 & 20 & 27 & 04 & 11 & 15 & 22 & 29 & 06 & 13 & 17 & 24 & 31 & 08 & 15 \\ 20 & 27 & 04 & 11 & 18 & 25 & 32 & 09 & 16 & 23 & 18 & 25 & 02 & 09 & 16 \\ 26 & 08 & 10 & 17 & 19 & 31 & 13 & 15 & 22 & 24 & 24 & 06 & 08 & 15 & 17 \\ 07 & 09 & 16 & 23 & 25 & 12 & 14 & 21 & 28 & 30 & 05 & 07 & 14 & 21 & 23 \\ 13 & 15 & 22 & 24 & 06 & 18 & 20 & 27 & 29 & 11 & 11 & 13 & 20 & 22 & 04 \\ 14 & 21 & 28 & 05 & 12 & 19 & 26 & 33 & 10 & 17 & 12 & 19 & 26 & 03 & 10 \end{bmatrix}$$ is not isomorphic to

$$\begin{bmatrix} 24 & 17 & 22 & 31 & 24 & 29 & 08 & 01 & 06 & 15 & 08 & 13 & 22 & 15 & 20 \\ 19 & 21 & 23 & 26 & 28 & 30 & 03 & 05 & 07 & 10 & 12 & 14 & 17 & 19 & 21 \\ 20 & 25 & 18 & 27 & 32 & 25 & 04 & 09 & 02 & 11 & 16 & 09 & 18 & 23 & 16 \\ 30 & 23 & 28 & 12 & 05 & 10 & 14 & 07 & 12 & 21 & 14 & 19 & 23 & 16 & 21 \\ 25 & 27 & 29 & 07 & 09 & 11 & 09 & 11 & 13 & 16 & 18 & 20 & 18 & 20 & 22 \\ 26 & 31 & 24 & 08 & 13 & 06 & 10 & 15 & 08 & 17 & 22 & 15 & 19 & 24 & 17 \\ 11 & 04 & 09 & 13 & 06 & 11 & 20 & 13 & 18 & 27 & 20 & 25 & 29 & 22 & 27 \\ 06 & 08 & 10 & 08 & 10 & 12 & 15 & 17 & 19 & 22 & 24 & 26 & 24 & 26 & 28 \\ 07 & 12 & 05 & 09 & 14 & 07 & 16 & 21 & 14 & 23 & 28 & 21 & 25 & 30 & 23 \\ 17 & 10 & 15 & 19 & 12 & 17 & 26 & 19 & 24 & 28 & 21 & 26 & 10 & 03 & 08 \\ 12 & 14 & 16 & 14 & 16 & 18 & 21 & 23 & 25 & 23 & 25 & 27 & 05 & 07 & 09 \\ 13 & 18 & 11 & 15 & 20 & 13 & 22 & 27 & 20 & 24 & 29 & 22 & 06 & 11 & 04 \\ 18 & 11 & 16 & 25 & 18 & 23 & 32 & 25 & 30 & 09 & 02 & 07 & 16 & 09 & 14 \\ 13 & 15 & 17 & 20 & 22 & 24 & 27 & 29 & 31 & 04 & 06 & 08 & 11 & 13 & 15 \\ 14 & 19 & 12 & 21 & 26 & 19 & 28 & 33 & 26 & 05 & 10 & 03 & 12 & 17 & 10 \end{bmatrix}$$

Remark 3.3. See[2] for the definition of a group.

Theorem 3.4. The set of minimal Bull Eye Composite Loubéré Magic Square $B_{n^2\times n^2}$ equipped with matrix binary operation of addition $\oplus$ forms an infinite additive abelian group.

Proof. Let $A, B \in B_{n^2\times n^2}$, be arbitrarily considered, we define $\oplus$ as follows:

$$A = \begin{bmatrix} 15 & 08 & 13 & 12 & 05 & 10 & 08 & 01 & 06 & 12 & 05 & 10 & 13 & 06 & 11 \\ 10 & 12 & 14 & 07 & 09 & 10 & 03 & 05 & 07 & 07 & 09 & 11 & 08 & 10 & 12 \\ 11 & 16 & 09 & 08 & 13 & 06 & 04 & 09 & 02 & 08 & 13 & 06 & 09 & 14 & 07 \\ 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 \\ 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 \\ 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 \\ 10 & 03 & 08 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 14 & 07 & 12 \\ 05 & 07 & 09 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 09 & 11 & 13 \\ 06 & 11 & 04 & 08 & 13 & 06 & 08 & 13 & 06 & 12 & 05 & 10 & 12 & 05 & 10 \\ 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 & 12 & 05 & 10 \\ 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 & 07 & 09 & 11 \\ 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 & 08 & 13 & 06 \\ 11 & 04 & 09 & 12 & 05 & 10 & 16 & 09 & 14 & 12 & 05 & 10 & 09 & 02 & 07 \\ 06 & 08 & 10 & 07 & 09 & 11 & 11 & 13 & 15 & 07 & 09 & 11 & 04 & 06 & 08 \\ 07 & 12 & 05 & 08 & 13 & 06 & 12 & 17 & 10 & 08 & 13 & 06 & 05 & 10 & 03 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 17 & 10 & 15 & 14 & 07 & 12 & 10 & 03 & 08 & 14 & 07 & 12 & 15 & 08 & 13 \\ 12 & 14 & 16 & 09 & 11 & 13 & 05 & 07 & 09 & 09 & 11 & 13 & 10 & 12 & 14 \\ 13 & 18 & 11 & 10 & 15 & 08 & 06 & 11 & 04 & 10 & 15 & 08 & 11 & 16 & 09 \\ 14 & 07 & 12 & 14 & 07 & 12 & 14 & 07 & 12 & 14 & 07 & 12 & 14 & 07 & 12 \\ 09 & 11 & 13 & 09 & 11 & 13 & 09 & 11 & 13 & 09 & 11 & 13 & 09 & 11 & 13 \\ 10 & 15 & 08 & 10 & 15 & 08 & 10 & 15 & 08 & 10 & 15 & 08 & 10 & 15 & 08 \\ 12 & 05 & 10 & 14 & 07 & 12 & 14 & 07 & 12 & 14 & 07 & 12 & 16 & 09 & 14 \\ 07 & 09 & 11 & 09 & 11 & 13 & 09 & 11 & 13 & 09 & 11 & 13 & 11 & 13 & 15 \\ 08 & 13 & 06 & 10 & 15 & 08 & 10 & 15 & 08 & 10 & 15 & 08 & 12 & 17 & 10 \\ 14 & 07 & 12 & 14 & 07 & 12 & 14 & 07 & 12 & 14 & 07 & 12 & 14 & 07 & 12 \\ 09 & 11 & 13 & 09 & 11 & 13 & 09 & 11 & 13 & 09 & 11 & 13 & 09 & 11 & 13 \\ 10 & 15 & 08 & 10 & 15 & 08 & 10 & 15 & 08 & 10 & 15 & 08 & 10 & 15 & 08 \\ 13 & 06 & 11 & 14 & 07 & 12 & 18 & 11 & 16 & 14 & 07 & 12 & 11 & 04 & 09 \\ 08 & 10 & 12 & 09 & 11 & 13 & 13 & 15 & 17 & 09 & 11 & 13 & 06 & 08 & 10 \\ 09 & 14 & 07 & 10 & 15 & 08 & 14 & 19 & 12 & 10 & 15 & 08 & 07 & 12 & 05 \end{bmatrix} \text{, then}$$

$$A \oplus B = \begin{bmatrix} 32 & 18 & 28 & 26 & 12 & 22 & 18 & 04 & 14 & 26 & 12 & 22 & 28 & 14 & 24 \\ 22 & 26 & 30 & 16 & 20 & 23 & 8 & 12 & 16 & 16 & 20 & 23 & 18 & 22 & 26 \\ 24 & 34 & 20 & 18 & 28 & 14 & 10 & 20 & 06 & 18 & 28 & 14 & 20 & 30 & 16 \\ 26 & 12 & 22 & 26 & 12 & 22 & 26 & 12 & 22 & 26 & 12 & 22 & 26 & 12 & 22 \\ 16 & 20 & 23 & 16 & 20 & 23 & 16 & 20 & 23 & 16 & 20 & 23 & 16 & 20 & 23 \\ 18 & 28 & 14 & 18 & 28 & 14 & 18 & 28 & 14 & 18 & 28 & 14 & 18 & 28 & 14 \\ 22 & 08 & 18 & 26 & 12 & 22 & 26 & 12 & 22 & 26 & 12 & 22 & 30 & 16 & 26 \\ 12 & 16 & 20 & 16 & 20 & 23 & 16 & 20 & 24 & 16 & 20 & 23 & 20 & 24 & 28 \\ 14 & 24 & 10 & 18 & 28 & 14 & 18 & 28 & 14 & 18 & 28 & 14 & 22 & 32 & 18 \\ 26 & 12 & 22 & 26 & 12 & 22 & 26 & 12 & 22 & 26 & 12 & 22 & 26 & 12 & 22 \\ 16 & 20 & 23 & 16 & 20 & 23 & 16 & 20 & 23 & 16 & 20 & 23 & 16 & 20 & 23 \\ 18 & 28 & 14 & 18 & 28 & 14 & 18 & 28 & 14 & 18 & 18 & 14 & 18 & 28 & 14 \\ 24 & 10 & 20 & 26 & 12 & 22 & 34 & 20 & 30 & 26 & 12 & 22 & 20 & 06 & 16 \\ 14 & 18 & 22 & 16 & 20 & 23 & 24 & 28 & 32 & 16 & 20 & 23 & 10 & 14 & 18 \\ 16 & 26 & 12 & 18 & 28 & 14 & 26 & 36 & 22 & 18 & 28 & 14 & 12 & 22 & 08 \end{bmatrix}$$

i. $B_{n^2 \times n^2}$ is **closed** with respect to $\oplus$: From the above defination, if $A, B \in B_{n^2 \times n^2}$, then $A \oplus B = C \in B_{n^2 \times n^2}$, as in the above.

ii. $\oplus$ is **associative**: For $A, B, C \in B_{n^2 \times n^2}, A \oplus (B \oplus C) = (A \oplus B) \oplus C$ whence the entries of A, B and C are integer numbers.

iii. The additive **identity element** I is

$$I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

, where each 0 except the first entry of the square is a series of 0s correspondingly n times.

iv. Each element of $B_{n^2 \times n^2}$ has an **inverse**: If $D \in B_{n^2 \times n^2}$, then $-D \in B_{n^2 \times n^2}$ where –D is the square with magic sum –M(S) formed as a result of scalar multiplication of entries in D having magic sum M(S) by -1.

For example, the inverse of $\begin{bmatrix} b & a & c \\ c & b & a \\ a & c & b \end{bmatrix}$ is $\begin{bmatrix} -b & -a & -c \\ -c & -b & -a \\ -a & -c & -b \end{bmatrix}$.

v. $\oplus$ is **commutative**: $A, B \in B_{n^2 \times n^2} \Rightarrow A \oplus B = B \oplus A$ whence the matrix binary operation of addition over set of integer numbers is commutative (inheritance).

Thus, $(B_{n^2 \times n^2}, \oplus)$ is an infinite additive abelian group.

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