Solving Fuzzy Linear Programming Problems Using Two-Level Programming Approach

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Abstract

Linear programming (LP) has been widely applied to solving real world problems. The conventional LP model requires the parameters to be known constants. In the real world, however, the parameters are seldom known exactly and have to be estimated. This paper deals a linear programming (FLP) problem with fuzzy parameters. The problem is considered by incorporating fuzzy numbers in the cost coefficients, required coefficients, and technological coefficients. Through the use of the α-level sets of fuzzy numbers, the FLP problem is converted to the corresponding α-parametric LP problem (α-PLP) and hence to interval linear programming (ILP) problem. A pair of two-level mathematical programs is formulated to calculate the lower bound (Lb) and upper bound (Ub) of the objective values of ILP problem. The two-level mathematical programs are then transformed into one-level nonlinear programs. Solving the pair of nonlinear programs produces the interval of the objective values of the problem. An illustrative numerical example is provided in the sake of the paper to clarify the proposed approach.

Keywords: Linear programming problem; Fuzzy numbers; α-cut; Interval confidence; Two-level programming approach; α-optimal solution.

1. Introduction

Linear programming (LP) is a mathematical modeling technique designed to optimize the usage of limited resources. It has been widely used to solve problem in military industries, agriculture, economic, and even behavioral and social sciences. Hartley (1992) and Lane et al., (1993) indicate that LP is the most frequently used technique in solving real world problems among all operations research techniques.

In fuzzy decision making problems, the concept of a maximizing decision was proposed by Bellman and Zadeh (1970). By adopting the concept to a mathematical programming problem. Tanaka et al., (1984) formulated the so-called fuzzy mathematical programming problem, and showed that a compromise solution of decision maker (DM) could be obtained through an iterative use of linear

programming technique. Zimmermann (1976) also presented a fuzzy approach to multi-objective linear programming problems.

Fuzzy linear programming problem with fuzzy coefficients was formulated by Negoita (1970) and called robust programming. Dubois and Prade (1982) investigated linear fuzzy constraints. Tanaka and Asal (1984) proposed a formulation of FLP with fuzzy coefficients, and gave a method to solve it on the basis of the inequality relation between fuzzy numbers. Shaocheng (1994) presented two kinds of linear programming with fuzzy numbers. They are called interval number and fuzzy number linear programmings, respectively. Guijun and Xiaoping (1998) defined interval-valued fuzzy numbers and interval-distribution numbers, and then their extended operations are given. Carlsson and Korhonen (1986), Chanas (1983), and Sakawa and Yano (1991) studied the fuzzy mathematical programming with fuzzy number coefficients. Wang and Qiao (1993) put forward some models of LP with fuzzy random variable coefficients, and studied the solution and distribution problem of a fuzzy random programming. Qiao et al., (1994) studied fuzzy random LP having random variable coefficients and decision vector of fuzzy pseudorandom variables. Dubey et al., (2012) exploited the equivalence between the interval. Valued fuzzy set and the intuitionistic fuzzy set (IFS) to study linear programming problems involving interval uncertainty method using IFS. Veeramani and Duraisamy (2012) discussed the fully fuzzy linear programming (FFLP) problems in which all the parameters and variables are triangular fuzzy numbers. They proposed a new approach of solving FFLP problem using the concept of nearest symmetric triangular fuzzy number approximation with preserve expected interval. Kumar and Bhtia (2012) studied the sensitivity analysis for fuzzy variable linear programming problems using ranking function. Nasseri and Ebrahimenjad (2012) defined a new dual problem for the linear programming problem with trapezoidal fuzzy variables as a linear programming problem with trapezoidal fuzzy variables and hence deduce the duality results such as a weak duality. Zhong et al., (2013) presented a new problems, which is named a revised interior point method. Jimenez (2007) proposed a method for solving linear programming problems where all the coefficients are, in general, fuzzy numbers. Sanei (2013) proposed a new method for solving a special kind of fuzzy number linear programming problem with bounded variables directly using linear ranking function. Nasseri et al., (2014) presented a new method for solving gully fuzzy linear programming problems.

In this paper, a linear programming problem with fuzzy parameters including, the cost coefficients, required coefficients, and technological coefficients is presented. The considered problems converted to the corresponding α-parametric LP problem and hence to the interval LP problem. A two-level programming approach is being used to solve the interval LP problem.

This paper is organized as in the following sections. In section 2, some preliminaries are introduced. In section 3, a linear programming problem with fuzzy parameters is introduced as a specific definition and properties. In section 4, a two-level programming approach for solving ILP problem is provided. In section 5, a numerical example is given. Finally, some concluding remarks are reported in section 6.

2. Preliminaries

In this section, some of the fundamental definitions and concepts of fuzzy numbers intiated by Bellman and Zadeh (1970) and interval of confidence introduced by Kaufmann and Gupta (1988) are reviewed.

Definition 1. Let R be the set of real numbers, the fuzzy number $\tilde{p}$ is a mapping $\mu_{\tilde{p}} : R \rightarrow [0, 1]$, with the following properties:

(1) $\mu_{\tilde{p}}(x)$ is upper semi continuous membership functions.

(2) $\tilde{p}$ is convex fuzzy set, i.e., $\mu_{\tilde{p}}(\lambda x^1 + (1-\lambda)x^2) \geq \min(\mu_{\tilde{p}}(x^1), \mu_{\tilde{p}}(x^2))$,

  for all $x^1, x^2 \in R$, $\lambda \in [0, 1]$.

(3) $\tilde{p}$ is normal, i.e., $\exists\, x_0 \in R$ for which $\mu_{\tilde{p}}(x_0) = 1$.

(4) Supp $(\tilde{P}) = \{x : \mu_{\tilde{p}}(x) > 0\}$ is the supper of a fuzzy set $\tilde{p}$.

Definition 2. An α-level set (or α-cut) of a fuzzy set $\tilde{p}$ of R is a non-fuzzy set denoted by $L_\alpha(\tilde{p})$ and defined by:

$$L_\alpha(\tilde{p}) = \{x \in R : \mu_{\tilde{p}}(x) \geq \alpha\},\ \alpha \in [0, 1]$$

Let $I(R) = \{[p^-, p^+] : p^-, p^+ \in R = (-\infty, \infty), p^- \leq p^+\}$ denote the set of all closed bounded interval numbers on R.

Definition 3. Suppose that $[p^-, p^+]$, $[q^-, q^+] \in I(R)$. We define

(i) $[p^-, p^+] (+) [q^-, q^+] = [p^- + q^-, p^+ + q^+]$.

(ii) $[p^-, p^+] (-) [q^-, q^+] = [p^- - q^-, p^+ - q^+]$.

(iii) $[p^-, p^+] (\cdot) [q^-, q^+] = [\min(p^- \cdot q^-, p^- \cdot q^+, p^+ \cdot q^-, p^+ \cdot q^+),$

$$\max(p^- \cdot q^-, p^- \cdot q^+, p^+ \cdot q^-, p^+ \cdot q^+)].$$

(iv) $[p^-, p^+] (/) [q^-, q^+] = [\min(p^- / q^-, p^- / q^+, p^+ / q^-, p^+ / q^+),$

$$\max(p^- / q^-, p^- / q^+, p^+ / q^-, p^+ / q^+)].$$

(v) The order relation $"\leq"$ in $I(R)$ is defined by $[p^-, p^+] (\leq) [q^-, q^+]$ if and only if $p^- \leq q^-$, $p^+ \leq q^+$.

Throughout this paper, $F_0(R)$ denote the set of all compact (i.e., bounded and closed fuzzy numbers on R, that is for any $\tilde{p} \in F_0(R)$, $\tilde{p}$ satisfies.

(a) There exists $x \in R$ such that $\mu_{\tilde{p}}(x) = 1$.

(b) For any $\alpha \in [0, 1]$, $\hat{p}_\alpha = [p_\alpha^-, p_\alpha^+]$ is a closed interval number on R.

Observed that $R \subset I(R) \subset F_0(R)$.

3. Problem formulation

A linear programming problem with fuzzy parameters is formulated as:

$$\min Z = \sum_{j=1}^{n} \tilde{c}_j x_j$$

subject to

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \,\tilde{\geq}\, \tilde{b}_i, \qquad i = 1, ..., r \qquad\qquad (1)$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \,\tilde{=}\, \tilde{b}_i, \qquad i = r + 1, ..., m$$

$$x_j \geq 0, \qquad\qquad j = 1, ..., s,$$

$$x_j \text{ unrestricted in sign}, \quad j = s + 1, ..., n,$$

where $\tilde{c}_j = (\tilde{c}_1, ..., \tilde{c}_n)$, $\tilde{b}_i = (\tilde{b}_1, ..., \tilde{b}_m)^T$, and $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ are fuzzy parameters. The superscript T means transposition. Assuming that these fuzzy parameters are characterized by fuzzy numbers. Let the corresponding membership functions be:

$(\mu_{\tilde{c}_1}(c_1), ..., \mu_{\tilde{c}_n}(c_n))$, $(\mu_{\tilde{b}_1}(b_1), ..., \mu_{\tilde{b}_m}(b_m))^T$, and $\mu_{\tilde{a}_{ij}}(a_{ij})$, $i = 1, ..., r$, $r + 1, ..., m$; $j = 1, ..., s$, $s + 1, ..., n$.

We introduce the α-level set of the fuzzy numbers $\tilde{c}, \tilde{b}$ and $\tilde{A}$ defined as the ordinary set $(\tilde{c}, \tilde{b}, \tilde{A})_\alpha$ is which the degree of their membership unction exceeds level α:

$$(\tilde{c}, \tilde{b}, \tilde{A})_\alpha = \{(c, b, A): \mu_{\tilde{c}_j}(c_j) \geq \alpha, \; j = 1, ..., n; \; \mu_{\tilde{b}_i}(b_i) \geq \alpha, \; i = 1, ..., r,$$

$$r + 1, ..., m; \; \mu_{\tilde{a}_{ij}}(a_{ij}) \geq \alpha, \; i = 1, ..., r, r + 1, ..., m;$$

$$j = 1, ..., s, s + 1, ..., n\} \qquad (2)$$

Now, suppose that DM considers that the degree of all of the membership functions of the fuzzy numbers involved in the linear programming problem should be greater than or equal to some value α. Then, for such a degree α. Model (1) can be interpreted as the following non fuzzy linear programming problem which depends on coefficient vector $(c, b, A) \in (\tilde{c}, \tilde{b}, \tilde{A})_\alpha$ (Sakawa (1993))

$$\min_x Z(x) = \sum_{j=1}^n c_j x_j$$

subject to

$$\sum_{j=1}^n a_{ij} x_j \geq b_i, \qquad i = 1, ..., r;$$

$$\sum_{j=1}^n a_{ij} x_j = b_i, \qquad i = r + 1, ..., m; \qquad (3)$$

$$(c, b, A) \in (\tilde{c}, \tilde{b}, \tilde{A})_\alpha,$$

$$x_j \geq 0, \qquad\qquad j = 1, ..., s;$$

$$x_j \text{ unrestricted in sign, } \; j = s + 1, ..., n,$$

Observed that there exists on infinite number of such a problem (3) depending on the coefficient vector $(c, b, A) \in (\tilde{c}, \tilde{b}, \tilde{A})_\alpha$ and the values of (c, b, A) are arbitrary for any $(c, b, A) \in (\tilde{c}, \tilde{b}, \tilde{A})_\alpha$ in the sense that the degree of all of the membership

functions for the fuzzy numbers in problem (3) exceeds level α. However, if possible, it would be desirable for DM to choose $(c, b, A) \in (\tilde{c}, \tilde{b}, \tilde{A})_{\alpha}$ in problem, (3) so as to minimize the objective function under the constraints. From such a point of view, for a certain degree α, it seems to be quite natural to have understood the linear programming problem with fuzzy parameters as the following non fuzzy α-parametric linear programming problem (Sakawa and Yano (1190)).

$$\min \ Z(x, c) = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i = 1, ..., r;$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = r+1, ..., m;$$

$$(c, b, A) \in (\tilde{c}, \tilde{b}, \tilde{A})_{\alpha},$$

$$x_j \geq 0, \qquad\qquad j = 1, ..., s;$$

$$x_j \text{ unrestricted in sign}, \quad j = s+1, ..., n. \tag{4}$$

It is assumed that DM chooses a degree of the α-level. It should be noted that the parameters (a, b, A) are treated as decision variables rather than constants.

From the properties of the α-level set for the vectors of fuzzy numbers $\tilde{c}, \tilde{b}$ and $\tilde{A}$, it should be noted that the Fuzzy parameters for $\tilde{c}, \tilde{b}$ and $\tilde{A}$ can be denoted, respectively, by the closed intervals $[(c_{\alpha}^-), (c_{\alpha}^+)], [(b_{\alpha}^-), (b_{\alpha}^+)]$ and $[(A_{\alpha}^-), (A_{\alpha}^+)]$, $\alpha \in [0, 1]$. Therefore, problem (4) may be rewritten as

$$\min \ Z_{\alpha} = \sum_{j=1}^{n} (c_j)_{\alpha} x_j$$

subject to

$$\sum_{j=1}^{n} (a_{ij})_{\alpha} x_j \geq (b_i)_{\alpha}, \qquad i = 1, ..., r;$$

$$\sum_{j=1}^{n} (a_{ij})_{\alpha} x_j = (b_i)_{\alpha}, \qquad i = r+1, ..., m; \tag{5}$$

$$x_j \geq 0, \qquad\qquad j = 1, ..., s;$$

$$x_j \geq 0 \text{ unrestricted in sign}, \quad j = s+1, ..., n,$$

where $(c_j)_\alpha \in [(c_j^-)_\alpha, (c_j^+)_\alpha]$, $(b_i)_\alpha \in [(b_i^-)_\alpha, (b_i^+)_\alpha]$, and $(a_{ij})_\alpha \in [(a_{ij}^-)_\alpha, (a_{ij}^+)_\alpha]$, respectively.

Clearly, different values of $(c_j)_\alpha$, $(b_i)_\alpha$ and $(a_{ij})_\alpha$ produce different objective values. To find the interval of the objective values, it suffies to find the LB and UB of the objective values of Model 5. Denote $G = \{(c_\alpha, b_\alpha, a_\alpha) : (c_j)_\alpha^- \leq (c_j)_\alpha \leq (c_j)_\alpha^+,$ $(b_i)_\alpha^- \leq (b_i)_\alpha \leq (b_i)_\alpha^+$, $(a_{ij})_\alpha^- \leq (a_{ij})_\alpha \leq (a_{ij})_\alpha^+$, $i = 1, ..., m; j = 1, ..., n\}$. The value of $(c_j)_\alpha$, $(b_i)_\alpha$ and $(a_{ij})_\alpha$ that attain the smallest value for Z can be determined from the following two-level mathematical programming model.

$$Z_\alpha^L = \min_{(c_\alpha, b_\alpha, a_\alpha) \in G} \min_x Z = \sum_{j=1}^n (c_j)_\alpha x_j$$

subject to

$$\left. \begin{array}{ll} \sum_{j=1}^n (a_{ij})_\alpha x_j \geq (b_i)_\alpha, & i = 1, ..., r; \\[2em] \sum_{j=1}^n (a_{ij})_\alpha x_j = (b_i)_\alpha, & i = r+1, ..., m; \\[2em] x_j \geq 0, & j = 1, ..., s; \\[1em] x_j \text{ unrestricted in sign}, & j = s+1, ..., n, \end{array} \right\} \quad (6)$$

where the inner program calculates the objective value for each $(c_j)_\alpha$, $(b_i)_\alpha$ and $(a_{ij})_\alpha$ specified by the outer program, while the outer program determines the values of $(c_j)_\alpha$, $(b_i)_\alpha$ and $(a_{ij})_\alpha$ that produces the smallest objective value. The objective value is the lower bound of the objective values for Model (5). By the same token, to find the values of $(c_j)_\alpha$, $(b_i)_\alpha$ and $(a_{ij})_\alpha$ that produce the largest objective value for Z, a two-program of Model (6) from "min" to "max".

$$Z_\alpha^U = \max_{(c_\alpha, b_\alpha, a_\alpha) \in G} \min_x Z = \sum_{j=1}^n (c_j)_\alpha x_j$$

subject to

$$\sum_{j=1}^n (a_{ij})_\alpha x_j \geq (b_i)_\alpha, \qquad i = 1, ..., r ;$$

$$\sum_{j=1}^n (a_{ij})_\alpha x_j = (b_i)_\alpha, \qquad i = r+1, ..., m ; \qquad (7)$$

$$x_j \geq 0, \qquad\qquad j = 1, ..., s ;$$

$$x_j \text{ unrestricted in sign}, \qquad j = s+1, ..., n ,$$

The objective value Z^+ is the upper bound of the objective values for Model (5).

4. Solution procedure

In this section, we discuss how to transform the two-level program into the conventional one-level program. With the pair of one-level program can be obtained.

4.1 Lower bound

Since both the inner program and outer program of (6) have the same minimization operation, they can be combined into a conventional one-level program with the constraints of the two programs considered at the same time.

$$Z_\alpha^L = \min Z = \sum_{j=1}^n (c_j)_\alpha x_j$$

subject to

$$\sum_{j=1}^n (a_{ij})_\alpha x_j \geq (b_i)_\alpha, \qquad i = 1, ..., r ;$$

$$\sum_{j=1}^n (a_{ij})_\alpha x_j = (b_i)_\alpha, \qquad i = r+1, ..., m ;$$

$$(c_j)_\alpha^- \leq (c_j)_\alpha \leq (c_j)_\alpha^+, \qquad j = 1, ..., n ;$$

$$(b_i)_\alpha^- \leq (b_i)_\alpha \leq (b_i)_\alpha^+, \qquad i = 1, ..., m ; \qquad (8)$$

$$(a_{ij})_\alpha^- \leq (a_{ij})_\alpha \leq (a_{ij})_\alpha^+, \quad i = 1, ..., m ; \ j = 1, ..., n ;$$

$$x_j \geq 0, \qquad\qquad j = 1, ..., s ;$$

$$x_j \text{ unrestricted in sign}, \qquad j = s+1, ..., n ,$$

this model is nonlinear program. Model (8) can be rewritten as

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$$Z_\alpha^L = \min Z = \sum_{j=1}^{s} (c_j)_\alpha \, x_j + \sum_{j=s+1}^{n} (c_j)_\alpha \, x_j$$

subject to

$$\sum_{j=1}^{s} (a_{ij})_\alpha \, x_j + \sum_{j=s+1}^{n} (a_{ij})_\alpha \, x_j \geq (b_i)_\alpha, \quad i = 1, ..., r \, ;$$

$$\sum_{j=1}^{s} (a_{ij})_\alpha \, x_j + \sum_{j=s+1}^{n} (a_{ij})_\alpha \, x_j = (b_i)_\alpha, \quad i = r+1, ..., m \, ;$$

$$(c_j)_\alpha^- \leq (c_j)_\alpha \leq (c_j)_\alpha^+, \qquad j = 1, ..., n \, ; \tag{9}$$

$$(b_i)_\alpha^- \leq (b_i)_\alpha \leq (b_i)_\alpha^+, \qquad i = 1, ..., m \, ;$$

$$(a_{ij})_\alpha^- \leq (a_{ij})_\alpha \leq (a_{ij})_\alpha^+, \quad i = 1, ..., m \, ; \; j = 1, ..., n \, ;$$

$$x_j \geq 0, \qquad\qquad\qquad j = 1, ..., s \, ;$$

$$x_j \text{ unrestricted in sign}, \qquad j = s+1, ..., n \, .$$

In searching for the minimal value of the objective function, the interval parameters $(c_j)_\alpha$, $j = 1, ..., n$, must reach its lower bound. Consequently, we have

$$\min Z = \sum_{j=1}^{s} (c_j^-)_\alpha \, x_j + \sum_{j=s+1}^{n} (c_j)_\alpha \, x_j \, .$$

We can reduce the number of nonlinear terms in Model 9, by using a variable transformation technique. So Model (9) become

$$Z_\alpha^- = \min Z = \sum_{j=1}^{s} (c_j^-)_\alpha \, x_j + \sum_{j=s+1}^{n} (c_j)_\alpha \, x_j$$

subject to

$$\sum_{j=1}^{s} U_{ij} + \sum_{j=s+1}^{n} (a_{ij})_\alpha \, x_j \geq (b_i^-)_\alpha, \; i = 1, ..., r \, ;$$

$$\sum_{j=1}^{s} U_{ij} + \sum_{j=s+1}^{n} (a_{ij})_\alpha \, x_j = (b_i^-)_\alpha, \; i = r+1, ..., m \, ;$$

$$(c_j)_\alpha^- \leq (c_j)_\alpha \leq (c_j)_\alpha^+, \qquad j = 1, ..., n \, ; \tag{10}$$

$$(b_i)_\alpha^- \leq (b_i)_\alpha \leq (b_i)_\alpha^+, \qquad i = 1, ..., m \, ;$$

$$(a_{ij})_\alpha^- \, x_j \leq U_{ij} \leq (a_{ij})_\alpha^+, \quad j = 1, 2, ..., n \, ;$$

$$(a_{ij})_\alpha^- \leq (a_{ij}) \leq (a_{ij})_\alpha^+, \quad i = 1, ..., m \, ; \; j = 1, ..., n \, ;$$

$$x_j \geq 0, \qquad\qquad\qquad j = 1, ..., s \, ;$$

$$x_j \text{ unrestricted in sign}, \qquad j = s+1, ..., n \, .$$

The lower bound of the objective value Z_α^L, is obtained the solving the mathematical program (10).

4.2 Upper bound

Using the duality theorem, the inner program of Model (7) is transformed into a maximization problem as:

$$Z_\alpha^U = \max_{(c_\alpha, b_\alpha, a_\alpha) \in G} \max_y Z = \sum_{j=1}^m (b_j)_\alpha \, y_j$$

subject to

$$\left. \begin{aligned} &\sum_{i=1}^m (a_{ij})_\alpha \, y_i \le (c_j)_\alpha, && j = 1, ..., s \,; \\ &\sum_{i=1}^m (a_{ij})_\alpha \, y_i = (c_j)_\alpha, && j = s+1, ..., n \,; \\ &y_i \ge 0, && i = 1, ..., r \,; \\ &y_i \text{ unrestricted in sign,} && i = s+1, ..., n \,. \end{aligned} \right\} \quad (11)$$

Now, Model (11), can be merged into a one-level program with the constraints at the two levels considered at the same time

$$Z_\alpha^U = \max_x Z = \sum_{i=1}^m (b_i)_\alpha \, y_i$$

subject to

$$\left. \begin{aligned} &\sum_{i=1}^m (a_{ij})_\alpha \, y_i \le (c_j)_\alpha, && j = 1, ..., s \,; \\ &\sum_{i=1}^m (a_{ij})_\alpha \, y_i = (c_j)_\alpha, && j = s+1, ..., n \,; \\ &(c_j)_\alpha^- \le (c_j)_\alpha \le (c_j)_\alpha^+, && j = 1, ..., n \,; \\ &(b_i)_\alpha^- \le (b_i)_\alpha \le (b_i)_\alpha^+, && i = 1, ..., m \,; \\ &(a_{ij})_\alpha^- \le (a_{ij})_\alpha \le (a_{ij})_\alpha^+, && i = 1, ..., m \,; \; j = 1, ..., n \,; \\ &y_i \ge 0, && i = 1, ..., r \,; \\ &y_i \text{ unrestricted in sign,} && i = s+1, ..., n \,. \end{aligned} \right\} \quad (12)$$

Similar to the case of lower bound discussed in Subsection 4.1, Model 12, can be rewritten as

$$Z_\alpha^U = \max Z = \sum_{i=1}^r (b_i)_\alpha\, y_i + \sum_{i=r+1}^m (b_i)_\alpha\, y_i$$

subject to

$$\sum_{i=1}^r (a_{ij})_\alpha\, y_i + \sum_{i=r+1}^m (a_{ij})_\alpha\, y_i \le (c_j)_\alpha, \qquad j=1,...,s \;;$$

$$\sum_{i=1}^r (a_{ij})_\alpha\, y_i + \sum_{i=r+1}^m (a_{ij})_\alpha\, y_i = (c_j)_\alpha, \qquad j=s+1,...,n \;;$$

$$(c_j)_\alpha^- \le (c_j)_\alpha \le (c_j)_\alpha^+, \qquad j=1,...,n\,;$$

$$(b_i)_\alpha^- \le (b_i)_\alpha \le (b_i)_\alpha^+, \qquad i=1,...,m\,;$$

$$(a_{ij})_\alpha^- \le (a_{ij})_\alpha \le (a_{ij})_\alpha^+, \quad i=1,...,m\,;\; j=1,...,n\,;$$

$$y_i \ge 0, \qquad\qquad i=1,...,r\,;$$

$$y_i \text{ unrestricted in sign,} \qquad i=s+1,...,n\,.$$

(13)

The objective function of Model (13) can be replaced by

$$\max\ Z = \sum_{i=1}^r (b_i^+)_\alpha\, y_i + \sum_{i=r+1}^m (b_i)_\alpha\, y_i\,.$$

Via the dual formulation, bound value assignment, and variable transformation, the two-level mathematical program of Model 7, is transformed into the following nonlinear program

$$Z_\alpha^U = \max Z = \sum_{i=1}^r (b_i)_\alpha^+\, y_i + \sum_{i=r+1}^m (b_i)_\alpha\, y_i$$

subject to

$$\sum_{i=1}^r v_{ij} + \sum_{i=r+1}^m (a_{ij})_\alpha\, y_i \le (c_j)_\alpha^+, \quad j=1,...,s\,;$$

$$\sum_{i=1}^r v_{ij} + \sum_{i=r+1}^m (a_{ij})_\alpha\, y_i = (c_j)_\alpha^+, \quad j=s+1,...,n\,;$$

$$(c_j)_\alpha^- \le (c_j)_\alpha \le (c_j)_\alpha^+, \qquad j=1,...,n\,;$$

$$(b_i)_\alpha^- \le (b_i)_\alpha \le (b_i)_\alpha^+, \qquad i=1,...,m\,;$$

$$(a_{ij})_\alpha^-\, x_j \le v_{ij} \le (a_{ij})_\alpha^+\, x_j\,;\; i=1,...,m\,;\; j=1,...,n\,;$$

$$(a_{ij})_\alpha^- \le (a_{ij})_\alpha \le (a_{ij})_\alpha^+, \qquad i=1,...,m\,;\; j=1,...,n\,;$$

$$y_i \ge 0, \qquad\qquad i=1,...,r\,;$$

$$y_i \text{ unrestricted in sign,} \qquad i=s+1,...,n\,.$$

(14)

The optimal solution Z_α^U is the upper bound of the objective values of the interval linear program. Together with Z_α^- solved from Subsection 4.1, $[Z_\alpha^L, Z_\alpha^U]$ constitutes the interval on which the objective values of the interval linear program lie.

5. Numerical example

Consider the following fuzzy linear programming problem

$$\min \ Z = \tilde{9}x_1 + 0x_2 + \tilde{8}x_3 - 2x_4 - \tilde{3}x_5 - \tilde{2}x_6 - 10x_7$$

subject to $\quad x_1 + 2x_2 - 2x_3 + \tilde{3}x_4 + 0x_5 + \tilde{4}x_6 + \tilde{1}x_7 = -\tilde{5}$,

$$-2x_1 + \tilde{2}x_2 - x_3 + \tilde{2}x_4 + \tilde{1}x_5 + \tilde{1}x_6 + 2x_7 = \tilde{1},$$

$$\tilde{2}x_1 + 0x_2 + 2x_3 + 2x_4 + \tilde{3}x_5 + 2x_6 - 2x_7 = \tilde{8},$$

$$x_1, x_2, x_4, x_5, x_6 \geq 0; \ x_3, x_7 \text{ unrestricted in sign.}$$

According to Model (5), the considered problem transformed into

$$\min \ Z_\alpha = [\alpha + 8, -\alpha + 10]z_1 + 0x_2 + [\alpha + 7, -\alpha + 9] - 2x_4 - [\alpha + 2, -\alpha + 4]x_5$$

$$- [\alpha + 1, -\alpha + 3]x_6 - 10x_7$$

subject to

$$x_1 + 2x_2 - 2x_3 + [\alpha + 2, -\alpha + 4]x_4 + 0x_5 + [\alpha + 3, -\alpha + 5]x_6$$

$$+ [\alpha + 0, -\alpha + 2]x_7 = -[\alpha + 4, -\alpha + 6],$$

$$-2x_1 + [\alpha + 1, -\alpha + 3]x_2 - x_3 + [\alpha + 1, -\alpha + 3]x_4 + [\alpha + 0, -\alpha + 2]x_5$$

$$+ [\alpha + 0, -\alpha + 2]x_6 + 2x_7 = [\alpha + 0, -\alpha + 2],$$

$$[\alpha + 1, -\alpha + 3]x_1 + 0x_2 + 2x_3 + 2x_4 + [\alpha + 2, -\alpha + 4]x_5 + 2x_6$$

$$-2x_7 = [\alpha + 7, -\alpha + 9],$$

$$x_1, x_2, x_4, x_5, x_6 \geq 0; \ x_3, x_7 \text{ unrestricted in sign.}$$

At $\alpha = 0$

$$\min \ Z_0 = [18, 10]x_1 + 0x_2 + [7, 9]x_3 - 2x_4 + [-4, -2]x_5 + [-3, -1]x_6 - 10x_7$$

subject to $\quad x_1 + 2x_2 - 2x_3 + [2, 4]x_4 + 0x_5 + [3, 5]x_6 + [0, 2]x_7 = [-6, -4]$,

$$-2x_1 + [1, 3]x_2 - x_3 + [1, 3]x_4 + [0, 2]x_5 + [0, 2]x_6 + 2x_7 = [0, 2],$$

$$[1, 3]x_1 + 0x_2 + 2x_3 + 2x_4 + [2, 4]x_5 + 2x_6 - 2x_7 = [7, 9],$$

$$x_1, x_2, x_4, x_5, x_6 \geq 0; \ x_3, x_7 \text{ unrestricted in sign.}$$

Based on Model (10), the lower bound of the objective value Z_0^- can be formulated as:

$$Z_0^L = \min\,(8x_1 + (c_7)_0 x_3 - 2x_4 - 4x_5 - 3x_6 - 10x_7)$$

subject to

$$x_1 + 2x_2 - 2x_3 + r_{14} + r_{16} + (a_{17})_0 = (b_1)_0,$$

$$-2x_1 + r_{22} - x_3 + r_{24} + r_{25} + r_{26} + 12x_7 = (b_2)_0,$$

$$r_{31} + 2x_3 + 2x_4 + r_{35} + 2x_6 - 2x_7 = (b_3)_0,$$

$$7 \le (c_3)_0 \le 9,$$

$$-6 \le (b_1)_0 \le -4, \qquad 0 \le (b_2)_0 \le 2, \qquad 7 \le (b_3)_0 \le 9,$$

$$2x_4 \le r_{14} \le 4x_4, \qquad 3x_6 \le r_{16} \le 5x_6, \qquad x_2 \le r_{22} \le 3x_2,$$

$$x_4 \le r_{24} \le 3x_4, \qquad 0x_5 \le r_{25} \le 2x_3, \qquad 0x_6 \le r_{26} \le 2x_6,$$

$$x_1 \le r_{31} \le 3x_1, \qquad 2x_5 \le r_{33} \le 4x_5, \qquad 0 \le (a_{17})_0 \le 2.$$

$$x_1, x_2, x_4, x_5, x_6 \ge 0;\ x_3, x_7 \text{ unrestricted in sign.}$$

This model can be solved using any available nonlinear programming package, for example, GINO (Assat and Wasil (1986); Liberman et al., (1986)) or SUPER GINO (Abadie (1978). We obtain $Z_0^- = 14$ at $x_1^* = 26$, $x_3^* = 38$, $x_7^* = 46$, $x_2^* = x_4^* = x_5^* = x_6^*$, $(c_3)_0 = 7$, $(b_1)_0 = -4$, $(b_2)_0 = 2$, $(b_3)_0 = 9$, and $(a_{17})_0 = 0$.

Based on Model (13), the upper bound of the objective value Z_0^+ can be formulated as:

$$Z_0^+ = \max\,((b_1)_0 y_1 + (b_2)_0 y_2 + (b_3)_0 y_3)$$

subject to

$$y_1 - 2y_2 + (r_{31})_0 y_3 \le 10,$$

$$2y_1 + (a_{22})_0 y_2 \le 0,$$

$$-2y_1 - y_2 + 2y_3 = (c_3)_0,$$

$$(a_{14})_0 y_1 + (a_{24})_0 y_2 + 2y_3 \le -2,$$

$$(a_{25})_0 y_2 + (a_{35})_0 y_3 \le -2,$$

$$(a_{16})_0 + (a_{26})_0 y_2 + 2y_3 \le -1,$$

$$(a_{17})_0 y_1 + 2y_2 - 2y_3 = 10,$$

$$7 \le (c_3)_0 \le 9,$$

$$-6 \le (b_1)_0 \le -4, \qquad 0 \le (b_2)_0 \le 2, \qquad 7 \le (b_3)_0 \le 9,$$

$$1 \leq (a_{31})_0 \leq 3, \qquad 1 \leq (a_{22})_0 \leq 3, \qquad 1 \leq (a_{14})_0 \leq 3,$$

$$1 \leq (a_{24})_0 \leq 3, \qquad 2 \leq (a_{25})_0 \leq 4, \qquad 2 \leq (a_{35})_0 \leq 4,$$

$$3 \leq (a_{16})_0 \leq 5, \qquad 0 \leq (a_{36})_0 \leq 2, \qquad 0 \leq (a_{17})_0 \leq 2,$$

y_1, y_2, y_3 unrestricted in sign.

Using GINO, we obtain $Z_0^+ = 29.8$ at $y_1^* = -1.4$, $y_2^* = -2.4$, $y_3^* = 1.9$, $(c_3)_0 = 9$, $(b_1)_0 = -6$, $(b_2)_0 = -1$, and $(b_3)_0 = 10$.

We conclude that the objective values lie in the range of [14, 29.8].

6. Concluding remarks

In this paper, we have used a two-level programming approach for solving fuzzy linear programming problems. the idea of this approach is to find the ℓb and $U b$ of the range by employing the two-level mathematical programming technique. Following the duality theorem, the two-level mathematical programs are transformed into a pair of one-level mathematical programs. However, GINO or SUPERGINO has bean used to obtain the results.

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