On a Composite Functional Equation Related to Abelian Groups

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A*bstract*— In this paper, we investigate a composite type functional equation

$$\boldsymbol{f\big(x\,f(y) - y\,f(x)\big) = f(x) - f(y) + x - y} \qquad \textbf{(1)}$$

on an Abelian group. This is a new composite functional equation introduced by authors and we are interested in finding the various properties of the function desired on an Abelian group which is uniquely divisible by 2.

K*eywords*— Composite functional equations, Abelian groups.

I. INTRODUCTION

The different types of functional equations like Additive, Quadratic are introduced by Cauchy and D' Alembert. These functional equations and its stability were discussed vividly in many research papers in the middle years of 20th century. Later, Cubic, Quartic and mixed type functional equations were introduced and its stability and many other properties were investigated by many authors [9, 10, 11, 14, 16, 19, 20, 21, 22]

Composite functional equations are very rare and very few mathematicians dealt and discussed its solutions and its regularity properties [4, 7, 17, 18, 24]. It is very hard to solve them than a non-composite one in general. In monograph [1] of J. Aczel, J. Dhombres studied the functional equation

$$f\big(x + f(y)\big) = f(x) + f(y) \qquad (2)$$

for x, y belonging to an Abelian group G. It was proved that $f: G \to G$ is a solution of the equation (2) if and only if f is additive and idempotent.

In 2001, in the American Mathematical Monthly [6] W.W. Chao proposed the following problem. Find every function $f: \mathbb{R} \to \mathbb{R}$ that is continuous at 0 and satisfies

$$f\big(x + 2f(y)\big) = f(x) + y + f(y) \qquad (3)$$

for all real numbers x and y.

In 2004, the solution of (3) was published by Henderson [12]. He showed first that every solution of (3) is additive and so $f(x) = cx$ where $c = 1$ or $c = \frac{-1}{2}$.

In 2006, Boros and Daroczy [4] generalized the problem. They proved that on an Abelian group with no elements of order 2, each solutions of the equation (3) is additive.

The composite functional equation

$$f\big(f(x) - f(y)\big) = f(x + y) + f(x - y) - f(x) - f(y) \qquad (4)$$

on an Abelian group was investigated by W. Frechner in the paper [24].

The Golab – Schinzel equation

$$f(x + f(x)y) = f(x)f(y) \qquad (5)$$

is one of the most important composite type functional equations. It has been considered for the first time by Golab and Schinzel [8]. Some applications of (5) and its further generalizations can be found in [2, 3, 5, 13, 15].

In this paper, we investigate a new composite type functional equation

$$f\big(x\,f(y) - y\,f(x)\big) = f(x) - f(y) + x - y$$

on an Abelian group. This composite functional equation is introduced by authors and we are interested in finding the various properties of the function desired on an Abelian group uniquely divisible by 2.

II. MAIN RESULT

Lemma 2.1 Let (G, +) be an Abelian group and if $f : G \to G$ is a solution of (1). Then $for\ each\ x \in G$

we have, (i) $f\left(f(x)\right) = f(-x)$,

and (ii) $f(nx) = n\,f(x) \quad for\ each\ x \in G, n \in \mathbb{N}$.

Proof. Replace $y = x$ in (1), we get

$$f(0) = f(x) - f(x) + x - x = 0, \quad \forall x \in G,$$

That is, $f(0) = 0$ for each $x \in G$. (6)

Now substitute $y = 0$ in (1) and using (6), we have

$$f\left(x\,f(0) - 0f(x)\right) = f(x) - f(0) + x - 0,$$

Which gives,

$$f(x) = -x. \tag{7}$$

Similarly, if we substitute $x = 0$ in (1) and using (6), we obtain

$$f(y) = -y. \tag{8}$$

Thus, $f\left(f(x)\right) = f(-x), \quad \forall\, x \in G$

which proves (i).

By using induction on n, we prove (ii). For $n = 1$ (ii) is obviously true. Assume the result is true for $n - 1$, now we will prove the result for n, by the induction hypothesis, we have,

$$(n-1)f(x) = f\left((n-1)x\right)$$

$$= f(nx - x) = -nx + x$$

$$= (-nx) + x$$

$$= (-nx) - (-x)$$

$$= f(nx) - f(x)$$

$$(n-1)f(x) + f(x) = f(nx)$$

$$n\,f(x) = f(nx) \quad for\ each\ n \in \mathbb{N}\ \ and\ \ x \in G. \tag{9}$$

This completes the proof of this lemma.

Lemma 2.2 Let $(G, +)$ be an abelian group and $f : G \to G$ is a solution of (1). Then

(i) $f\left(f(x) - f(y)\right) = f(y - x) = x - y$

(ii) $f\left(x\,f(y) - y\,f(x)\right) = -f\left(y\,f(x) - x\,f(y)\right)$

(iii) $f\left(x\left(f(x) + f(-x)\right)\right) = -\left(f(x) + f(-x)\right)$

(iv) $f\left(-y\left(f(y) - f(-y)\right)\right) = f(y) + f(-y)$

(v) $f\left(f(y)(f(nx) + nx)\right) = f(nx) + nx$

(vi) $f\left(-f(x)\right) = f\left(f(-x)\right)$

$for\ each\ x, y \in G,\ \ n \in N.$

Proof. By using the Lemma 2.1 (i) and subtracting (7) and (8), we obtain,

$$f(x) - f(y) = -x + y$$

$$f(x) - f(y) = y - x$$

$$f\left(f(x) - f(y)\right) = f(y - x) = x - y$$

$$for\ each\ x, y \in G.$$

Which proves (i). Now to prove (ii), from (1) we have

$$f\left(x\,f(y) - y\,f(x)\right) = f(x) - f(y) + x - y$$

$$= \left(f(x) + x\right) - \left(f(y) + y\right)$$

$$= -\left(\left(f(y) + y\right) - \left(f(x) + x\right)\right)$$

$$= -\left(f\left(y\,f(x) - x\,f(y)\right)\right)$$

$$for\ each\ x, y \in G.$$

Now we prove (iii), put $y = -x$ in (1), we have

$$f\left(x\,f(-x) + x\,f(x)\right) = 2x + f(x) - f(-x)$$

$$f\left(x\left(f(x) + f(-x)\right)\right) = -2\,f(x) + f(x) - f(-x)$$

$$= \; -f(x) - f(-x)$$

$$= -\big(f(x) + f(-x)\big) \qquad for\ each\ x \in G.$$

Similarly we can prove,

$$f\Big(-y\big(f(y) - f(-y)\big)\Big) = \; f(y) + f(-y)$$

$$for\ each\ y \in G.$$

Now we will prove (v) by induction on $n, for\ each\ n \in \mathbb{N}$, put $n = 1\ in$ (1) we have

$$f\big(f(y) - y\, f(1x)\big) = \; f(1x) - f(y) + 1x - y$$

$$= \; f(1x) + 1x - y + y$$

$$f\big(f(y)(f(x) + x)\big) = \; f(x) + x.$$

Hence the result is true for n = 1, Assume the result is true for n-1, now to prove the result for n, then by the given hypothesis, we have,

$$f\big(f(y)(f(n-1)x + (n-1)x)\big) = f\big((n-1)x\big) + (n-1)x$$

$$f\Big(f(y)\big((n-1)f(x) + (n-1)x\big)\Big) = (n-1)f(x) + (n-1)x$$

$$f\Big(f(y)\big((n-1)f(x) + nx - x\big)\Big) = (n-1)f(x) + nx - x$$

$$f\Big(f(y)\big((n-1)f(x) + nx + f(x)\big)\Big) = (n-1)f(x) + nx + f(x)$$

$$f\big(f(y)(nf(x) + n)\big) = n\, f(x) + n$$

$$f\big(f(y)(f(nx) + nx)\big) = f(nx) + nx$$

$$for\ each\ x \in G\ and\ n \in \mathbb{N}.$$

Now let $\quad f\big(-f(x)\big) = f(x) = -x = -f(-x)$

$$= f\big(f(-x)\big) \qquad for\ each\ x \in G,$$

Which proves (vi).

This completes the proof of this lemma.

From now on we will assume that the group G is uniquely divisible by 2, (that is the map $G \ni x \to x + x \in G$ is bijective). This allow us to split f into its even and odd parts that is defined by

$$f_e(x) \; = \; \frac{f(x) + f(-x)}{2},$$

$$f_o(x) \; = \; \frac{f(x) - f(-x)}{2}, \qquad \forall\, x \in G$$

Clearly, f_e is even, f_o is odd and $f = f_e + f_o$.

Lemma 2.3 Let $(G, +)$ be an Abelian group uniquely divisible by 2 and $f : G \to G$ is a solution of (1). Then

(i) (a). $f_o\big(f(x)\big) \quad = -\, f_o(x) \quad \forall\, x \in G$

(b). $f_o\big(f(x)\big) \quad = \quad x \quad \forall\, x \in G$

(ii) $f_o\big(f(x) - f(y)\big) = \; f_o\big(f(x-y)\big)$

(iii) (a). $f\big(f_o(x)\big) \quad = \; f_o\big(f(x)\big) \quad \forall\, x \in G$

(b). $f_o(-x) \quad = \; -\, f_o(x) \quad \forall\, x \in G$

(iv) $f_o\big(f_o(x)\big) \quad = \; -f_o(x) \quad \forall\, x \in G$

(v) $f_o(nx) \quad = \; n.\ f_o(x) \quad \forall\, x \in G$

(vi) (a). $f\big(f(y) - f(x)\big) + f\big(f(-x) - f(-y)\big)$

$$= 2\big(f_o(x) - f_o(y)\big)$$

(b). $f\big(f(x) - f(y)\big) + f\big(f(-y) - f(-x)\big)$

$$= 2\big(f_o(y) - f_o(x)\big) \quad \forall\, x, y \in G.$$

Proof. By using the definition of odd function and also using Lemma 2.1 (i), we obtain

$$f_o\big(f(x)\big) \; = \; \frac{f\big(f(x)\big) - f\big(f(-x)\big)}{2}$$

$$= \; \frac{f(-x) - f(x)}{2}$$

$$= \; -\left(\frac{f(x) - f(-x)}{2}\right)$$

$$= \; -f_o(x) \qquad \text{for each } x \in G$$

$$= -\frac{1}{2}(-x-x) \;= x \quad \text{for each } x \in G$$

Also using the definition of odd function we observe that,

$$f_o(-x) = x \qquad (10)$$

$$f_o(x) = -x, \qquad (11)$$

Which proves (i). Now, in order to prove (ii), consider the left hand side of Lemma 2.3 (ii) and using (10) and (11) and also using the definition of odd function, we obtain

$$f_o\big(f(x)-f(y)\big) = \frac{f\big(f(x)-f(y)\big)-f\big(f(y)-f(x)\big)}{2}$$

$$= \frac{f(y)-f(x)-f(x)+f(y)}{2} = f(y)-f(x)$$

$$= -(y-x) = f_o(y-x)$$

$$= f_o\big(f(x-y)\big) \;\; \text{for each } x,y \in G.$$

Now, $f\big(f_o(x)\big) = -f_o(x) = \frac{1}{2}\big(f(-x)-f(x)\big)$

$$= f_o(-x) \;\; \text{for each } x \in G,$$

From this we proved (iii),

i.e) $f_o(-x) = -f_o(x)$ and $f\big(f_o(x)\big) = f_o\big(f(x)\big)$ for each $x \in G$.

In the similar way we prove that,

$$f_o\big(f_o(x)\big) = \frac{1}{2}\, f\big(f_o(x)\big) - f\big(-f_o(x)\big)$$

$$= -f_o(x) \qquad \text{for each } x \in G.$$

From the proof of (iii) (b), we have

$$f_o(-x) = -f_o(x) \quad \text{for each } x \in G$$

That is, $f_o(-1.x) = -1\,.\,f_o(x)$, then obviously we have,

$$f_o(-n.x) = -n.f_o(x) \quad \text{for each } x \in G.$$

Replace $-n$ by n, we reach (v).

Now take the L. H. S of the equation,

$$f\big(f(y)-f(x)\big)+f\big(f(-x)-f(-y)\big)$$

$$= -\big(f(y)-f(x)\big)-\big(f(-x)-f(-y)\big)$$

$$= \big(f(x)-f(-x)\big)-\big(f(y)-f(-y)\big)$$

$$= 2\big(f_o(x)-f_o(y)\big) = 2(y-x).$$

This proves the part of (vi) (a), in the same manner we can prove that,

$$f\big(f(x)-f(y)\big)+f\big(f(-y)-f(-x)\big) = 2\big(f_o(y)-f_o(x)\big)$$

$$\text{foreach } x,y \in G.$$

This completes the proof of this lemma.

Lemma 2.4 Let $(G,+)$ be an Abelian group uniquely divisible by 2 and $f : G \to G$ is a solution of (1). Then

(i) (a). $f_e\big(f(x)\big) = -f_e(x) \quad \forall x \in G$

(b). $f_e\big(f(x)\big) = 0 \quad \forall x \in G$

(ii) $f_e(f(x)-f(y)) = f(x-y) \quad \forall x,y \in G$

(iii) (a). $f\,(f_e\,(x)\,) = f_e\big(f(x)\big) \quad \forall x \in G$

(b). $f_e(-x) = f_e(x) \quad \forall\, x \in G$

(iv) $f_e\big(f_e(x)\big) = 0 \quad \forall x \in G$

(v) $f_e(nx) = n.f_e(x) \quad \forall x \in G$

(vi) (a). $f\big(f(y)-f(x)\big)-f\big(f(-x)-f(-y)\big)$

$$= 2\big(f_e(x)-f_e(y)\big) \;\; \forall\, x,y \in G$$

(b). $f\big(f(x)-f(y)\big)-f\big(f(-y)-f(-x)\big)$

$$= 2\,\big(f_e(y)-f_e(x)\big) \;\; \forall\, x,y \in G.$$

Proof. By using the definition of even function and by using Lemma 2.1 (i), we obtain

$$f_e(f(x)) = \frac{f(f(x)) + f(f(-x))}{2}$$

$$= \frac{-f(x) - f(-x)}{2}$$

$$= -\left(\frac{f(x) + f(-x)}{2}\right)$$

$$= -f_e(x) \qquad for\ each\ x \in G$$

$$= -\frac{1}{2}(-x + x) = 0 \quad for\ each\ x \in G$$

Also using the definition of even function, we observe that,

(12) $$f_e(-x) = 0$$

(13) $$f_e(x) = 0,$$

Which proves (i).

Now, in order to prove (ii), consider the left hand side of (ii) and using the definition of even function and also using (7) and (8), we obtain

$$f_e(f(x) - f(y)) = \frac{f(f(x) - f(y)) - f(f(y) - f(x))}{2}$$

$$= \frac{f(y) - f(x) - f(x) + f(y)}{2} = f(y) - f(x)$$

$$= -(y - x)$$

$$= f(y - x) \qquad for\ each\ x, y \in G.$$

Now, $$f(f_e(x)) = -f_e(x)$$

$$= -\frac{1}{2}(f(x) + f(-x))$$

$$= 0 \qquad for\ each\ x \in G,$$

From this we arrive (iii),

i.e.) $$f_e(-x) = -f_e(x) \quad and$$

$$f(f_e(x)) = f_e(f(x)) \qquad for\ each\ x \in G.$$

In the similar way we can prove that,

$$f_e(f_e(x)) = \frac{1}{2}\left(f(f_e(x)) + f(-f_e(x))\right)$$

$$= \frac{1}{2}(-f_e(x) + f_e(x))$$

$$= 0 \qquad for\ each\ x \in G.$$

Using the proof of Lemma 2.4 (iii), obviously we reach (v).

Now, $$f(f(y) - f(x)) - f(f(-x) - f(-y))$$

$$= -(f(y) - f(x)) + (f(-x) - f(-y))$$

$$= (f(x) + f(-x)) - (f(y) + f(-y))$$

$$= 2(f_e(x) - f_e(y)).$$

This proves the part of (vi) (a), in the similar manner we can prove,

$$f(f(x) - f(y)) - f(f(-y) - f(-x))$$

$$= 2(f_e(y) - f_e(x)) \qquad foreach\ x, y \in G.$$

This completes the proof of this lemma.

With the help of this lemma we may prove the following corollary.

Corollary 2.5 Let $(G, +)$ be an Abelian group uniquely divisible by 2 and $f : G \to G$ is a solution of (1). Each of the following conditions are equivalent:

(a) $f\ is\ surjective;$
(b) $f\ is\ injective;$
(c) $f_o\ is\ injective;$
(d) $f_o\ is\ surjective;$
(e) $f(x) = -x \qquad for\ each\ x \in G.$

Proof. Let us assume (e) then immediately it implies that (a), (b), (c) and (d) are true. We will now try to prove the converse implications.

If f is surjective, then by using Lemma 2.1 (i), we obtain $f(x) = -x$ $for\ each\ x \in G$, similarly we can prove $f(x) = -x$ if f is injective.

If f_o is injective, then using Lemma 2.3 (i), we get that $f(x) = -x \; for \; each \; x \in G$.

Finally, if f_o is surjective, then using the Lemma 2.3 (v), we observe that $f_o(x) = -x \; for \; each \; x \in G$ and by Lemma 2.3 (iii), we obtain $f = f_o$.

In the next theorem we will solve equation (1) under flexible conditions.

Theorem 2.6 Let $(G,+)$ be an Abelian group uniquely divisible by 2 and $f : G \to G$ is a solution of (1) and satisfies $f(G) \subset -f(G)$ if and only if f is additive and $f \circ f = f$.

Proof. To prove the converse part let us find a $y \in G$ such that $f(x) = -f(y)$. Then using the Lemma 2.1 (i) and Lemma 2.2 (vi) we obtain,

$$-f(x) = f(y) = f(-f(y))$$

$$= f(f(-y)) = f(-f(-x)) = f(-x).$$

Then f is odd. Using f is odd in Lemma 2.3 (iii) we obtain $f = f_o$ is additive. The condition $f \circ f = f$ has already been proved in Lemma 2.1 (i).

Obviously we can prove the converse implication, it is a straightforward calculation.

Lemma 2.7 Let $(G,+)$ be an Abelian group uniquely divisible by 2 and $f : G \to G$ is a solution of (1). Then

(i) $f_e(f_o(x)) = 0 \quad for \; each \; x \in G;$
(ii) $if \; f_o \; is \; additive \; then$ $f_o(f_e(x)) = 0 \; for \; each \; x \in G;$
(iii) $f(f_e(x)) + f(f_o(x)) = -f(x) = x \quad \forall x \in G;$
$f_e(f(x)) = f(f_e(x)) = f_e(x) \qquad \forall x \in G.$

Proof. From Lemma 2.3 (iii) and (iv), we obtain

$$f_o(x) + f_e(-f_o(x)) = f_o(-f_o(x)) + f_e(-f_o(x))$$

$$= f(-f_o(x)) = f_o(x), \qquad \forall x \in G,$$

This proves that $f_e \circ f_o = 0$. Further, if f_o is additive then using Lemma 2.3 (i) and (iv), we obtain

$$f_o(x) + f_o(-f_e(x)) = f_o(-f_o(x)) + f_o(-f_e(x))$$

$$= f_o(-f(x)) = f_o(x), \qquad \forall \, x \in G.$$

This leads to $f_o \circ f_e = 0.$

Now, $$f_e(f_o(x)) = \tfrac{1}{2}\left(f_o(x) + f_o(-x)\right)$$

$$= \tfrac{1}{4}\left(f(x) - f(-x) + f(-x) - f(x)\right)$$

$$= 0.$$

This proves (i). We will try to prove (ii), Let us take the left hand side of (ii)

$$f_o(f_e(x)) = \tfrac{1}{2}\left(f_e(x) - f_e(-x)\right)$$

$$= \tfrac{1}{4}\left(f(x) + f(-x) - f(-x) - f(x)\right)$$

$$= 0.$$

Now we prove (iii) using the definition of even and odd function and using the Lemma 2.1 (i), we obtain

$$f(f_e(x)) + f(f_o(x)) = -f_e(x) - f_o(x)$$

$$= -f(x) = x \quad for \; each \; x \in G,$$

This proves (iii).

Now to prove (iv), using Lemma 2.3 (iv), we observe that

$$f_e(f(x)) = f_e(x) = f(f_e(x)),$$

This completes the proof.

With the help of the above Lemma we can deduce the following corollary.

Corollary 2.8 Let $(G,+)$ be an Abelian group uniquely divisible by 2 and $f : G \to G$ is a solution of (1). Then

(i) $f \; is \; even, \; if \; f_e(x) \neq 0 \; \; for \; x \neq 0$;
(ii) $f \; is \; even,$ $if \; f_o \; is \; additive \; and \; f_e \; is \; surjective$;
(iii) $if \; f(G) \cap -f(G) = \{0\} \; then \; f \; is \; even.$

Proof. Using the Lemma 2.7 (i) and (ii), we can easily prove the result (i) and (ii) of this corollary. In order to prove (iii), we use the Lemma 2.3 (iii) and also Theorem 2.6, we obtain

$f_o(G) \subset f(G)$, but already we have $f_o(G) = -f_o(G)$. From this we arrive $f_o = 0$.

Theorem 2.9 Let us take $f : \mathbb{R} \to \mathbb{R}$ is a monotonic function. Then f is a solution of (1) if and only if one of the following possibilities holds:

(i) $f(x) = 0 \quad for\ each\ x \in \mathbb{R};$

(ii) $f(x) = -x \quad for\ each\ x \in \mathbb{R}.$

Proof. Given that f is a monotonic function, which implies that, f is strictly decreasing or strictly increasing then we obtain, f is injective. Also using the corollary 2.7 we get $f(x) = -x\ for\ each\ x \in \mathbb{R}$. Suppose f is constant on an interval of a positive length. Then assume (a, b) be an open interval such that f is constant on (a, b). Further by using Lemma 2.3 (i) and (ii), we arrive that f_o is constantly equal to be zero on the interval $(a - b, b - a)$. Use the Lemma 2.1 (ii) we can prove $f_o = 0$ on $\mathbb{R}$. Then using the monotonicity of f we arrive, $f = f_e = 0$. which completes the proof of this theorem.

III. Conclusions

We obtained very interesting result for the Composite functional equation (1), some of the result gives the idempotent properties. Some peculiar properties appeared in the above results will help the future reader to bring out effective application to various sciences.

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