M-Gonal number$\pm$ 3 = A Perfect square

M.A.Gopalan[#1], V.Geetha[*2]

1. Professor, Department of Mathematics, Shrimathi Indira Gandhi College, Trichy, Tamilnadu, India
2. Assiatant Professor, Department of Mathematics, Cauvery College for Women, Trichy, Tamilnadu, India

Abstract:
We search for the ranks of Triangular, Pentagonal, Hexagonal number, Heptagonal numbers such that each of these M-gonal number - 3= a perfect square and the ranks of Triangular, Pentagonal, Hexagonal, Heptagonal, Octagonal and Nanogonal such that each of these M-gonal number + 3= a perfect square.

Keywords:
Triangular number, Pentagonal number, Hexagonal number, Heptagonal number, Octagonal number and Nanogonal number

I. Introduction:

Number is the essence of mathematical calculation. Variety of numbers have variety of range and richness, Many numbers exhibit fascinating properties, they form sequences, they form patterns and so on [1]. In [3] explicit formulas for the ranks of triangular numbers which are equal to Pentagonal, Octagonal, Decagonal and Dodecagonal numbers in turn are presented. In this context one may refer [4-8]. In this communication, we make an attempt to obtain the ranks of Triangular, Pentagonal, Hexagonal number, Heptagonal numbers such that each of these M-gonal number - 3= a perfect square and the ranks of Triangular, Pentagonal, Hexagonal, Heptagonal, Octagonal and Nanogonal such that each of these M-gonal number + 3= a perfect square. Also the recurrence relations satisfied by the ranks of each of these M-gonal numbers are presented.

Method of Analysis

Pattern 1:

Denoting the rank of the nth Triangular number to be A, the identity,

$$\text{Triangular number} - 3 = x^2 \quad (1)$$

is written as

$$y^2 = 8x^2 + 25 \quad (2)$$

where $y = 2A + 1$ (3)

whose initial solution is $x_0 = 1, y_0 = 3$ (4)

Let $(\tilde{x}_0, \tilde{y}_0)$ be the general solution of the Pellian

$$y^2 = 8x^2 + 1 \quad (5)$$

where $x_s = \frac{5}{2\sqrt{8}}\left(\left(3+\sqrt{8}\right)^{s+1} - \left(3-\sqrt{8}\right)^{s+1}\right)$

$$y_s = \frac{5}{2}\left(\left(3+\sqrt{8}\right)^{s+1} + \left(3-\sqrt{8}\right)^{s+1}\right), \quad s = 0,1,....$$

Inview of (3), the ranks of Triangular number is given by

$$A_s = \frac{1}{4}\left[5\left(\left(3+\sqrt{8}\right)^{s+1} + \left(3-\sqrt{8}\right)^{s+1}\right) - 2\right],\ s = 0,1,2...$$

and the corresponding recurrence relation is found to be

$$A_{s+2} = 6A_{s+1} - A_s + 2$$

In a similar manner, we present below the ranks of Pentagonal, Hexagonal and Heptagonal numbers in tabular form

S. No	**M-Gonal number**	**General form of ranks**
1	Pentagonal number (B)	$B_{2s-1} = \frac{1}{12}\left(\left(5+\sqrt{24}\right)^{2s}\left(13+2\sqrt{24}\right) + \left(5-\sqrt{24}\right)^{2s}\left(13-2\sqrt{24}\right) + 2\right),\ s = 1,2...$
2	Hexagonal number (C)	$C_{2s-1} = \frac{1}{8}\left[5\left(\left(3+\sqrt{8}\right)^{2s} + \left(3-\sqrt{8}\right)^{2s}\right) + 2\right], s = 1,2...$
3	Heptagonal number (D)	$D_{2s-1} = \frac{1}{20}\left(\left(19+3\sqrt{40}\right)^{2s}\left(13+\sqrt{40}\right) + \left(19-3\sqrt{40}\right)^{2s}\left(13-\sqrt{40}\right) + 6\right)\ s = 1,2...$

The recurrence relations satisfied by the ranks of each the M-Gonal numbers presented in the table above are as follows

S.NO	**RECURRENCE RELATIONS**
1	$B_{2s+3} = 98B_{2s+1} - B_{2s-1} - 16$
2	$C_{2s+3} = 6C_{2s+1} - C_{2s-1} - 1$
3	$D_{2s+3} = 1442D_{2s+1} - D_{2s-1} - 432$

Pattern 2:

Denoting the rank of the nth Pentagonal number to be B, the identity,

$$\text{Pentagonal number} + 3 = x^2 \quad (6)$$

is written as

$$y^2 = 24x^2 - 71 \quad (7)$$

where $y = 6B - 1$ (8)

whose initial solution is $x_0 = 2, y_0 = 5$ (9)

Let $\left(\tilde{x}_0, \tilde{y}_0\right)$ be the general solution of the Pellian

$$y^2 = 24x^2 + 1 \quad (10)$$

where $\tilde{x}_s = \frac{1}{2\sqrt{24}}\left(\left(5+\sqrt{24}\right)^{s+1} - \left(5-\sqrt{24}\right)^{s+1}\right)$

$$\tilde{y}_s = \frac{1}{2}\left(\left(5+\sqrt{24}\right)^{s+1} + \left(5-\sqrt{24}\right)^{s+1}\right), \quad s = 0,1,....$$

Applying Brahmagupta's lemma [2] between the solutions (x_0, y_0) and $(\tilde{x}_s, \tilde{y}_s)$ the sequence of values of x and y satisfying equation (7) is given by

$$x_s = \frac{1}{2\sqrt{24}}\left(\left(5+\sqrt{24}\right)^{s+1}\left(5+2\sqrt{24}\right) - \left(5-\sqrt{24}\right)^{s+1}\left(5-2\sqrt{24}\right)\right)$$

$$y_s = \frac{1}{2}\left(\left(5+\sqrt{24}\right)^{s+1}\left(5+2\sqrt{24}\right) + \left(5-\sqrt{24}\right)^{s+1}\left(5-\sqrt{24}\right)\right), \quad s = 0,1,....$$

Inview of (8), the ranks of Pentagonal number is given by

$$B_{2s-1} = \frac{1}{12}\left(\left(5+\sqrt{24}\right)^{2s}\left(5+2\sqrt{24}\right) + \left(5-\sqrt{24}\right)^{2s}\left(5-2\sqrt{24}\right) + 2\right), \quad s = 1,2...$$

and the corresponding recurrence relation is found to be

$$B_{2s+3} = 98B_{2s+1} - B_{2s-1} - 16$$

In a similar manner, we present below the ranks of Pentagonal, Hexagonal and Heptagonal numbers in tabular form

S. No	**M-Gonal number**	**General form of ranks**
1	Triangular number (A)	$A_s = \frac{1}{4}\left[5\left(\left(3+\sqrt{8}\right)^{s+1} + \left(3-\sqrt{8}\right)^{s+1}\right) - 2\right], \; s = 1,2...$
2	Hexagonal number (C)	$C_{2s-1} = \frac{1}{8}\left[\left(3+\sqrt{8}\right)^{2s}(7+3\sqrt{8}) + \left(3-\sqrt{8}\right)^{2s}(7-3\sqrt{8}) + 2\right], s = 1,2...$
3	Heptagonal number (D)	$D_{2s-1} = \frac{1}{20}\left(\left(19+3\sqrt{40}\right)^{2s}\left(7+2\sqrt{40}\right) + \left(19-3\sqrt{40}\right)^{2s}\left(7-2\sqrt{40}\right) + 6\right) \; s = 1,2...$
4.	Octagonal number (E)	$E_{2s-1} = \frac{1}{6}\left(\left(2+\sqrt{3}\right)^{2s}\left(2+2\sqrt{3}\right) + \left(2-\sqrt{3}\right)^{2s}\left(2-2\sqrt{3}\right) + 2\right), \; s = 1,2...$
5.	Nanogonal number (F)	$F_s = \frac{1}{28}\left[(15+2\sqrt{56})^{s+1}(9+2\sqrt{56}) + (15-2\sqrt{56})^{s+1}(9-2\sqrt{56}) + 10\right], \; s = 0,1,2...$

The recurrence relations satisfied by the ranks of each the M-Gonal numbers presented in the table above are as follows

S.NO	**RECURRENCE RELATIONS**
1	$A_{s+2} = 6A_{s+1} - A_s + 2$
2	$C_{2s+3} = 34C_{2s+1} - C_{2s-1} - 8$

3	$D_{2s+3} = 1442D_{2s+1} - D_{2s-1} - 432$
4.	$E_{2s+3} = 140E_{2s+1} - E_{2s-1} - 4$
5	$F_{s+2} = 30F_{s+1} - F_s - 10$

Conclusion:

To conclude, one may search for the other M-gonal numbers satisfying the relation under consideration.

References:

1. L.E.Dikson, "History of Theory of Numbers", Chelsa Publishing Company, New York, 2, 1952.
2. T.S.Bhanumathy, Ancient Indian Mathematics, New Age Publishers Ltd, New Delhi, 1995.
3. M.A.Gopalan and S.Devibala, "Equality of Triangular Numbers with Special m-gonal numbers", Bulletin of Allahabad Mathematical Society, 21, 25-29, 2006.
4. M.A.Gopalan, Manju Somanath and N.Vanitha, "Equality of Centered Hexagonal Number with Special M-Gonal numbers", Impact J.Sci.Tech.Vol.1(2), 31-34(2007).
5. M.A.Gopalan, Manju Somanath and N.Vanitha, "M-Gonal number – 1 = A Perfect square", Acta Cinencia Indica, Vol.XXXIII M, No.2, 479- 480, 2007.
6. M.A.Gopalan., Manju Somanath, N.Vanitha, "On pairs of m-gonal numbers with unit difference", Reflections des ERA-JMS, Vol.4, Issue 4, Pp.365-376, 2009.
7. M.A.Gopalan., K.Geetha and Manju Somanath, "Equality of centered hexagonal number with special M-gonal number", Global journal of Mathematics and Mathematical Science, Vol.3, No.1, PP.41-45, 2013.
8. M.A.Gopalan., K.Geetha and Manju Somanath,"On pairs of M-gonal numbers with unit difference", International Journal of Engineering Science and Mathematics, Vol.2, Issue.2, PP.219-222, June 2013.