On the Nonstandard approach to the Numerical Solution of Ordinary Differential Equations

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Abstract

A set of new discrete models have been created using Nonstandard methods. A new renormalized denominator function has been derived. This has been applied to some special initial value problems in ordinary differential equation. Result from a suitable numerical solver created from these schemes showed that , the schemes are suitable and they carry along the dynamics of the original equation.

Keywords

Nonstandard methods, Renormalized denominator functions, Initial value problems Non-local approximation, Discrete models.

I. INTRODUCTION

The concept of numerical instability as analyzed in the works of Micken's 1994 has thrown more light on the fact that a lot of standard methods exhibit various form of instability primarily due to the denominator h. In particular a lot of schemes derived using these standard methods fail to preserve the dynamics of the original phenomena as represented by the original differential equation.

Micken's has suggested a five point rule that can help numerical modelers to avoid numerical instability and produce schemes that carry along the dynamics of the original differential equation. These rules are both advisory and instructional as some merely point to pitfalls that must be avoided while two of the rules basically required that the schemes be created to satisfy certain conditions. This paper is concerned with deriving schemes that obey the dynamics of the original differential equation.

***Definition*:** A finite difference scheme is called Non-standard finite difference method, if at least one of the following instructional conditions is met (Angueluv and Lubuma 2000):

i. In the discrete derivative, the traditional denominator is replaced by a non-negative function φ such that, $\varphi(h) = h + O(h^2)$, as $h \to 0$. (1)

ii. Non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable function of several points of the mesh.
For example the terms below may be approximated in the following ways

$y^2 \approx a y_k^2 + b y_k\, y_{k+1}$ a+b = 1 , $a,b \in \mathbb{R}$ (2)

$y^3 \approx a y_k^3 + (1-a) y_k^2\, y_{k+1}$ $a \in \mathbb{R}$ e.t.c (3)

The major consequence of these results is that such scheme does not allow numerical instabilities to occur.
In this work we shall apply these rules to create schemes that preserve the dynamical behavior of the original equation. We will consider a first order differential equation that can be expressed in the form:

$y' = f(x,y)$, $y(0) = y_0$ (4)

We shall proceed by deriving a new renormalized denominator function φ and then use it to create new schemes of the form:

. $y_{n+1} - y_n = \varphi\, f(x_n, y_n)$ (5)

II. DERIVATION OF A RENORMALIZED DENOMINATOR FUNCTION φ (h).

Let the function F(x) be defined continuous and differentiable in the domain of the solutions of equation (4). Suppose F(x) is polynomial of the form

$F(x) = \propto x^k + \beta$ $\propto, \beta \in \mathbb{R}$ $k \in \mathbb{N}$ (6)

Then $F'(x) = k \propto x^{k-1}$ (7)

$F(x_{n+1}) - F(x_n) = \propto (x_{n+1}^k - x_n^k)$ (8)

Using the relationship

$$F'(x_n) = y'(x_n) = f(x_n, y_n) \quad (9)$$

$$\Rightarrow k \propto (x_n^{k-1}) = f(x_n, y_n) \quad (10)$$

$$\propto = \frac{f(x_n, y_n)}{k(x_n^{k-1})} \quad (11)$$

If the theoretical and the analytical solution coincides then

$$y_{n+1} - y_n = F(x_{n+1}) - F(x_n) \quad (12)$$

$$y_{n+1} - y_n = \propto (x_{n+1}^k - x_n^k) \quad (13)$$

Substituting equation (11) into (13)

$$y_{n+1} - y_n = \propto (x_{n+1}^k - x_n^k) \quad (14)$$

$$= \frac{f(x_n, y_n)}{k(x_n^{k-1})} (x_{n+1}^k - x_n^k) \quad (15)$$

$$= \left[\frac{(x_{n+1}^k - x_n^k)}{k(x_n^{k-1})}\right] f(x_n, y_n) \quad (16)$$

Comparing the above with equation (5)

$$y_{n+1} - y_n = \varphi f(x_n, y_n) \quad (17)$$

Then $\varphi = \left[\frac{(nh+h)^k - (nh)^k}{k(nh)^{k-1}}\right]$ (18)

***A. Proposition*:**

The condition of $\varphi \to h + O(h^2)$ as $h \to 0$

is satisfied by the function

$$\varphi(h) = \left[\frac{(nh+h)^k - (nh)^k}{k(nh)^{k-1}}\right] .$$

Proof:

Substituting $x_n = nh, x_{n+1} = (n+1)h$

Comparing the above with equation (3)

$$\varphi = \left[\frac{(nh+h)^k - (nh)^k}{k(nh)^{k-1}}\right] \quad (19)$$

When k= 1 $\quad \varphi = nh + h - nh = h$ (20)

When k= 2 $\quad \varphi = \frac{(nh+h)^2 - (nh)^2}{2nh} = h + \frac{h^2}{2nh}$

When k = m $\quad \varphi = h + h\left[\frac{\sum_{r=2}^{m} \binom{m}{r} n^{m-r}}{m(n^{m-1})}\right]$ (21)

It can be shown that the series $\left[\frac{\sum_{r=2}^{m} \binom{m}{r} n^{m-r}}{m(n^{m-1})}\right]$ is a decreasing sequence for all m,r >3

Note that

From the above for each iteration $\varphi \to h + O(h^2)$ as $h \to 0$

φ does not depend on the value of $\propto$ and or β

From the theory of non-standard scheme, $\varphi(h)$ may be used as a renormalized denominator function for the integration of any initial value problem arising from ordinary differential equations .It may be sufficient to use the value of this $\varphi(h)$ for k ≤ 4 in this case (see equations (17) to (21) above)

This renormalized denominator functions

(i) $\varphi = \sin(h)$

(ii) $\varphi = h\left[\frac{(kh+\delta)}{(k+1)h+\delta}\right]$, $\delta \in R$

(iii) $\varphi = h + h\left[\frac{\left(6k^2+4k+1\right)}{4k^3}\right]$ $k \neq 0$

(iv) $\varphi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\lambda \in R, \lambda \neq 0$

will be used to construct new schemes for a sample of initial value problems from first order ordinary differential equation. All the above satisfies the condition of Rule 2 of the Nonstandard modeling rules . See Mickens (1994,2003) , Ibijola et al. (2013).

B. Derivation

In this examples we will replace y' by $\frac{(y_{k+1}-y_k)}{\varphi}$ and approximate terms in y non-locally

(1). $xy' - 2y = 2x^4$, $y(2) = 8$

Analytic solution is $y(x) = x^4 - 2x^2$

$$x\left(\frac{y_{k+1}-y_k}{\varphi}\right) - 2(ay_k + by_{k+1}) = 2x^4$$

$$y_{k+1} = \frac{(x-2a\varphi)y_k + 2\varphi x^4}{x - 2b\varphi}$$ **Scheme A** (22)

(2). $(x^2+1)y' = xy$, $y(\sqrt{15}) = 2$

$$y(x) = \frac{(x^2+1)^{1/2}}{2}$$

$$\frac{y_{k+1}-y_k}{\varphi} = \frac{x(ay_k + by_{k+1})}{x^2+1}$$

$$y_{k+1} = \frac{y_k(ax\varphi + x^2 + 1)}{(x^2+1) - bx\varphi}$$ **Scheme B** (23)

(3). $\frac{dy}{dx}(e^{2x}y) = (2x - e^{2x}y^2)$, y(0)=2

$$y(x) = e^{-x}(2x^2+4)^{1/2}$$

Replace y^2 by $(ay_k^2 + by_ky_{k+1})$

$$\frac{y_{k+1}-y_k}{\varphi} = \frac{2x - e^{2x}(ay_k^2 + by_ky_{k+1})}{e^{2x}y_k}$$

$$y_{k+1}(e^{2x}y_k + b\varphi y_k) = e^{2x}y_k^2 + \frac{2\varphi x - a\varphi e^{2x}y_k^2}{e^{2x}y_k}$$

$$y_{k+1} = \frac{e^{2x}{y_k}^2}{e^{2x}y_k+b\varphi y_k} + \frac{2\varphi x-a\varphi e^{2x}{y_k}^2}{e^{2x}y_k\,(e^{2x}y_k+b\varphi y_k)}$$ **Scheme C** (24)

(4). $xy' + y = 2x,\ \ y(x) = \frac{-2}{x} + x\,,\ y(1) = -1$

$$x\frac{(y_{k+1}-y_k)}{\varphi} + y_k = 2x_k$$

$$y_{k+1} = \frac{2\varphi+(x_k-\varphi)y_k}{x_k}$$ **Scheme D** (25)

(5). $y' = (-2x+y)^2 - 7\,,\ \ y(0) = 1$

$$y(x) = 2x + \frac{3(1-e^{6x})}{(1+e^{6x})}$$

$$\frac{(y_{k+1}-y_k)}{\varphi} = 4x_k^2 - 4x_ky_k + ay_k^2 + by_ky_{k+1} - 7$$

$$y_{k+1} = \frac{4\varphi x_k(x_k-y_k) + y_k(1+a\varphi y_k) - 7\varphi}{(1-b\varphi y_k)}$$ **Scheme E** (26)

TABLE I : SCHEMES OF SELECTED INITIAL VALUE PROBLEMS

1	Equation $xy' - 2y = 2x^4$, $y(2) = 8$ Analytic Solution $y(2) = 8$ $y(x) = x^4 - 2x^2$	**SCHEME A** $y_{k+1} = \frac{(x-2a\varphi)y_k+2\varphi x^4}{x-2b\varphi}$
	For Scheme A1 $\varphi = \sin(h)$ **For Scheme A2** $\varphi = h\left[\frac{(kh+\delta)}{(k+1)h+\delta}\right]$, $\delta \in R$	**For Scheme A3** $\varphi = h + h\left[\frac{(6k^2+4k+1)}{4k^3}\right]$ **For Scheme A4** $\varphi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\lambda \in R$
2	Equation $(x^2+1)y' = xy$, $y(\sqrt{15}) = 2$ Analytic Solution $y(x) = \frac{1}{2}(x^2+1)^{1/2}$	**SCHEME B** $y_{k+1} = \frac{x\varphi\, y_k+(x^2+1)y_k}{x^2+1}$
	For Scheme B1 $\varphi = \sin(h)$ **For Scheme B2** $\varphi = h\left[\frac{(kh+\delta)}{(k+1)h+\delta}\right]$,, $\delta \in R$	**For Scheme B3** $\varphi = h + h\left[\frac{(6k^2+4k+1)}{4k^3}\right]$ **For Scheme B4** $\varphi = \frac{(e^{\lambda h}-1)}{\lambda}$,$\lambda \in R$
3	Equation $y'(e^{2x}y) = (2x - e^{2x}y^2)$, y(0)=2 Analytic Solution $y(x) = e^{-x}\,(2x^2+4)^{1/2}$	**SCHEME C** $y_{k+1} = \frac{e^{2x}{y_k}^2}{e^{2x}y_k + b\varphi y_k} + \frac{2\varphi x - a\varphi e^{2x}{y_k}^2}{e^{2x}y_k\,(e^{2x}y_k + b\varphi y_k)}$
	For Scheme C1 $\varphi = \sin(h)$ **For Scheme C2** $\varphi = h\left[\frac{(kh+\delta)}{(k+1)h+\delta}\right]$, $\delta \in R$	**For Scheme C3** $\varphi = h + h\left[\frac{(6k^2+4k+1)}{4k^3}\right]$ **For Scheme C4** $\varphi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\lambda \in R$
4	Equation $xy' + y = 2x$ y(1)= -1 Analytic Solution $y(x) = x - \frac{2}{x}$	**SCHEME D** $y_{k+1} = \frac{2\varphi+(x_k-\varphi)y_k}{x_k}$
	For Scheme D1 $\varphi = \sin(h)$ **For Scheme D2** $\varphi = h\left[\frac{(kh+\delta)}{(k+1)h+\delta}\right]$, $\delta \in R$	**For Scheme D3** $\varphi = h + h\left[\frac{(6k^2+4k+1)}{4k^3}\right]$ **For Scheme D4** $\varphi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\lambda \in R$
5	Equation $y' = (-2x+y)^2 - 7\,, y(0) = 1$	**SCHEME E**

	Analytic Sol. $y = 2x + 3(1 - e^{6x})/\,(1 + e^{6x})$	$y_{k+1} = \dfrac{4\varphi x_k(x_k - y_k) + y_k(1 + a\varphi y_k) - 7\varphi}{(1 - b\varphi y_k)}$
	For Scheme E1 $\varphi = \sin(h)$ **For Scheme E2** $\varphi = h\left[\frac{(kh+\delta)}{(k+1)h+\delta}\right]$, $\delta \in R$	**For Scheme E3** $\varphi = h + h\left[\frac{(6k^2+4k+1)}{4k^3}\right]$ **For Scheme E4** $\varphi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\lambda \in R$

III. RESULTS OF NUMERICAL EXPERIMENT

A suite of software program written for the schemes generate the following graphs for the tested initial value problems

A. Graphical Representation

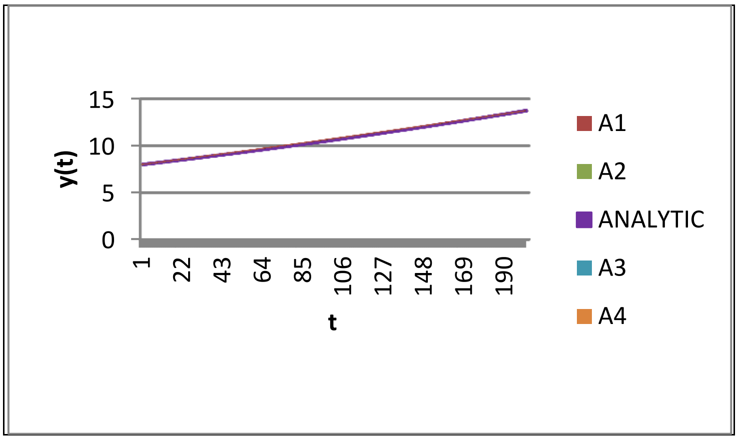

Fig 1: Schemes of $xy' - 2y = 2x^4$, $y(2) = 8$, h=0.1

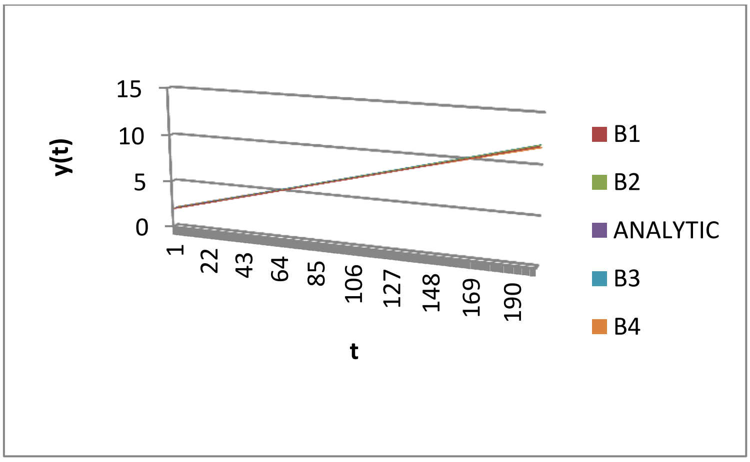

Fig 2: Schemes of $(x^2 + 1)y' = xy$, $y(\sqrt{15}) = 2$, h=0.01

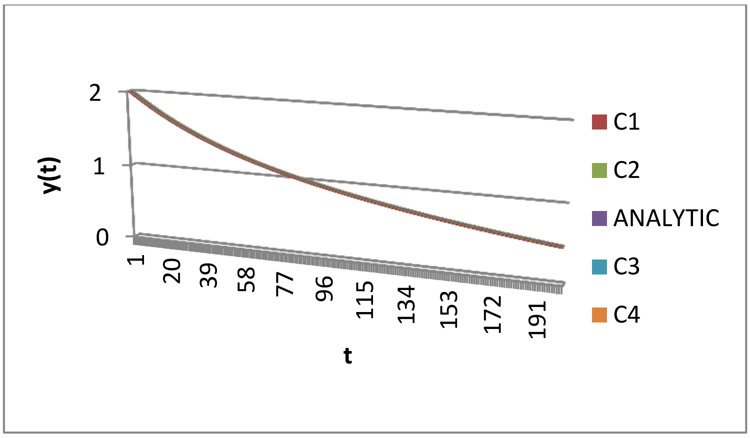

Fig 3: Schemes of $y'(e^{2x}y) = (2x - e^{2x}y^2)$,y(0)=2, h=0.1

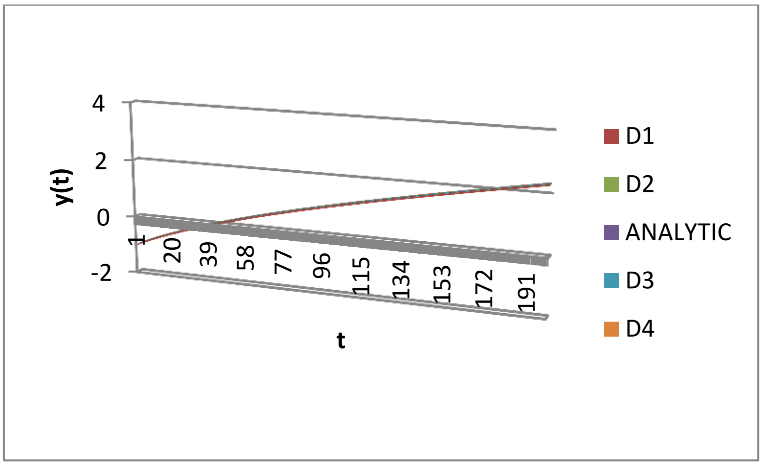

Fig 4: Schemes of $xy' + y = 2x,\ \ y(x) = \frac{-2}{x} + x\ ,\ \ y(1) = -1$, h=0.01

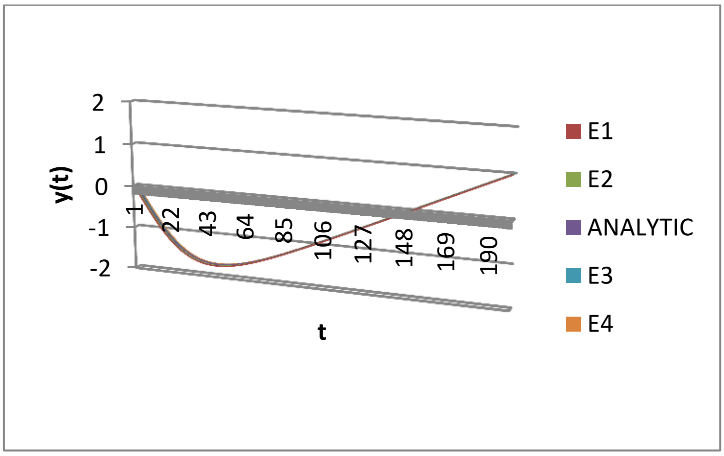

Fig 5: Schemes of $y' = (-2x + y)^2 - 7\ , y(0) =\ 1$, h=0.1

B. Observation:

Further observation has shown that

(i) $\varphi = \sin(h)$ perform better for scheme E

(ii) $\varphi = h\left[\frac{(kh+\delta)}{(k+1)h+\delta}\right]$, $\delta \in R$ perform better for scheme D

(iii) $\varphi = h + h\left[\frac{(6k^2+4k+1)}{4k^3}\right]$ $k \neq 0$ perform better for higher values of h in all cases but best for B and C

(iv) $\varphi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\lambda \in R$, $\lambda \neq 0$ perform better for scheme A

All the curves of the schemes behave in the same manner like the analytic solution as shown in the 3D graphs .The error functions are however different in each case. It is worth noting here that our new denominator function perform better in some cases mainly because of the opportunity to vary the term $h\left[\frac{\sum_{r=2}^{m}\binom{m}{r}n^{m-r}}{m(n^{m-1})}\right]$ while the second denominator performed better in some cases because of the opportunity to vary the term $\left[\frac{(kh+\delta)}{(k+1)h+\delta}\right]$. The performance of the fourth denominator function also improve depending on the choice of λ . We also note that these terms and expressions give the denominators their power to obey the second non-standard modeling rules . From these analysis we wish to conclude that it is sufficient to have a function $\varphi = h + [\propto]h$ where $0 < [\propto] \ll 1$ is a decreasing function in order to have a properly renormalized denominator function(The condition of $\varphi \to h + O(h^2)\ as\ h \to 0$ is satisfied).

It will be recalled that earlier works already shown that the renormalization of the denominator functions and the derivatives, contribute more to the ability of the schemes to carry along the dynamical behavior of the original equation than the non-local approximation of the other terms in the differential equation see(Mickens 1994, Angueluv and Lubuma 2000)

From the results of these experiments on the sampled equations it can be seen that all the schemes are stable with respect to monotonicity of solution i.e the orbit of the schemes behave exactly like the orbit of the original equation.

Further works in this area may be undertaken to derive a technique for selecting the best denominator for a particular type of differential equation .

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