A Note on Chromatic Transversal Weak Domination in Graphs

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Abstract

Let G = (V, E) be a graph with chromatic number k. A weak dominating set D of G is called a chromatic transversal weak dominating (ctwd) set if D intersects every colour class of any k-colouring of G. The minimum cardinality of chromatic transversal weak dominating set is called the chromatic transversal Weak domination number of G and it is denoted by $\gamma_{ctwd}(G)$. We calculate chromatic transversal weak domination number for some standard graphs and we charecta- rize $\gamma_{ctwd}(G) = n$.

Key words: Domination, Weak domination, Chromatic Number.

Introduction

By a graph G = (V, E) we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [5]. One of the fastest growing areas within graph theory is the study of domination and related subset problems. A comprehensive treatment of fundamentals of domination is given in the book by Haynes et al. [1]. Surveys of several advanced topics in domination can be seen in the book edited by Haynes et al. [1]. Another important area of research within graph theory is graph colourings which deals with fundamental problem of partitioning a set of objects into classes according to certain rules. A set $D \subseteq V$ is called a dominating set of G if every vertex in V-D is adjacent to a vertex in D. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. Sampathkumar.E and Puspha Latha. L[2] introduced concept of weak domination. A subset of V is weak dominating set of G if every vertex $v \in V - D$ is weakly dominated by some $u \in D$. The minimum cardinality of a weak dominating set in G is called the weak domination number of G and is denoted by $\gamma_w(G)$. Ayyaswamy. S.K and Benedict Michaelraj. L[4] introduced concept of chromatic transversal domination. In this paper we introduce a graph theoretic parameter which combines the concept of weak domination and vertex colouring. A vertex colouring of a graph G is a partition of Vinto independent sets and the minimum order of such partition is called the chromatic number of G and is denoted by $\chi(G)$. If $\zeta = \{V_1, V_2, V_3, \dots, V_k\}$ is a k-colouring of G

then a subset D of V is called transversal of ζ if $D \cap V_i \neq \Phi$ for all i = 1, 2, 3, ..., k. The minimum cardinality of a chromatic transversal weak dominating set in G is called the chromatic transversal weak domination number of G and is denoted by $\gamma_{ctwd}(G)$.

Definition: 1 A set D subset of V is said to be **chromatic transversal weak** dominating (ctwd) - set of G if D is a weak dominating set and D is transversal of every chromatic partition of G. The minimum cadinality of a chromatic transversal weak dominating (ctwd) - set is a chromatic transversal weak domination (ctwd) number and it is denoted by γ_{ctwd} (G)

Example: 1 It is easy to verify that $\gamma_{ctwd}(K_n) = n$.

Theorem: 1 Let $G = K_{m,n}$, $2 \le m \le n$. Then

$$\gamma_{ctwd} \left(K_{m,n} \right) = \begin{cases} 2 & \text{if } m = n \\ \max \left\{ m, n \right\} + 1 & \text{if } m \neq n \end{cases}$$

Proof: Let D be a γ_w set of G.

Case (i): Let m = n. Then $\gamma_w(G) = 2$, since G is bipartite $\chi(G) = 2$. As G is bipartite graph it has a unique χ -partition say (X,Y), then there exists a weak dominating set D of cardinality $\gamma_w(G)$ such that $D \cap X \neq \Phi$ and $D \cap Y \neq \Phi$. In any two vertices, one can taken from each part will be a minimum chromatic transversal weak dominating (ctwd) set of G. Hence $\gamma_{ctwd}(K_n) = 2$.

Case (ii): Let m > n. Then $\gamma_w(G) = m$. But $\chi(\langle D \rangle) = 1$. Therefore $D \cup \{u\}$ is a bipartite, it has a unique χ - partition. Hence $\gamma_{ctwd}(K_{m,n}) = m+1$. Hence if $m \neq n$, then $\gamma_{ctwd}(K_{m,n}) = \max\{m,n\}+1$.

Theorem: 2 For $n \ge 1$. Then

$$\gamma_{ctwd} \left(P_n \right) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \\ \frac{n}{3} + 1 & \text{if } n \equiv 0 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 1 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof: The result is obviously true for n = 1, 2, 3. Therefore let $n \ge 4$. As P_n is bipartite it has a uniqe χ - partition says (X, Y). Let $P_n : V_1, V_2, \dots, V_n$. Without losse of generality $V_1, V_3, \dots \in X, V_2, V_4, \dots \in Y$. Let n = 3k + r, where $0 \le r \le 2$. As $n \ge 4$, we have $k \ge 2$.

Case (i): Let r = 0. Then n = 3k. The set $D = \{V_1, V_4, \dots, V_{n-2}, V_n\}$ is a minimum weak dominating set having $\frac{n}{3} + 1$ vertices. D is also a transversal. Since $v_1 \in X$ and $v_4 \in Y$. Thus D is a minimum chromatic transversal weak dominating set. Hence $\gamma_{ctwd} = \frac{n}{3} + 1$. **Case (ii):** Let r = 1. Then n = 3k + 1. In this case the set $D = \{V_1, V_4, \dots, V_{n-3}, V_n\}$ will be minimum weak dominating set having $\lceil \frac{n}{3} \rceil$ vertices. D is also a transversal, since $v_1 \in X$ 1, $v_4 \in Y$. Thus D is a minimum chromatic transversal weak dominating set. Hence $\gamma_{ctwd} = \frac{n}{3}$.

Case (iii): Let $r = 2\mathbf{r} = 2$. Then n = 3k + 2. The set $D = \{V_1, V_4, \dots, V_{n-2}, V_n\}$ is a minimum weak dominating set having $\lceil \frac{n}{3} \rceil + 1$ vertices. D is also a transversal, since $v_1 \in X$ 1, $v_4 \in Y$. Thus D is a minimum chromatic transversal weak dominating set. Hence $\gamma_{ctwd} = \lceil \frac{n}{3} \rceil + 1$.

Theorem: 3 For $n \ge 3$. Then

$$\gamma_{ctwd}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

Proof: Let $C_n: v_1, v_2, ..., v_n$.

Case (i): Let *n* be odd. Corresponding to each vertex $v \in C_n$ there exists a χ -partition having *v* as a part. Therefore the minimum chromatic transversal weak dominating set will contain each vertex of C_n . Hence $\gamma_{ctwd}(C_n) = n$.

Case(ii): Let *n* be even. Then we write n = 3k + r, $0 \le r \le 2$. If *k* is odd, then r = 1. The set $\{V_2, V_5, ..., V_{n-2}, V_n\}$ will be a weak dominating set containing $\left\lceil \frac{n}{3} \right\rceil$ vertices and this is also a transversal of a unique χ -partition. If *k* is even, then r = 0 or 2. For r = 0, the set $\{V_2, V_5, ..., V_{n-1}\}$ is a minimum chromatic transversal weak dominating set containing $\left\lceil \frac{n}{3} \right\rceil$ vertices. For r = 2. The set $\{V_2, V_5, ..., V_{n-1}, V_n\}$ is a minimum chromatic transversal weak dominating set containing $\left\lceil \frac{n}{3} \right\rceil$ vertices. For r = 2. The set $\{V_2, V_5, ..., V_{n-1}, V_n\}$ is a minimum chromatic transversal weak dominating set $\gamma_{ctwd}(C_n) = \left\lceil \frac{n}{3} \right\rceil$

Theorem: 4 For $n \ge 3$. Then

$$\gamma_{ctwd} \left(W_n \right) = \begin{cases} n = \gamma & \text{if } n \text{ is even} \\ \left\lceil \frac{n-1}{3} \right\rceil + 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof: Let $W_n: v_1, v_2, ..., v_n$.

Case (i): When *n* is even, as in the case of C_n , every vertex in W_n can be made as a part of some χ - partition. Therefore $\gamma_{ctwd} = n$.

Case (ii): When *n* is odd, W_n has a unique χ - partition with $\left\lceil \frac{n-1}{3} \right\rceil$ parts. Any $\left\lceil \frac{n-1}{3} \right\rceil$ vertices, taken from each part will be a minimum chromatic transversal weak dominating set of W_n . Hence $\gamma_{ctwd}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1$.

Theorem: 4 $\gamma_{ctwd}(P) = 5$, where P is the Petersen graph. **Proof:** We know that $\gamma(P) = 3$ and $\chi(P) = 3$

Claim: Any independent set S of three vertices in P can be made as a part or subset of a part of a χ - partition of P.



Figure (a)

Let the vertex set of P be $\{v_1, v_2, v_3, v_4, v_5, w_1, w_2, w_3, w_4, w_5\}$, where v_i 's form the outer cycle and w_i 's form the inner cycle. Neither all the three vertices of S are in the outer cycle. Without loss of generality, let us assume that two are in the inner cycle and one is in the outer cycle. Assume that $v_1, v_2 \in D$ (a similar case may be discussed for v_1, v_4). Then S will be one of the following three sets $\{v_1, v_3, w_2\}, \{v_1, v_3, w_4\},$ $\{v_1, v_3, w_5\}$. Each set can be made as a part or a subset of a part of a χ - partition on P. Hence the claim. Let D be weak dominating set of P with three vertices. Then at least one vertex will be in the outer cycle and one in the inner cycle. Suppose only one vertex is in the outer cycle and the remaining two are in the inner cycle. The two vertices in the inner cycle are non-adjacent for otherwise D will not be a weak dominating set. Moreover for the same reason, D will be a independent set. By the claim D will not be a transversal of every χ - partition of P. Hence $\gamma_{ctwd} \ge 4$. Let as now take the case of a weak dominating set D of P with four vertices. Then all the four vertices will not be in the outer cycle as these can weakly dominate only four vertices of in the inner cycle. So at least one must be in the inner cycle. Now we prove that D contains an independent set of three vertices.

Case (i): Suppose three vertices are in the outer cycle and one in the inner cycle. The three outer vertices will weakly dominate the remaining two vertices in the outer

cycle. So D to be a weak dominating set, the fourth vertex is not adjacent to any of these three outer vertices. Thus D contains an independent subset of three vertices. **Case (ii):** Suppose two vertices are in the outer cycle and two are in the inner cycle. Atleast one of these pairs is non-adjacent. This pair together with a vertex from the other pair will form an independent subset of D. In both the cases D has an independent subset containing three vertices. So D will not be a chromatic transversal weak dominating set. Hence $\gamma_{ctwd}(P) \ge 5$. The set $\{v_1, v_2, v_3, v_4, v_5\}$ is a chromatic transversal weak dominating set of P. Hence $\gamma_{ctwd}(P) = 5$.

Proposition: 1 For $n \ge 2$. Then $\gamma_{ctwd}(K_{1,n-1}) = n$

Observation: 1 Chromatic transversal weak dominating set exists for all graphs since the vertex set V is a trivial chromatic transversal weak dominating set.

Observation: 2 $\gamma_{ctsd}(G) \leq \gamma_{ctwd}(G)$

Theorem: 5 Given a positive integer $k \ge 1$, there exists a graph G such that (i). $\gamma_{ctwd}(G) - \gamma(G) = k$ (ii). $\gamma_{ctsd}(G) - \gamma(G) = k$ (iii). $\gamma_{ctwd}(G) - \gamma_{ctsd}(G) = k$ **Proof:** For (i) Let $G = K_{k+1, k}$ be a graph. Then $\gamma_{ctwd}(G) = k + 2$ and $\gamma(G) = 2$. Therefore $\gamma_{ctwd}(G) - \gamma(G) = k + 2 - 2 = k$. For (ii) Let $G = K_{k+2, k+1}$ be a graph. Then

$$\begin{split} \gamma_{ctsd}(G) &= \min\{k+2, k+1\} + 1 = k + 1 + 1 = k + 2 & \text{and} & \gamma(G) = 2. & \text{Therefore} \\ \gamma_{ctsd}(G) - \gamma(G) &= k + 2 - 2 = k. & \text{For (iii)} & \text{Let} & G = K_{k+5, 5} & \text{be a graph.} & \text{Then} \\ \gamma_{ctwd}(G) &= \min\{k+5,5\} + 1 = k + 5 + 1 = k + 6 & \text{and} & \gamma_{ctsd}(G) = \min\{k+5,5\} + 1 = 5 + 1 = 6. \\ \text{Therefore} & \gamma_{ctwd}(G) - \gamma_{ctsd}(G) = k + 6 - 6 = k. \end{split}$$

Theorem: 6 Given a positive integer $k \ge 2$, there exists a graph G such that $\gamma_{ctwd}(G) = k$

Proof: Let G be a complete bipartite graph $K_{k-2, k-1}$. Then $\gamma_{ctwd}(G) = \max\{k-2, k-1\}+1$. Therefore $\gamma_{ctwd}(G) = k$.

Theorem: 7 Let G be a graph with $\Delta(G) = 1$. Then $\gamma_{ctwd}(G) + \Delta(G) = n$ if and only if $G = 2K_2 \cup (n-4)K_1$.

Proof: If $G = 2K_2 \cup (n-4)K_1$, then $\Delta(G) = 1$ and $\gamma_{ctwd}(G) = n-1$. Therefore $\gamma_{ctwd}(G) + \Delta(G) = n-1+1=n$. Conversely suppose G is a graph with $\Delta(G) = 1$ satisfying that $\gamma_{ctwd}(G) + \Delta(G) = n$, then each component of G is K_1 or K_2 with at least $2K_2$ components and $\gamma_{ctwd}(G) = n-1$ if there are more than $2K_2$ components, then G can be chromatic transversal weakly dominated by less than n-1. Hence G has exactly $2K_2$ components and every other component is K_1 . Hence the proof.

Theorem: 8 Let G be a graph with $\Delta(G)=1$. Then $\gamma_{ctwd}(G)+\Delta(G)=n+1$ if and only if $G = 2K_2 \cup (n-2)K_1$. **Proof:** If $G = k_2 \cup (n-2)k_1$, then $\Delta(G)=1$ and $\gamma_{ctwd}(G)=n$, therefore $\gamma_{ctwd}(G)+\Delta(G)=n+1$.

Conversely, suppose G is a graph with $\Delta(G) = 1$ satisfying that $\gamma_{ctwd}(G) + \Delta(G) = n+1$, then each component of G is K_1 or K_2 with at least one K_2 component and $\gamma_{ctwd}(G) = n$. If there is more than one K_2 component, then G can be chromatic transversal weakly dominated by less than n vertices. Hence G has exactly one K_2 component and every other component is K_1 . Hence the proof.

Characterization Theorems - I

Theorem: 9 A ctwd- set D is minimal if and only if for every $u \in D$, one of the following is true.

- (i). *u* is a weak isolate in *D*.
- (ii). There exists $v \in V D$ such that $N_w(v) \cap D = \{u\}$

(iii). There exists a χ - partition $\zeta = \{V_1, V_2, \dots, V_{\chi}\}$ such that $D \cap V_i = \{u\}$ for some *i*.

Proof: Suppose *D* is a minimal *ctwd*- set of *G*. Then $D - \{u\}$ is not a *ctwd*- set for every $u \in D$ and hence either $D - \{u\}$ is not a weak dominating set or not a transversal of some χ - partition of *G*.

Case (1): Suppose $D - \{u\}$ is not a weak dominating set. Then there exists a vertex $v \in V - D \cup \{u\}$, that is not weak dominated by any vertex of $D - \{u\}$. If u = v, u is a weak isolate of D. Let $v \in V - D$. v is not weak dominated by any vertex of a $D - \{u\}$, but v is weak dominated by some vertex of D. So, v is adjacent only to the vertex $u \in D$. Hence $N_w(v) \cap D = \{u\}$. So, condition 1 or 2 is satisfied.

Case (2): Suppose $D - \{u\}$ is not a transversal of some χ -partition of G. D being a transversal of every χ -partition $\zeta = \{V_1, V_2, ..., V_{\chi}\}$ such that D is a transversal of ζ whereas $D - \{u\}$ is not. So there exists a part V_i for which $D \cap V_i \neq \Phi$ but $D - \{u\} \cap V_i = \Phi$. This implies that $D \cap V_i \leq \Phi$. So 3 is satisfied.

Conversely suppose D is a *ctwd*-set and for every vertex, one of the three stated condition holds. We show that D is a minimal *cted*-set. Suppose D is not a minimal *ctwd*-set. Then there exists a vertex $u \in D$ such that D-u is a *ctwd*-set. D and D-u are weak dominating sets. Hence u is weak dominated by at least one vertex of D-u that is condition 1 does not exist. Also every vertex in V-D is weak dominated by at least one vertex of u are transversals of every χ - partition of G. So the condition 3 is also not satisfied. So our assumption is wrong. Hence D is a minimal *ctwd*-set of G.

Corollary: 1 In a graph G without isolated vertices there exists a γ_{ctwd} set D such that for every $u \in D$ either 2 or 3 of the theorem is satisfied.

Proof: Among all γ_{ctwd} of G, let D be one such that $\langle D \rangle$ has maximum size. Suppose to the contrary that D contains a vertex u that does not have the desired property. By the above theorem, u is an isolated vertex in $\langle D \rangle$, Since G contains no isolated vertex, u is adjacent to a vertex $w \in V(G) - D$.



Also every vertex of V(G)-D is adjacent to some other vertex of D as well. Consequently $D-\{w\}\cup\{v\}\}$ is a minimum dominating set of G whose induced subgraph contains at least one edge incident with w and hence has greater size than $\langle D \rangle$. This produces a contradiction. Hence the result

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