CR–Submanifold of a Lorentzian Para Sasakian Manifold

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Abstract:

In this paper, we study CR-submanifold of a Lorentzian Para – Sasakian manifold. The integrability condition and parallel horizontal distribution also have been discussed.

Key Words: Differentiable manifold, vector-field, 1-form, Lorentzian metric, sub-manifold, distribution, tangential, normal, horizontal and vertical components.

1. LORENTZIAN PARA-SASAKIAN MANIFOLD

Let $\tilde{M}$ be a differntiable manifold of dimension n (odd). Let there exist a tensor F of type (1, 1), a vector field U, a 1-form u and a Lorentzian metric g satisfying for X, Y, Z, ..., tangent to $\tilde{M}$,

1.1 a) $F^2X = X + u(X)U$,

b) $u(U) = -1$,

c) $u(X) = -g(X, U)$,

d) $g(FX, FY) = -g(X,Y) + u(X)u(Y)$,

e) $(\tilde{\nabla}_X F)Y = -g(X,Y)U + u(X)u(Y)U + [X + u(X)U]u(Y)$

then ($\tilde{M}$, F, U, u, g) is calld a Lorentzian Para-Sasakian structure and $\tilde{M}$ is called Lorentian Para-Sasakian manifold.

From the definition, we obtain

f) $FU=0$,

g) $u(FX)=0$,

h) $\tilde{\nabla}_X U = FX$,

i) $\text{rank}(F) = n-1$.

2. CR-SUMANIFOLD OF A LORENTZIAN PARA-SASAKIAN MANIFOLD

Definition 2.1 : Let M be a Riemannian sub-manifold of a Lorentzian Para-Sasakian manifld $\tilde{M}$. The submanifold M is called a CR-submanifold of $\tilde{M}$ if.

2.1 a) M is tangent to U,

2.2 a) There exists a differentiable distribution

$$D : x \to D_x \subseteq T_x(M), \forall x \in M, \text{ such that}$$

$$FD_x \subseteq T_x(M)$$

b) The orthogonal complementary distribution of D, $D^1 : x \to D_x^1 \subseteq T_x(M), \forall x \in M$ such that $FD_x^1 \subseteq T_x^1(M)$

From the definition, it is clear that

$T(M) = D \oplus D^1$,

$$T^1(M) = FD^1 \oplus \mu,$$

Where μ denotes the orthogonal complement of FD^1. Let g be the induced metric on M. Let $\tilde{\nabla}$ and ∇ denote the covariant differentiation in $\tilde{M}$ and M respectively, then we have Gauss formula.

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \forall X, Y \in T(M)$$

Again Weingartan formula is

$$\tilde{\nabla}_X N = -A_N X + \nabla^1_X N, \ \forall X \in T(M) \text{ and } N \in T^1(M).$$

From (2.5) and (2.6)

2.7 $$g(h(X, Y), N) = g(A_N X, Y).$$

We put

2.8 $$X = PX + QX, \ \forall X \in T(M),$$

Where $PX \in D$, $QX \in D^1$.

2.9 $$FN = BN + CN, \ \forall N \in T^1(M),$$

where BN (respectively CN) is the tangential (respectively normal), component of FN.

3. INTEGRABILITY OF DISTRIBUTIONS ON CR-SUBMANIFOLD

First of all, we establish two lemmas.

Lemma (3.1) Let M be a CR-submanifold of a Lorentzian Para-Sasakian manifold $\tilde{M}$. Then

3.1 $$P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y - g(X, Y)PU + u(Y)PX + 2u(X)u(Y)PU,$$

3.2 $$Q\nabla_X FPY - QA_{FQY}X = Bh(X, Y) + u(Y)QX - g(X, Y)QU + 2u(X)u(Y)QU,$$

3.3 $$\nabla^1_X FQY = FQ\nabla_X Y + Ch(X, Y) - h(X, FPY),$$

3.4 $$\nabla_X FPY - A_{FQY} = FP\nabla_X Y + Bh(X, Y) - g(X, Y)U + u(Y)X + 2u(X)u(Y)U,$$

3.5 $$\nabla_X FPY + \nabla_Y FPX - A_{FQY}X - A_{FQX}Y = FP\nabla_X Y + FP\nabla_Y X + 2Bh(X, Y) - 2g(X, Y)U + u(Y)X + u(X)Y + 4u(X)u(Y)U.$$

Proof: From (1.1)(e)

$$(\tilde{\nabla}_X F)Y = -g(X, Y)U + u(Y)X + 2u(X)u(Y)U,$$

3.6 $$\tilde{\nabla}_X FY - F(\tilde{\nabla}_X Y) = -g(X, Y)U + u(Y)X + 2u(X)u(Y)U.$$

Using (2.5), (2.8), (2.9) in (3.6),

$$\tilde{\nabla}_X F(PY+QY) - F(\tilde{\nabla}_X Y) = -g(X,Y)U + u(Y)X + 2u(X)u(Y)U.$$

$$\tilde{\nabla}_X FPY + \tilde{\nabla}_X FQY - F[\nabla_X Y + h(X,Y)]$$
$$= -g(X,Y)(PU+QU) + u(Y)(PX+QX)$$
$$+ 2u(X)u(Y)(PU+QU)$$

$$\nabla_X FPY + h(X,FPY) + [-A_{FQY}X + \nabla^1_X FQY]$$
$$- F[P\nabla_X Y + Q\nabla_X Y] - F[h(X,Y)]$$
$$= -g(X,Y)(PU+QU) + u(Y)(PX+QX)$$
$$+ 2u(X)u(Y)(PU+QU)$$

3.7 $$\nabla_X FPY + h(X,FPY) - A_{FQY}X + \nabla^1_X FQY$$
$$- FP\nabla_X Y - FQ\nabla_X Y - Bh(X,Y) - Ch(X,Y)$$
$$= -g(X,Y)(PU+QU) + u(Y)(PX+QX)$$
$$+ 2u(X)u(Y)(PU+QU)$$

Equating horizontal parts on botn the sides of (3.7),

$$P\nabla_X FPY - PA_{FQY}X - FP\nabla_X Y = -g(X,Y)PU$$
$$+ u(Y)PX$$
$$+2u(X)u(Y)PU,$$

3.8 $$P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y - g(X,Y)PU$$
$$+ u(Y)PX + 2u(X)u(Y)PU$$

Equating vertical parts on both the sides of (3.7)

$$Q\nabla_X FPY - QA_{FQY}X - Bh(X,Y) = -g(X,Y)QU$$
$$+ u(Y)QX$$
$$+2u(X)u(Y)QU,$$

3.9 $$Q\nabla_X FPY - QA_{FQY}X = Bh(X,Y) - g(X,Y)QU + U(Y)QX$$
$$+2u(X)u(Y)QU,$$

Equating normal parts on both the sides of (3.7)

$$h(X,FPY) + \nabla^1_X FQY - FQ\nabla_X Y - Ch(X,Y) = 0,$$

3.10 $$\nabla^1_X FQY = FQ\nabla_X Y + Ch(X,Y) - h(X,FPY)$$

Adding (3.8) and (3.9)

3.11 $$\nabla_X FPY - A_{FQY}X = FP\nabla_X Y + Bh(X,Y) - g(X,Y)U$$
$$+u(Y)X+2u(X)u(Y)U.$$

Interchanging X and Y in (3.11)

3.12 $\nabla_Y FPX - A_{FQX} Y = FP\nabla_Y X + Bh(X,Y) - g(X,Y)U + u(X)Y$

+2u(X)u(Y)U.

Adding (3.11) and (3.12)

3.13 $\nabla_X FPY + \nabla_Y FPX - A_{FQY} X - A_{FQX} Y$

$= FP\nabla_X Y + FP\nabla_Y X + 2Bh(X,Y)$

–2g(X,Y)U+u(Y)X+u(X)Y

+ 4u(X)u(Y)U.

Lemma (3.2) : Let M be a CR-submanifold of a Lorentzian Para-Sasakian manifold $\tilde{M}$, then $\forall Y,\, Z \in D^1$

3.14 $FP[Y,Z] = A_{FY} Z - A_{FZ} Y + u(Y)Z - u(Z)Y.$

$\forall Y,\, Z \in D^1$

Proof : For $\forall Y,\, Z \in D^1$

3.15 $\tilde{\nabla}_Y FZ = (\tilde{\nabla}_Y F) Z + F(\tilde{\nabla}_Y Z).$

Using (1.1) (e), (2.5), (2.6) in (3.15)

3.16 $-A_{FZ} Y + \nabla^1_Y FZ = -g(Y,Z)U + u(Y)u(Z)U$

+[Y+u(Y)U]u(Z)

$+ F[\nabla_Y Z + h(Y,Z)].$

Using (2.8) and (2.9)

3.17 $-A_{FZ} Y + \nabla^1_Y FZ = -g(Y,Z)U + u(Z)Y + 2u(Y)u(Z)U$

$+ F[P\nabla_Y Z + Q\nabla_Y Z]$

+ Bh(Y,Z)+Ch(Y,Z).

Using (3.3) in (3.17),

$-A_{FZ} Y + (FQ\nabla_Y Z + Ch(Y,Z) - 0 = -g(Y,Z)U + u(Z)Y$

+2u(Y)u(Z)U

$+ FP\nabla_Y Z + FQ\nabla_Y Z$

+ Bh(Y,Z)+Ch(Y,Z).

3.18 $FP\nabla_Y Z = -A_{FZ} Y + g(Y,Z)U - u(Z)Y - 2u(Y)u(Z)U$

-Bh(Y,Z).

Interchanging Y and Z in (3.18)

3.19 $FP\nabla_Z Y = -A_{FY} Z + g(Y,Z)U - u(Y)Z - 2u(Y)u(Z)U$

-Bh(Y,Z).

Subtracting (3.19) from (3.18)

$$FP[Y,Z] = A_{FY} Z - A_{FZ} Y + u(Y)Z - u(Z)Y.$$

Theorem (3.1): Let M be a CR-submanifold of Lorentzian Para-Sasakian manifold $\tilde{M}$. The distribution D is integrable if and only if.

3.20 $$h(X,FY) = h(Y,FX), \forall X,Y \in D.$$

Proof: As $Y \in D$

$$\Rightarrow QY = 0, PY = Y$$

3.21 $$\nabla^1_X FQY = 0.$$

From (3.3) and (3.21)

$$FQ\nabla_X Y = h(X,FPY) - Ch(X,Y),$$

3.22 $$FQ\nabla_X Y = h(X,FY) - Ch(X,Y).$$

Interchanging X an Y.

3.23 $$FQ\nabla_Y X = h(Y,FX) - Ch(Y,X).$$

Subtracting (3.23) from (3.22)

3.24 $$FQ[X,Y] = h(X,FY) - h(Y,FX).$$

Thus D is integrable iff $X,Y \in D \Rightarrow [X,Y] \in D$

$$\Rightarrow Q[X,Y] = 0.$$

Thus D is integrable if and only if

$$h(X,FY) = h(Y,FX)$$

Theorem (3.2): Let M be a CR submanifold of a Lorentzian Para-Sasakian manifold $\tilde{M}$. The Distribution D^1 is integrable if and only if.

3.25 $$A_{FY}Z - A_{FZ}Y = u(Z)Y - u(Y)Z, \forall Y,Z \in D^1$$

Proof: From (3.14) $\forall Y,\ Z \in D^1$

3.26 $$FP(Y,Z) = A_{FY}Z - A_{FZ}Y + u(Y)Z - u(Z)Y$$

D^1 is integrable if and only if $Y,\ Z \in D^1 \Rightarrow [Y,Z] \in D^1$

$$\Rightarrow P[Y,Z] = 0.$$

Thus,

3.27 $$FP[Y,Z] = 0.$$

From (3.26) and (3.27) D^1 is integrable if and only if $A_{FY}Z - A_{FZ}Y = u(Z)Y - u(Y)Z.$

Corollary: Let M be a U – horizontal CR-submanifold of Lorentzian Para-Sasakian manifold $\tilde{M}$. The distribution D^1 is integrable if and only if)

3.28. $$A_{FY}Z = A_{FZ}Y,\ \forall Y,\ Z \in D^1$$

Proof: Since M is U-horizontal $\Rightarrow U \in D$

Thus $\forall Y \in D^1$

g(Y,U) = 0.

3.29 u(Y)=0; similarly u(Z)=0

Thus from above theorem, D^1 is integrable if and only if

$$A_{FY}Z = A_{FZ}Y$$

4. PARALLEL HORONZOLTAL DISTRIBUTIONS OF CR-SUBMANIFOLDS

Definition (4.1) The U horizontal distribution D is called parallel if $X, Y \in D \Rightarrow \nabla_X Y \in D,$

Theorem (4.1): Let M be U-horizontal CR-submanifold of a Lorentzian Para-Sasakian manifold $\tilde{M}$. The distribution D is parallel if and only if.

4.1 $$h(X, FY) = h(FX, Y) = Fh(X, Y), \forall X, Y \in D$$

Proof : Since M is U-horizontal, $U \in D$

4.2 $$QU = 0$$

Also $X, Y \in D \Rightarrow QX = 0 = QY,\ Y = PY \in D, FPY \in D.$

Thus,

4.3 $$FQY = 0,\ A_{FQY} = 0,\ FPY \in D,\ QX = 0.$$

Let us assume that D is parallel, then from equation (4.3)

4.4 $$\nabla_X FPY \in D,\ Q\nabla_X FPY = 0.$$

From (4.2), (4.3), (4.4) and (3.2)

4.5 $$Bh(X, Y) = 0 \Rightarrow Fh(X, Y) = Ch(X, Y).$$

Again $X, Y \in D$ and D being parallel.

4.6 $$\nabla_X Y \in D \Rightarrow Q\nabla_X Y = 0.$$

From (3.3) using (4.2), (4.3) and (4.6)

4.7 Ch(X,Y) = h(X,FY).

Using symmetry of h

4.8 Ch(X,Y) = h (X,FY) = h (Y,FX).

From (4.5) and (4.8)

h (X,FY) = h(Y,FX) = Fh(X,Y).

Conversely, let us assume that

h (X,FY) = h(Y,FX) = Fh(X,Y), hold good then with this condition and first equation of (4.3), equation (3.3) gives $\nabla_X Y \in D$. Thus D is a parallel distribution.

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