Minimum Zero-Centre-Pandiagonal Composite Type II (a) Loubéré Magic Squares over Multi Set of Integer Numbers as a Semiring

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ABSTRACT

This pioneer work explicates that the set of the Minimum Zero-Centre-Pandiagonal Composite Type II (a) Loubéŕe Magic Squares over Multi Set of Integer Numbers Z_L (as we denote it) forms an additive abelian group if equipped with the matrix binary operation of addition $\oplus$ (as we denote it) and if it is enclosed with the integer number operation of multiplication $\otimes$ (as we denote it), it forms a multiplicative semigroup with identity. That is, $(Z_L, \oplus, \otimes)$ forms a semiring for it satisfies all the axioms of a semiring.

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1. INTRODUCTION

Magic Square has been investigated over 3886 years. Mathematicians introduced it long time before astronomers picked up interest in it. Magic Square is lost in mathematics and is attributed to astronomers for many centuries. Recently, mathematicians tend to bring it back to mathematics again arguably after the introduction of Ring Theory. The oldest magic square recorded and originated in China is the Lo Shu Magic Square. De La Loubéré, a French diplomat, designed a procedure of constructing the Lo Shu Magic Square. We term all the Lo Shu Magic Squares constructed with the procedure as Loubéré Magic Squares.

By Zero-Centre-Pandiagonal Composite Type II (a) Loubéŕe Magic Squares over Multi Set of Integer Numbers, we understand the set of magic squares formed by the De La Loubéŕe Procedure of repeating- pattern- sequence such that each of its cell is a Loubéŕe Magic Square of Type II(a) and such that the diagonal magic squares across the face-centre magic square contain only zero entries.

We showcased that the Minimum Zero-Centre-Pandiagonal Composite Type II (a) Loubéŕe Magic Squares over Multi Set of Integer Numbers Z_L (as we denote it) forms an additive abelian group if equipped with the matrix binary operation of addition $\oplus$ (as we denote it) and if it is enclosed with the integer number operation of multiplication $\otimes$ (as we denote it), it forms a multiplicative semigroup with identity. $(Z_L, \oplus, \otimes)$ satisfies the distributivity of $\otimes$ over $\oplus$ from either sides and the neutral element of $\oplus$ is $\otimes$ absorbing.

Arbitrarily, $3^2 \times 3^2$ Minimum Zero-Centre-Pandiagonal Composite Type II (a) Loubéŕe Magic Squares over Multi Set of Integer Numbers is considered – not squares such that $n \geq 5$ – to economize space, and not $1^2 \times 1^2$, the triviality, for it is isomorphic to the aforementioned underlining multi set.

The sequence a, a, a, ... , n number, b, b, b, ... , n number, c, c, c, ... n number, ... n times rather than a, b, c, ... n times, a, b, c, ... n times, a, b, c, ... n times, ... n number is considered, though the later will also give an analogous result; but presenting both the two is a babyish tautology.

2. PRELIMINARIES

Definition 2,1. A multi set is a set in which repetition of elements is relevant/important. For example, $\{a, a, a, b, b, b, c, c, c\} \neq \{a, b, c\} \neq \{a, b, c, a, b, c, a, b, c\}$ for their Loubéré Magic Squares are not isomorphic.

Definition 2.2. A Composite Loubéŕe Magic Square is a magic square such that each of its cell (grid) is a Loubéŕe Magic Square. See also [1].

Definition 2.3. Main Row or Column is the column or row of the Loubéré Magic Squares containing the first term and the last term of the arithmetic sequence in the square.

Definition 2.4. A Loubéŕe Magic Square of type I is a magic square of arithmetic sequence entries such that the entries along the main column or row have a common difference and the main column or row is the central column or central row respectively.

2.5 Loubéré Procedure (NE-W-S or NW-E-S, the cardinal points). Consider an empty $n \times n$ square of grids (or cells). Start, from the central column or row at a position $\left\lceil \frac{n}{2} \right\rceil$ where $\lceil \quad \rceil$ is the greater integer number less than or equal to, with the number 1. The fundamental movement for filling the square is diagonally up, right (clock wise or NE or SE) or up left (anti clock wise or NW or SW) and one step at a time. If a filled cell (grid) is encountered, then the next consecutive number moves vertically down ward one square instead. Continue in this fashion until when a move would leave the square, it moves due N or E or W or S (depending on the position of the first term of the sequence) to the last row or first row or first column or last column. See also [2] for such a procedure.

Definition 2.6. Loubéŕe Magic Squares of type II are magic squares constructed with Loubéŕe Procedure with repeating – pattern - sequence.

Remark 2.7. If we use a repeating pattern sequence a, a, a, ... n times, b, b, b,...n times, c, c, c, ... n times, ... n number; we get Type II(a) Loubéŕe Magic Square and if instead we use a,b,c,... n number, a, b, c,... n number, ... n times; we changed the orientation of the diagonal and we get the Type II(b) Loubéŕe Magic Square.

2.8. Semigroup

A non empty set S equipped with an operation * is known as a semigroup if the following properties are satisfied.

i. S is closed with respect to *.i.e., $a * b \in S, \forall a, b \in S;$

ii. $*$ is associative in S. i.e., $a * (b * c) = (a * b) * c, \forall a, b, c \in S;$

If in addition to the axioms above, the following axiom is satisfied, we call $(S,*)$ a semigroup with identity where $(S,*)$ is a denotation of a semigroup.

iii. $\exists e \ni a * e = e * a = a, \forall a \in S;$

For the definition of semigroup, see also [3].

2.9.Abelian Group

If in addition to the axioms above (2.8), the following axioms are satisfied, we call $(S,*)$ an abelian group.

iv. $\forall a \in S, \exists b \in S \ni a * b = b * a = e$; and

v. $\forall a, b \in S;\ a * b = b * a.$

2.10. Semiring [4]

A semiring is an algebraic structure, consisting of a nonempty set R on which we have two operations, addition $\oplus$ and multiplication$\otimes$ such that the following conditions hold:

i. Addition is associative and commutative and has a neutral element. That is to say, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ and $a \oplus b = b \oplus a$ for all $a, b, c \in R$ and there exists a special element of R, usually denoted by 0, such that $a \oplus 0 = 0 \oplus a$ for all $a \in R$. It is very easy to prove that this element is unique.

ii. Multiplication is associative and has a neutral element. That is to say, $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ for all $a, b, c \in R$ and there exists a special element of R, usually denoted by 1, such that $a \otimes 1 = a = 1 \otimes a$ for all $a \in R$. It is very easy to prove that this element too is unique. In order to avoid trivial cases, we will always assume that $1 \neq 0$, thus insuring that R has at least two distinct elements.

iii. Multiplication distributes over addition from either side. That is to say, $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$ and $(a \oplus b)c = a \otimes c \oplus b \otimes c$ for all $a, b, c \in R$.

iv. The neutral element with respect to addition is multiplicatively absorbing. That is to say, $a \otimes 0 = 0 = 0 \otimes a$ for all $a \in R$.

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3. MINIMUM ZERO-CENTRE-PANDIAGONAL COMPOSITE TYPE II (A) LOUBERE MAGIC SQUARES OVER MULTI SET OF INTEGER NUMBERS AS A SEMIRING

Let the set of the Minimum Zero-Centre-Pandiagonal Composite Type II (a) Loubéŕe Magic Squares over Multi Set of Integer Numbers be denoted by Z_L , let the matrix binary operation of addition be denoted by $\oplus$, and let the integer number operation of multiplication be denoted by $\otimes$. Then, we present the following theorems:

Theorem 3.1. $(Z_L, \oplus)$ forms an additive abelian group.

Proof. Arbitrarily considering $Z_{L_{3^2 \times 3^2}}$(a denotation of the aforementioned squares of $3^2 \times 3^2$) $-$ we define $\oplus$ as follows: Let $A, B \in Z_{L_{3^2 \times 3^2}}$ where

$$A = \begin{bmatrix} f & d & e & c & a & b & 0 & 0 & 0 \\ d & e & f & f & a & c & 0 & 0 & 0 \\ e & f & d & b & c & a & 0 & 0 & 0 \\ c & a & b & 0 & 0 & 0 & f & d & e \\ a & b & c & 0 & 0 & 0 & d & e & f \\ b & c & a & 0 & 0 & 0 & e & f & d \\ 0 & 0 & 0 & f & d & e & c & a & b \\ 0 & 0 & 0 & d & e & f & a & b & c \\ 0 & 0 & 0 & c & a & b & b & c & a \end{bmatrix} \text{ and } B = \begin{bmatrix} z & x & y & w & u & v & 0 & 0 & 0 \\ x & y & z & u & v & w & 0 & 0 & 0 \\ y & z & x & v & w & u & 0 & 0 & 0 \\ w & u & v & 0 & 0 & 0 & z & x & y \\ u & v & w & 0 & 0 & 0 & x & y & z \\ v & w & u & 0 & 0 & 0 & y & z & x \\ 0 & 0 & 0 & z & x & y & w & u & v \\ 0 & 0 & 0 & x & y & z & u & v & w \\ 0 & 0 & 0 & y & z & x & v & w & u \end{bmatrix}. \text{ Then:}$$

$$A \oplus B = \begin{bmatrix} f & d & e & c & a & b & 0 & 0 & 0 \\ d & e & f & f & a & c & 0 & 0 & 0 \\ e & f & d & b & c & a & 0 & 0 & 0 \\ c & a & b & 0 & 0 & 0 & f & d & e \\ a & b & c & 0 & 0 & 0 & d & e & f \\ b & c & a & 0 & 0 & 0 & e & f & d \\ 0 & 0 & 0 & f & d & e & c & a & b \\ 0 & 0 & 0 & d & e & f & a & b & c \\ 0 & 0 & 0 & c & a & b & b & c & a \end{bmatrix} \oplus \begin{bmatrix} z & x & y & w & u & v & 0 & 0 & 0 \\ x & y & z & u & v & w & 0 & 0 & 0 \\ y & z & x & v & w & u & 0 & 0 & 0 \\ w & u & v & 0 & 0 & 0 & z & x & y \\ u & v & w & 0 & 0 & 0 & x & y & z \\ v & w & u & 0 & 0 & 0 & y & z & x \\ 0 & 0 & 0 & z & x & y & w & u & v \\ 0 & 0 & 0 & x & y & z & u & v & w \\ 0 & 0 & 0 & y & z & x & v & w & u \end{bmatrix}$$

$$= \begin{bmatrix} f+z & d+x & e+y & c+w & a+u & b+v & 0 & 0 & 0 \\ d+x & e+y & f+z & f+u & a+v & c+w & 0 & 0 & 0 \\ e+y & f+z & d+x & b+v & c+w & a+u & 0 & 0 & 0 \\ c+w & a+u & b+v & 0 & 0 & 0 & f+z & d+x & e+y \\ a+u & b+v & c+w & 0 & 0 & 0 & d+x & e+y & f+z \\ b+v & c+w & a+u & 0 & 0 & 0 & e+y & f+z & d+x \\ 0 & 0 & 0 & f+z & d+x & e+y & c+w & a+u & b+v \\ 0 & 0 & 0 & d+x & e+y & f+z & c+u & a+v & b+w \\ 0 & 0 & 0 & c+y & a+z & b+x & b+v & c+w & a+u \end{bmatrix} = C$$

i. $Z_{L_{3^2 \times 3^2}}$ is **closed** with respect to $\oplus$: From the above definition, if $A, B \in Z_{L_{3^2 \times 3^2}}$, then $A \oplus B = C \in Z_{L_{3^2 \times 3^2}}$. This is more vivid by letting (say) $p = a + u, q = b + v, r = c + w, s = d + x, t = e + y$ etc.

ii. $\oplus$ is **associative**: For $A, B, C \in Z_{L_{3^2 \times 3^2}}$, $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ whence $i \oplus \big(j \oplus (i \oplus j)\big) = (i \oplus j) + (i \oplus j)$ where $i \in A,\ j \in B$ and $i \oplus j \in C$

iii. The **identity element** 0 is $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

iv. Given $X = \begin{bmatrix} z & x & y & w & u & v & 0 & 0 & 0 \\ x & y & z & u & v & w & 0 & 0 & 0 \\ y & z & x & v & w & u & 0 & 0 & 0 \\ w & u & v & 0 & 0 & 0 & z & x & y \\ u & v & w & 0 & 0 & 0 & x & y & z \\ v & w & u & 0 & 0 & 0 & y & z & x \\ 0 & 0 & 0 & z & x & y & w & u & v \\ 0 & 0 & 0 & x & y & z & u & v & w \\ 0 & 0 & 0 & y & z & x & v & w & u \end{bmatrix}$, there exists an **inverse** of it, Y express as

$$Y = \begin{bmatrix} -z & -x & -y & -w & -u & -v & 0 & 0 & 0 \\ -x & -y & -z & -u & -v & -w & 0 & 0 & 0 \\ -y & -z & -x & -v & -w & -u & 0 & 0 & 0 \\ -w & -u & -v & 0 & 0 & 0 & -z & -x & -y \\ -u & -v & -w & 0 & 0 & 0 & -x & -y & -z \\ -v & -w & -u & 0 & 0 & 0 & -y & -z & -x \\ 0 & 0 & 0 & -z & -x & -y & -w & -u & -v \\ 0 & 0 & 0 & -x & -y & -z & -u & -v & -w \\ 0 & 0 & 0 & -y & -z & -x & -v & -w & -u \end{bmatrix}$$ such that $X \oplus Y = Y \oplus X = 0$.

v. $\oplus$ is **commutative**: $A, B \in Z_{L_{3^2 \times 3^2}} \Rightarrow A \oplus B = B \oplus A$ whence the matrix binary operation of addition over set of integer numbers is commutative (inheritance).

Thus, $(Z_L, \oplus)$ forms an additive abelian group.

Theorem 3.2. $(Z_L, \otimes)$ forms a multiplicative semigroup with identity.

Proof. Let $D, E \in Z_{L_{3^2 \times 3^2}}$. We define $\otimes$ as follows: Let $D = \begin{bmatrix} l & j & k & i & g & h & 0 & 0 & 0 \\ j & k & l & g & h & i & 0 & 0 & 0 \\ k & l & j & h & i & g & 0 & 0 & 0 \\ i & g & h & 0 & 0 & 0 & l & j & k \\ g & h & i & 0 & 0 & 0 & j & k & l \\ h & i & g & 0 & 0 & 0 & k & l & j \\ 0 & 0 & 0 & l & j & k & i & g & h \\ 0 & 0 & 0 & j & k & l & g & h & i \\ 0 & 0 & 0 & k & l & j & h & i & g \end{bmatrix}$ and

$$E = \begin{bmatrix} r & p & q & o & m & n & 0 & 0 & 0 \\ p & q & r & m & n & o & 0 & 0 & 0 \\ q & r & p & n & o & m & 0 & 0 & 0 \\ o & m & n & 0 & 0 & 0 & r & p & q \\ m & n & o & 0 & 0 & 0 & p & q & r \\ n & o & m & 0 & 0 & 0 & q & r & p \\ 0 & 0 & 0 & r & p & q & o & m & n \\ 0 & 0 & 0 & p & q & r & m & n & o \\ 0 & 0 & 0 & q & r & p & n & o & m \end{bmatrix}.$$

Then $$D \otimes E = \begin{bmatrix} l \times r & j \times p & k \times q & i \times o & g \times m & h \times n & 0 & 0 & 0 \\ j \times p & k \times q & l \times r & g \times m & h \times n & i \times o & 0 & 0 & 0 \\ k \times q & l \times r & j \times p & h \times n & i \times o & g \times m & 0 & 0 & 0 \\ i \times o & g \times m & h \times n & 0 & 0 & 0 & l \times r & j \times p & k \times q \\ g \times m & h \times n & i \times o & 0 & 0 & 0 & j \times p & k \times q & l \times r \\ h \times n & i \times o & g \times m & 0 & 0 & 0 & k \times q & l \times r & j \times p \\ 0 & 0 & 0 & l \times r & j \times p & k \times q & i \times o & g \times m & h \times n \\ 0 & 0 & 0 & j \times p & k \times q & l \times r & g \times m & h \times n & i \times o \\ 0 & 0 & 0 & k \times q & l \times r & j \times p & h \times n & i \times o & g \times m \end{bmatrix} = F$$

i. $Z_{L_{3^2\times 3^2}}$ is **closed** with respect to $\otimes$: For $D, E \in Z_{L_{3^2\times 3^2}}$,$F \in Z_{L_{3^2\times 3^2}}$ from the above definition. This is vivid by intimate look at the pattern of elements in F.

ii. **Associativity**: $\otimes$ is associative for $A, B, C \in Z_{L_{3^2\times 3^2}} \Rightarrow A \otimes (B \otimes C) = (A \otimes B) \otimes C$ because $x \otimes (y \otimes (xy)) = (x \otimes y) \otimes (xy)\ \forall x \in A, y \in B$ and $xy \in C$.

iii. The **identity element** is $\begin{bmatrix} 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1 \end{bmatrix}$

Thus, $(Z_L, \otimes)$ forms a multiplicative semigroup with identity.

Remark 3.3. For all $a, b, c \in Z_L$, $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$ and $(a \oplus b)c = a \otimes c \oplus b \otimes c$.

The neutral element with respect to addition is multiplicatively absorbing. That is to say, $a \otimes 0 = 0 = 0 \otimes a$ for all $a \in R$ whence

$$\begin{bmatrix} f&d&e&c&a&b&0&0&0\\ d&e&f&f&a&c&0&0&0\\ e&f&d&b&c&a&0&0&0\\ c&a&b&0&0&0&f&d&e\\ a&b&c&0&0&0&d&e&f\\ b&c&a&0&0&0&e&f&d\\ 0&0&0&f&d&e&c&a&b\\ 0&0&0&d&e&f&a&b&c\\ 0&0&0&c&a&b&b&c&a \end{bmatrix} \otimes \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix} =$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix} \otimes \begin{bmatrix} f&d&e&c&a&b&0&0&0\\ d&e&f&f&a&c&0&0&0\\ e&f&d&b&c&a&0&0&0\\ c&a&b&0&0&0&f&d&e\\ a&b&c&0&0&0&d&e&f\\ b&c&a&0&0&0&e&f&d\\ 0&0&0&f&d&e&c&a&b\\ 0&0&0&d&e&f&a&b&c\\ 0&0&0&c&a&b&b&c&a \end{bmatrix}$$

Theorem 3.4. $(Z_L, \oplus, \otimes)$ forms a semiring.

Proof. This follows immediately from Theorem 3.1, Theorem 3.2 and Remark 3.3 above.

4. CONCLUSION

Every Minimum Zero-Centre-Pandiagonal Composite Type II (a) Loubéré Magic Squares under discussion in this work has about 4 miscellany effects of rotations and/or reflections. Considering 1 out of the 4 effects is arbitrary. Although the Magic Squares presented here are not the basic (obvious) ones, yet they are Pandiagonal Loubéré – with unique magic sums and products. We recommend that if you are apt in searching for an example of any algebraic structures be it semigroup (Fibonacci), group (Symmetric), semiring (the one we presented), field (Loubéré), vector space (Magic Squares in general) or not enter Loubéré Magic Squares, you will find one.

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