# Construction of Angles Multiples of the Approximate Trisectant of $\frac{\pi}{3}$ 

Babayo A.M. ${ }^{1}$, G.U.Garba ${ }^{2}$<br>1. Department of Mathematics and Computer Science, Faculty of Science, Federal University Kashere, P.M.B.0182, Gombe State, Nigeria.

2. Department of Mathematics, Faculty of Science, Ahmadu Bello University Zaria, Kaduna State, Nigeria.


#### Abstract

The quest that angle 60 degrees cannot be trisected, an application of the extension of algebraic field equation, outlined a joyous pose for its approximate can be constructed using the unmarked straight one edge ruler and compass. The trisectant of $\mathbf{6 0}$ degrees is 20 degrees. We explicate - via the approximate trisectant - that all miscellany angles $a_{10 \mathrm{~m}}=\mathrm{ma}_{10}$ where $\mathrm{m} \in \mathbb{Z}^{+}$, the set of positive integer numbers and $a_{m}$ is the angle $m$ degrees, can be constructed for they satisfy $\mathrm{a}_{20 \mathrm{~m}}=\left(20^{\circ} \div 2\right) \mathrm{b}+\left(20^{\circ} \times 2\right) \mathrm{c}$ where $b=0$ or 1 or 2 or 3 and $\mathbf{c}$ is non-negative integer number less or equal to 9 . By approximate trisectant; we understand, however in this context, that it is the angle constructed as a result of approximate trisection.


Keywords: Trisectant, Algebraic Field

## 1. INTRODUCTION

Though widespread is the theorem that angle 60 degrees cannot be trisected using straight edge ruler and compass, yet but rarely such a theorem on the construction of the approximate trisectant because of the stimulating factor of the theorem. We present a new method of construction of such an approximate trisectant.

We name all existing methods of construction of set of fundamental generators \{angle $0^{\circ}$, angle $30^{\circ}$, angle $45^{\circ}$ \} as Method 1 and this new method as Me2. The set of generators equipped with the construction operations $\left({ }_{s},-x\right.$ and $\div$ ) have deducibility: $7.5^{\circ}, 15^{\circ}, 75^{\circ}, 90^{\circ}, 60^{\circ}, 105^{\circ}, 120^{\circ}, 135^{\circ}, 127.5^{\circ}, 97.5^{\circ}, 67.5^{\circ}, 112.5^{\circ}, 22.5^{\circ}, 67.5^{\circ}, 131.25^{\circ}$, etc. Though - and $\div$ are not algebraic operations, yet they are construction operations, and constructible angles form a field [5].

In the Me2, we showcased that we don't need information about the construction of the angle 60 degrees to construct the angle 30 degrees, and no any primitive angle construction is of interest in the construction of angle 40 degrees. So our new set of generators of fundamental angles is \{angle $0^{\circ}$, angle $20^{\circ}$, angle $30^{\circ}$, angle $45^{\circ}$ \} where angle 30 degrees is constructed using the Me 2 , and is having deducibilitya $\mathrm{a}_{10 \mathrm{~m}}$ and deducibility $\mathrm{a}_{20 \mathrm{~m}}$. In the method, unmarked straight one edge ruler and compass is required. For marked ruler construction, see 'Lagsamladem Method in [1].

## 2. PRELIMINARIES

## Definition 2.1.

1. The compass can be opened arbitrarily wide, but it has no markings on it. It can only be opened to widths that are apt for construction .
2. The ruler is infinitely long, but has no markings on it and has one edge. It can be used to draw a line segment between two points or extend an existing line.

The following definitions can be figured from [3, 4].

## Definition 2.2.

1. The line $L$ is given by $L:=a x+b y+c$ where $a, b, c \in E \subseteq \mathbb{R}^{2}$
2. The circle C is given by $C:=(x-a)^{2}+(y-b)^{2}=r^{2}$ where $a, b, r \in E \subseteq \mathbb{R}^{2}$

Definition 2.3. A point is constructible if and only if it is an intersection of lines and circles, or it is the point 0 or 1 .
Lemma 2.1. If $p, q \in \mathbb{R}$ are constructible angles, so are $p+q, p-q, p q, \frac{p}{q}$ and $\sqrt{p}$. [5]
Proof. This is easy to see this via concrete practical of construction. It is useful in proving that the set of all constructible numbers form a field.

Theorem 2.2. The set of all constructible numbers forms a field.
Proof. This follows directly from Lemma 2.1.
Theorem 2.3. It is impossible to trisect the angle 60 degrees using ruler and compass constructions.
Proof. We want to construct angle $20^{\circ}=\cos ^{-1}$ a, the trisectant of 60 degrees .
From Lemma 2.1, $a=\frac{b}{2}=\cos 20^{\circ} \quad$ Putting
$\theta=20^{\circ}$ and $\cos 60^{\circ}=\frac{1}{2}$ in the trigonometric identity $\cos (3 \theta)=4 \cos ^{2} \theta-3 \cos \theta$, we have $b^{a}-3 b-1=0$. This cubic equation is irreducible [3] (by radicals), and its degree of the field extension $[\mathbb{Q}(b) ; \mathbb{Q}]=3$. The result follows immediately.

Theorem 2.4 (Archimedes). Every angle can be trisected using a marked ruler and compass.
Proof. The new set of generators is $\left\{0^{\circ}, 30^{\circ}, 20^{\circ}\right\}$, it is enough to trisect an acute angle a, for if $3 b=a$, then $3(b+0)=a+0,3(b+30)=a+90,3(b+20)=a+60$. Considering the formula $a_{10 \mathrm{~m}}$ and $a_{20 \mathrm{~m}}$, the result follows immediately.

Remark 2.1. The $\div$ construction operation is analogous to bisection, we can use + many multiples for $\times .+$ and - are a complete set of operations on Me 2 . Though it is impossible to trisect the angle 60 degrees using an unmarkedruler and compass constructions, yet we can trisect it via marked ruler and compass, see 'Lagsamladeem Method in [1]. About 425 BC, Hippias of Elis was able to square the circle despite the following theorem.

Theorem 2.5 (Lindemann). It is impossible to square the circle with ruler and compass.

## 3.THE ME2 METHOD OF CONSTRUCTION OF ANGLES MULTIPLES OF THE APPROXIMATE TRISECTANT OF 60 DEGREES

In this method, we can directly (not via operations) construct fundamental angles (approximately): $20^{\circ}, 30^{\circ}$ and $40^{\circ}$. The complete set of generators, by analogy, is $\left\{0^{\circ}, 20^{\circ}, 30^{\circ}, 40^{\circ}\right\}$.

### 3.1Construction of Fundamental Angles $20^{\circ}{ }_{v} \mathbf{3 0}^{\circ}$ and $40^{\circ}$

i. Draw a line using an un-marked straight one edged ruler;
ii. Pick a starting point on the line;
iii. Draw a semi circle with centre the starting point and of an arbitrary radius;
iv. Draw another semi circle on the line to intersect it;
v. If the diameter of the second semi circle is chosen to equal to the radius of the first semi circle; by drawing an arc to cut the second semi circle with its radius and joining the arc with the centre of the first semi circle, we have the construction of
approximate angle
20

angle 20
degrees

degrees;
vi. If the diameter of the first circle is chosen to equal the radius of the second; by drawing an arc to cut the first semi circle with its radius and joining the arc with the centre of the second semi circle, we have the construction of approximate angle
degrees;

vii. If the two adjacent semi circles have equal radii; by drawing an arc to cut any of the semi circles with radius of one of them and joining the arc with the centre of any of the two, we have the construction of approximate angle 30

degrees;
Remarks 3.1. This method is simple, finer and newest. The method has analogous operations as that of the method 1.
Proposition 3.1. All angles multiples of ten degrees can be constructed by approximation with the Me2 Method for they satisfy the recurrence relation $a_{10 m}=m a_{10}$ where $m \in \mathbb{Z}^{+}$, the set of positive integers and $a_{m}$ is the angle $m$ degrees, and all angles multiples of twenty degrees can be constructed by approximation using $a_{20 n}=\left(20^{\circ} \div 2\right) b+\left(20^{\circ} \times 2\right) c$ where $0 \leq \mathrm{b} \leq 3$ and $0 \leq \mathrm{c} \leq 9, \mathrm{~b}, \mathrm{c}, \mathrm{n} \in \mathbb{Z}^{+}$.

Proof. Clearly $m=1,2, \ldots, 36 \mathrm{in} a_{10 \mathrm{~m}}=m a_{10}$ is enough for the first part. Since b takes 4 values and c takes 9 , and $0^{\circ} \leq \mathrm{x} \leq 360^{\circ}$ where $\mathrm{x}=\mathrm{ma}_{10}$ are 36 in number, all the 36 angles multiples of angle 10 degrees are covered.Substituting $n=1,2,3, \ldots 18$ in $a_{20 n}=\left(20^{\circ} \div 2\right) b+\left(20^{\circ} \times 2\right) c$ where $0 \leq \mathrm{b} \leq 3$ and $0 \leq \mathrm{c} \leq 9, \mathrm{~b}, \mathrm{c}, \mathrm{n} \in \mathbb{Z}^{+}$is also an enough proof of the second part.

Remarks 3.2. Lemma 2.1 is an enough proof of proposition 3.1. Construction of approximate angles are relevant where exact angles are non-constructible. If we set $m=\frac{1}{2}$, then $a_{5}$ is also constructible by approximation and again by Lemma 2.1, all its multiples are constructible by approximation.

## REFERENCES

[1] Olagunju, Sam Olu, Olademo, J.O.Ani, "Construction of Any Angle from $1^{0}$ to $360^{0}$ without Bisection," Journal of Physical Sciences and Innovation, 4:1, 2012.
[2] Peter Brown, Micheal Evans, David Hunt, Janine Mclntosh Bill Pender, Jacqui Ramagge, "The Improving Mathematics Education in Schools(TIMES) Project Construction a Guide for Teachers-Year 7-8," Australian Mathematical Sciences Institute, Module 13:10, 2011.
[3] Ian Sterwart, Galois Theory, Chapman and Hall ICRC, $3{ }^{\text {rd }}$ Edition, 2003.
[4] Leonard Eugene Dickson, New First Course in the Theory of Equations, John Wiley and Sons, Inc; 50 ${ }^{\text {th }}$ Edition, 1964.
[5] Matt Haddock, "Polynomials and Their Applications to Ruler and Compass Constructions, MA 213 Second Year Essay," 2008.

