Almost Jordan Generalized Derivations in Prime Rings

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Abstract— **Let R be a prime ring with char$\neq$ 2, D(. , .) be a symmetric bi-derivation of R, 0$\neq$ d be trace of D (. , .) , f –(α , β)$_r$ – d be right almost Jordan generalized derivation of R, β is surjective and a $\in$ R. If af(x) = 0 for all x $\in$R, then a = 0 and let R be a prime ring with char R $\neq$ 2,3 , D (. , . , .) be permuting tri-derivation of R , 0 $\neq$ d be the trace of D (. , . , .). (f– d)$_r$ be right almost Jordan generalized derivations of R. If [f(x) , r] = 0 for all x , r $\in$ R, then R is commutative ring.**

Keywords— **Ring , Prime ring, derivation, Jordan derivation , Generalized derivation, Jordan generalized derivation, symmetric bi-derivation, permuting tri-derivation , Almost Jordan generalized derivation.**

1. Introduction

Throughout the present paper R will be a ring and Z(R) will be its center. A ring R is prime, if xRy = { 0 } implies x = 0 or y = 0. We shall write [x, y] for xy-yx.

An additive map d: R→ R is called derivation(resp. Jordan derivation) if d(xy) = d (x) y+ x d (y) (resp. d(x^2) = d (x) x+ x d(x)) holds for all x ,y ϵ R. An additive map f: R→ R is called Generalized derivation(resp. Jordan Generalized derivation) if there exists a derivation d of R such that f(xy) = f(x) y + x d (y) (resp. F(x^2) = F (x) x+ x d(x)) holds for all x, y ϵ R. Hence the concept of generalized derivation covers both the concepts of a derivation and left centralizer (i.e an additive map f satisfying f (xy) = f (x) y for all x , y ϵ R) and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and left jordan centralizer (i.e an additive map f satisfying f (x^2) =f (x) x for all x $\in$ R)

In [4] and [1]. Maksa defined bi- derivation in ring theory mutually to partial derivations and examined some properties of this derivation.A map D (. , .) : R × R → R is said to be symmetric if D (x , y) = D (y , x) for all x , y $\in$ R. A map d: R→ R defined by d (x) = D (x , x) is called the trace of D (. , .) where D (. , .) : R × R → R is a symmetric map. It is clear that if D (. , .) is bi-additive (i.e additive in all arguments), then the trace d of D(. , .) satisfies the identity d (x + y) = d (x) + d (y) + 2 D(x , y) for all x , y $\in$ R. A symmetric bi-additive map D(. , .) : R × R → R is called symmetric bi-derivation if D(xz , y) = D(x , y)z + xD(z , y) for all x, y, z ϵ R . For any y $\in$ R, the map x → D (x , y) is derivation. Let D(. , .) is a symmetric bi-additive map on R. D (0 , y) = 0 for all y$\in$ R and D (-x , y) = - D(x , y) for all x , y $\in$ R. The trace of D (. , .) is an even function.

A map D (. , . , .) : R × R × R → R is called permuting if D (x , y , z) = D (x , z , y) = D (z , x , y) = D (z , y , x) = D (y , z , x) =D(y , x , z) holds for all x , y , z $\in$ R. A map d: R→ R defined by d (x) = D (x , x ,x) is called the trace of D (. , . , .) where D (. , . , .): R × R → R is a permuting map. It is obvious that if D (. , . , .) : R × R × R → R permuting tri-additive (i.e additive in all three arguments), then the trace d of D (. , . , .) satisfies the identity d (x + y) = d (x) + d (y) + 3 D (x , x , y) + 3D (x , y , y) for all x , y ϵ R. A permuting tri-additive map D (. , . , .) : R × R → R is called permuting tri -derivation if D(xw, y,z) = D(x , y , z)w + xD(w , y , z) for all x, y, z ,w ϵ R . The trace of D (. , . , .) is an odd function. Let D (. , . , .) be a permuting tri – derivation of R. In this case , for any fixed a $\in$ R and for all x , y $\in$ R , a map D_1(. , .) : R × R → R defined by D_1(x , y) =D(a,x,y) and a map d_2 :R → R defined by d_2(x) = D(a,a,x) are a symmetric bi-derivation (in this meaning permuting 2-derivation is a symmetric bi-derivation) and a derivation, respectively.

In this paper , we will take ring R as a prime ring with right and symmetric Martindale ring of quotients Q_r (R) and Q_s (R), extended centroid C and central closure R_C = RC. Let us review some important facts about these rings (see [10] , [6] and [5] for details).

The ring $Q_r(R)$ can be characterized by the following four properties.

(i) $R \subseteq Q_r(R)$,
(ii) For $q \in Q_r(R)$ there exists a non-zero ideal I of R such that $qI \subseteq R$,
(iii) If $q \in Q_r(R)$ and $qI = \{0\}$ for some non-zero ideal I of R, then q = 0,
(iv) If I is a non-zero ideal of R and $q \in Q_r(R)$ and $\varphi : I \to R$ is a right R –module map then there exists $q \in Q_r(R)$ such that $\varphi(x) = qx$ for all $x \in I$

The ring $Q_s(R)$ consists of those $q \in Q_r(R)$ for which $Iq \subseteq R$ for some non-zero ideal I of R. The extended centroid C is a field and it is the center of both $Q_r(R)$ and $Q_s(R)$. Thus, one can view the ring R as a subring of algebras R_C, $Q_r(R)$ and $Q_s(R)$over C . The extended centroid of R_C is equal to C, whence R_C is equal to its central closure.

In [8], Hvala gave a relation, using generalized derivation defined by Bresar, between prime rings and its extended centroid in ring theory. Many Authors have investigated comparable results on prime or semi-prime rings with generalized derivations (see[9],[7],[11],[3]…..)

In this paper, we prove some results about that what happens if we take trace of symmetric bi-derivation or permuting tri-derivation instead of derivation in definition of Jordan generalized derivation. Also we apply these results to very well-known results.

2. Jordan Generalized Derivation Determined By Trace of Symmetric Bi-Derivation

Definition 2.1 : Let R be a ring, $D(\,.\,,\,.\,) : R \times R \to R$ be a symmetric bi-derivation and d be trace of $D(\,.\,,\,.\,)$. An additive map $f: R \to R$ is called right Jordan generalized derivation determined by d, if $f(x^2) = f(x)x + x\,d(x)$ holds for all $x \in R$ and denoted by $(f-d)_r$

Example 2.2 : Let $R=\left\{ \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix} \middle| \, a_1\, b_1 \in I \right\}$ ring where I is the ring of integers, a map $D(.,.):R\times R \to R$,defind by $D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ d_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a_1c_1 & 0 \end{pmatrix}$ and a map $f: R \to R$ defined by $f\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_1+b_1 & 0 \\ 0 & 0 \end{pmatrix}$. $D(\,.\,,\,.\,)$is a symmetric bi-derivation. A map $d: R \to R$, $d(x)=D(x,x)$ is defined by $d\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ d_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a_1^2 & 0 \end{pmatrix}$ is the trace of $D(\,.\,,\,.\,)$. f is an additive map and $f(x^2) = f(x)x + x\,d(x)$ holds for all $x \in R$. Thus , f is a right Jordan generalized derivation determined by d. But f is not a Jordan derivation.

Lemma 2.3 : Let $f: R \to R_C$ be an additive map a satisfying $f(x^2) = f(x)x$ for all $x \in R$. Then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$ for all $x \in R$

If we consider the definition of Jordan generalized derivation introduced by us in the Definition 2.1 and Lemma2.3 , it is important to give the following remark

***Remark 2.4*:** Let R be a prime ring with char$\neq$ 2, $D(.\,,.)$ be a symmetric bi-derivation , d be a trace of $D(.\,,.)$ and $(f-d)_r$ be a right Jordan generalized derivation of R. Replacing x by –x in definition 2.1, we get $x\,d(x)=0$. If d = 0, then $f(x^2) = f(x)$ holds for all $x \in R$. From Lemma2.3, there exist $q \in Q_r(R_C)$ such that $f(x) = qx$ for all $x \in R$. If $d \neq 0$ then $R = \{0\}$. So that $(f-d)_r$ right Jordan generalized derivation has not got any meaning in prime ring

We can generalize above definition as follows

***Definition 2.5*:** Let R be a ring, $D(\,.\,,\,.\,) : R \times R \to R$ be a symmetric bi-derivation and d be trace of $D(\,.\,,\,.\,)$. An additive map $f: R \to R$ is called right Jordan generalized α-derivation determined by d, if there exists a function $\alpha: R \to R$ such that $f(x^2) = f(x)\,\alpha(x) + x\,d(x)$ for all $x \in R$ and denoted by $(f-\alpha-d)_r$.

***Example 2.6*:** Let $R=\left\{ \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix} \middle| \, a_1\, b_1 \in I_2 \right\}$ ring where I_2 is the ring of integers modulo 2, a map $D(.,.):R \times R \to R$, defined by $D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ d_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a_1c_1 & 0 \end{pmatrix}$, a map $f: R \to R$ defined by $f\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_1^2 & 0 \\ 0 & 0 \end{pmatrix}$. and a map $\alpha: R \to R$ defined by $\alpha\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_1^2 & 0 \\ b_1^2 & 0 \end{pmatrix}$, $D(\,.\,,\,.\,)$is a symmetric bi-derivation. A map $d: R \to R$, $d(x)=D(x,x)$ is defined by $d\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ d_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a_1^2 & 0 \end{pmatrix}$ is the trace of $D(\,.\,,\,.\,)$. f is an additive map and $f(x^2) = f(x)\,\alpha(x) + x\,d(x)$ holds for all $x \in R$. Thus , f is a right Jordan generalized α- derivation determined by d. But f is not a Jordan derivation.

***Remark 2.7*:** Let R be a prime ring with char$\neq$ 2, $D(.\,,.)$ be a symmetric bi-derivation , d be a trace of $D(.\,,.)$ and $(f-\alpha-d)_r$

be a right Jordan generalized α- derivation of R. Suppose that α is an odd function. Replacing x by –x in definition 2.5, we get xd(x)=0. Since R be a prime ring , d=0 or x=0 . If d=0, then $f(x^2) = f(x)\alpha(x)$ for all x ϵ R. If d≠ 0 then R = { 0 }.

Definition 2.8: Let R be a ring, D(. , .) : R × R → R be a symmetric bi-derivation and d be trace of D(. , .). An additive map f: R → R is called right Jordan generalized (α,β)-derivation determined by d, if there exists a function α: R → R and β: R → R such that $f(x^2) = f(x)\alpha(x) + \beta(x)d(x)$ for all x ϵ R .

Example 2.9: Let R= $\left\{ \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix} \middle| a_1\, b_1 \in I_2 \right\}$ ring where I_2 is the ring of integers modulo 2, a map D(.,.): R ×R→R, defined by $D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ d_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a_1c_1 & 0 \end{pmatrix}$, a map f: R → R defined by $f\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_1^2 & 0 \\ 0 & 0 \end{pmatrix}$. a map α: R → R defined by $\alpha\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_1^2 & 0 \\ b_1^2 & 0 \end{pmatrix}$, and a map β: R → R defined by $\beta\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} b_1^2 & 0 \\ a_1^2 & 0 \end{pmatrix}$. D(. , .)is a symmetric bi-derivation. A map d: R → R, d(x)=D(x,x) is defined by $d\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\right) = D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ d_1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a_1^2 & 0 \end{pmatrix}$ is the trace of D(. , .). f is right Jordan generalized (α,β)-derivation determined by d. But f is not a Jordan derivation.

Definition 2.10 :Let R be a ring, D(. , .) : R × R → R be a symmetric bi-derivation and d be trace of D(. , .). An even function f: R → R is called almost right Jordan generalized (α,β)-derivation determined by d, if there exists even function α: R → R and β: R → R such that $f(x^2) = f(x)\alpha(x) + \beta(x)d(x)$ for all x ϵ R and denoted by $f - (\alpha, \beta)_r - d$.

Now , Let A be any of the rings R, R_C , R_C + C , Q_r(R), Q_s(R) , *Q_r(R_C)and Q_s(R_C)*. We shall give make an extensive use of the following result

Lemma 2.11:[2 Lemma 1] If a_i , $b_i \in$ A satisfy $\sum a_ixb_i = 0$ for all x ϵ R.then the a_i's as well as b_i's are C-dependent, unless all $a_i = 0$ or $b_i = 0$.

Lemma 2.12 :Let R be a prime ring with char≠ 2 and let d_1 and d_2 be traces of symmetric bi-derivations D_1(. , .) and D_2 (. , .) respectively. If the identity

$$d_1(x)d_2(x) = d_2(x)\, d_1(x), \quad \forall x \in R$$

holds and $d_1 \neq 0$ then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$ for all $x \in R$

Proposition 2.13:Let R be a prime ring with char≠ 2 ,D_1(. , .) , D_2 (. , .) , D_3(. , .) and D_4 (. , .) be symmetric bi-derivations of R, $0 \neq d_1$, $0 \neq d_2$, $0 \neq d_3$ and $0 \neq d_4$ be the traces of D_1(. , .) , D_2 (. , .) , D_3(. , .) and D_4 (. , .) respectively, $f_1 - (\alpha, \beta)_r - d_1$, $f_2 - (\alpha, \beta)_r - d_2$, $f_3 - (\alpha, \beta)_r - d_3$ and $f_4 - (\alpha, \beta)_r - d_4$ be right almost jordan generalized derivations of R. If the identity

$$f_1(x)f_2(x) = f_3(x)\, f_4(x), \quad \forall x \in R \qquad (1)$$

holds, $0 \neq f_1$ and β is surjective , then there exists $\lambda \in C$ such that $f_3(x) = \lambda f_1(x)$ for all $x \in R$

Proof **:** Let x , z ∈ R. Replacing x by xz in f_2 and f_4 of (1), we get

$$f_1(x)f_2(x)\,\alpha(z) + f_1(x)\beta(x)\,d_2(z) = f_3(x)\,f_4(x)\,\alpha(z) + f_3(x)\beta(x)\,d_4(z)$$

From (1) we have

$$f_1(x)\beta(x)\,d_2(z) = f_3(x)\beta(x)\,d_4(z) \qquad \forall x, z \in R \qquad (2)$$

Replacing β(x) by $\beta(x)d_4(z)$ in (2) we get

$$f_1(x)\beta(x)\,d_4(z)d_2(z) = f_3(x)\beta(x)\,d_4(z)d_4(z)$$

from (2)

$$f_1(x)\beta(x)\,d_4(z)d_2(z) = f_3(x)\beta(x)\,d_2(z)d_4(z)$$

Hence

$$f_1(x)\beta(x)(d_4(z)d_2(z) - d_2(z)d_4(z)) = 0$$

Since $f_1 \neq 0$, β is surjective and R is a prime ring , we get

$d_4(z)d_2(z) = d_2(z)d_4(z)$ for all z∈ R.

From Lemma2.13, since $d_4 \neq 0$, there exists $\lambda \in C$ such that $d_2(z) = \lambda d_4(z)$ for all $z \in R$.

Using last relation in (2), we get

$f_1(x)\beta(x)\,\lambda\, d_4(z) = f_3(x)\beta(x)\,d_4(z)$ for all $x, z \in R$.

That is $(\lambda f_1(x) - f_3(x))\,\beta(x)\,d_4(z) = 0$.

Since $d_4 \neq 0$, β is surjective and R is a prime ring ,

$f_3(x) = \lambda f_1(x)$ for all $x \in R$ and $\lambda \in C$.

Corallary(2.14) *:* Let R be a prime ring with char≠ 2 ,D_1(. , .) and D_2 (. , .) be symmetric bi-derivations of R, $0 \neq d_1$ and $0 \neq d_2$ be the traces of D_1(. , .) and D_2 (. , .) respectively , $f_1 - (\alpha, \beta)_r - d_1$ and $f_2 - (\alpha, \beta)_r - d_2$ be right almost jordan generalized derivations of R. If the identity $f_1(x)f_2(x) = f_2(x)\,f_1(x), \ \forall x \in R$

holds, $0 \neq f_1$ and β is surjective , then there exists $\lambda \in C$ such that $f_2(x) = \lambda f_1(x)$ for all $x \in R$

Lemma 2.15: Let R be a prime ring with char≠ 2, D(. , .) be a symmetric bi-derivation of R, 0≠ d be trace of D (. , .) , $f -(\alpha , \beta)_r - d$ be right almost Jordan generalized derivation of R, β is surjective and $a \in R$. If af(x) = 0 for all $x \in R$, then a = 0

Proof : Let af(x) = 0 for all $x \in R$.

Replacing x by x^2 we get

af(x) α(x) + a β(x) d(x) = 0.

Using hypothesis, we have

a β(x) d(x)= 0.

Since R is a prime ring , a = 0 or d = 0

Since d≠0 , a = 0

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3. Jordan Generalized Derivation Determined By Trace of Permuting Tri-Derivation

Definition 3.1: Let a be a ring, D(. , . , .) : R × R × R → R be a permuting tri-derivation and d be trace of D(. , . , .) . An additive map f: R → R is called right jordan generalized derivation determined by d , if $f(x^2) = f(x)x + x\,d(x)$ holds for all x ϵ R. An additive map f: R → R is called left Jordan generalized derivation determined by d , if $f(x^2) = x f(x) + d(x)x$ holds for all x ϵ R . Also, an additive map f: R → R is called Jordan generalized derivation determined by d , if it is both a right Jordan generalized and a left Jordan generalized derivation.

Example 3.2: Let $R = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \middle| \ a1 , b1 , c1 \ \epsilon \ I \right\}$ ring where I is the ring of integers, a map D(. , . , .) : R × R × R → R defined by

$$D\left(\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 b_1 c_1 & 0 & 0 \end{pmatrix}$$

and a map f: R → R defined by

$$f\left(\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

, D is a permuting tri-derivation.. A map d : R → R , d (x) = D(x ,x ,x) is defined by

$$d\left(\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \right) = D\left(\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1^3 & 0 & 0 \end{pmatrix}$$

is the trace of D(. , . , .) . f is an additive map and $f(x^2) = f(x)x + x\,d(x)$ holds for all x ϵ R. But f is not a Jordan derivation

Remark 3.3: Let R be a prime ring with char R ≠ 3,4 D(. , . , .) : R × R × R → R be a permuting tri-derivation , d be a trace of D(. , . , .) and f be a right Jordan generalized derivation determined by d . Therefore $f(x^2) = f(x)x + x\,d(x)$ for all x ϵ R. Replacing x by x + z in this relation , we get

$$\begin{aligned} f(x^2) &= f((x+z)(x+z)) \\ &= f(x^2 + xz + zx + z^2) = f(x^2) + f(xz) + f(zx) + (z^2) \\ &= f(x)x + x\,d(x) + f(x)z + x\,d(z) + f(z)x + z\,d(x) + f(z)z + z\,d(z) \end{aligned}$$

And

$$\begin{aligned} f(x^2) &= f((x+z)(x+z)) \\ &= f(x+z)(x+z) + (x+z)\,d(x+z) \\ &= (f(x) + f(z))(x+z) + (x+z)(d(x) + d(z) + 3D(x,x,z) + 3D(x,z,z)) \end{aligned}$$

$$= f(x)x + f(x)z + f(z)x + f(z)z + x\,d(x) + x\,d(z) + x\,3D(x,x,z) + x\,3D(x,z,z) + z\,d(x) + z\,d(z) + z\,3D(x,x,z) + z\,3D(x,z,z)$$

Comparing these relations, and Since char R ≠ 3 we get

$$x D(x,x,z) + x D(x,z,z) + z D(x,x,z) + z D(x,z,z) = 0$$

Replace z by x , we get

$4x d(x) = 0$ for all x ϵ R

Since R is a prime ring and Char R ≠ 4 , x =0 or d (x) =0 for all x ϵ R

If d = 0 then $f(x^2) = f(x)x$ holds for all x ϵ R.

From Lemma 2.3 there exists $q \in Q_r(R_C)$ such that $f(x) = qx$ for all $x \in R$ In this case, we must take f be not additive when R is a prime char $R \neq 2,3,4$

we can give definition of Jordan generalized derivation according to above the definition as follows

Definition 3.4: Let a be a ring, $D(.,.,.) : R \times R \times R \to R$ be a permuting tri-derivation and d be trace of $D(.,.,.)$. A map $f: R \to R$ is called right almost Jordan generalized derivation determined by d , if $f(x^2) = f(x)x + x\,d(x)$ holds for all $x \in R$ and denoted by $(f-d)_r$. A map $f: R \to R$ is called left almost Jordan generalized derivation determined by d , if $f(x^2) = x\,f(x) + d(x)x$ holds for all $x \in R$ and denoted by $(f-d)_l$. Also, a map $f: R \to R$ is called almost Jordan generalized derivation determined by d , if it is both a right almost Jordan generalized and a left almost Jordan generalized derivation and denoted by $(f-d)$.

Lemma 3.5: Let R be a prime ring with char $R \neq 2,3$ and let d_1 and d_2 be traces of permuting tri-derivations $D_1(.,.,.)$ and $D_2(.,.,.)$ respectively. If the identity $d_1(x)d_2(x) = d_2(x)d_1(x)$ holds for all $x \in R$. and $d_1 \neq 0$, then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$ for all $x \in R$

Proposition 3.6 : Let R be a prime ring with char $R \neq 2,3$, $D_1(.,.,.)$, $D_2(.,.,.)$, $D_3(.,.,.)$ and $D_4(.,.,.)$ be permuting tri-derivations of R , $0 \neq d_1$,$0 \neq d_2$,$0 \neq d_3$ and $0 \neq d_4$ be the traces of $D_1(.,.,.)$, $D_2(.,.,.)$, $D_3(.,.,.)$ and $D_4(.,.,.)$ respectively ,$(f_1-d_1)_r$ $(f_2-d_2)_r$,$(f_3-d_3)_r$ and $(f_4-d_4)_r$ be right almost jordan generalized derivations of R. If the identity

$$f_1(x)f_2(x) = f_3(x)f_4(x), \quad \forall x \in R \qquad (3)$$

holds, $0 \neq f_1$, then there exists $\lambda \in C$ such that $f_3(x) = \lambda f_1(x)$ for all $x \in R$

Proof : Let $x, z \in R$ Replacing $f_2(x)$ by $f_2(xz)$ and $f_4(x)$ by $f_4(xz)$ in (3)

$$f_1(x)f_2(xz) = f_3(x)f_4(xz),$$

$$f_1(x)(f_2(x)z + x\,d_2(z)) = f_3(x)(f_4(x)z + x\,d_4(z))$$

$$f_1(x)f_2(x)z + f_1(x)x\,d_2(z) = f_3(x)f_4(x)z + f_3(x)x\,d_4(z)$$

using (3), we get

$$f_1(x)x\,d_2(z) = f_3(x)x\,d_4(z) \quad \forall x,z \in R \qquad (4)$$

Replace x by $x\,d_4(z)$ in (4) and using (4) we get

$$f_1(x)x\,d_4(z)d_2(z) = f_1(x)x\,d_2(z)d_4(z)$$

Thus

$$f_1(x)x(d_4(z)d_2(z) - d_2(z)d_4(z)) = 0$$

Since $f_1 \neq 0$ and R be a prime ring , we get

$$d_4(z)d_2(z) = d_2(z)d_4(z) \quad \forall z \in R$$

Since $d_4 \neq 0$, there exists $\lambda \in C$ such that $d_2(z) = \lambda d_4(z)$ for all $z \in R$.

Using last relation in (4), we get

$f_1(x)x\lambda d_4(z) = f_3(x)x\,d_4(z)$ for all $x, z \in R$.

i.e., $(\lambda f_1(x) - f_3(x))x\,d_4(z)=0$.

Since $d_4 \neq 0$ and R is a prime ring , we get

$f_3(x) = \lambda f_1(x)$ for all $x \in R$ and $\lambda \in C$

Corollary 3.7: Let R be a prime ring with char $R \neq 2,3$, $D_1(.,.,.)$ and $D_2(.,.,.)$ be permuting tri-derivations of R , $0 \neq d_1$ and $0 \neq d_2$ be the traces of $D_1(.,.,.)$ and $D_2(.,.,.)$ respectively ,$(f_1-d_1)_r$ and $(f_2-d_2)_r$ be right almost Jordan generalized derivations of R. If the identity

$$f_1(x)f_2(x) = f_2(x)f_1(x), \quad \forall x \in R$$

holds, $0 \neq f_1$, then there exists $\lambda \in C$ such that $f_2(x) = \lambda f_1(x)$ for all

Lemma 3.8: Let R be a prime ring ,$D(.,.,.)$ be permuting tri-derivation of R , $0 \neq d$ be the trace of $D(.,.,.)$. $(f-d)_r$ be right almost Jordan generalized derivations of R. and $a \in R$, then

(i) If $af(x) = 0$ for all $x \in R$ then $a = 0$

(ii) If $[a, f(x)] = 0$ for all $x \in R$ and char $R \neq 2,3$ then $a \in Z(R)$

Proof :

(i) Let $af(x) = 0$ for all $x \in R$

Replacing x by x^2 , we get

$$a\,f(x^2) = 0$$

$$a(f(x)x + x\,d(x)) = 0$$

$$af(x)x + ax\,d(x) = 0$$

$$ax\,d(x) = 0$$

Snce R is a prime ring and $d \neq 0$, we get $a = 0$

(ii) Let $[a, f(x)] = 0$ for all $x \in R$

Replacing x by x^2 , $x \in R$ and using hypothesis we get

$a f(x^2) - f(x^2) a = 0$

$a f(x) x + ax d(x) - f(x) a + x d(x) a = 0$

$f(x) [a, x] + [a, x] d(x) + x [a, d(x)] = 0$ (5)

Replacing x by x + z in (5) and using (5),

$f(x+z) [a, x+z] + [a, x+z] d(x+z) + (x+z) [a, d(x+z)] = 0$

By expanding the above equation, we have

$f(x)ax + f(x)az - f(x)xa - f(x)za + f(z)ax + f(z)az - f(z)xa - f(z)za + axd(x) + axd(z) + 3\,axD(x,x,z) + 3\,axD(x,z,z) + azd(x) + azd(z) + 3\,azD(x,x,z) + 3\,azD(x,z,z) - xad(x) - xad(z) - 3\,xaD(x,x,z) - 3\,xaD(x,z,z) - zad(x) - zad(z) - 3zaD(x,x,z) - 3zaD(x,z,z) + xad(x) + xad(z) + 3\,xaD(x,x,z) + 3\,xaD(x,z,z) - xd(x)a - xd(z)a - 3xD(x,x,z)a - 3xD(x,z,z)a + zad(x) + zad(z) + 3zaD(x,x,z) + 3zaD(x,z,z) - zd(x)a - zd(z)a - 3zD(x,x,z)a - 3zD(x,z,z)a = 0$

$[a, x]D(x, x, z) + [a, x]D(x, z, z) + x [a, D(x, x, z)] + x [a, D(x, z, z)] + [a, z]D(x, x, z) + [a, z]D(x, z, z) + z [a, D(x, x, z)] + z [a, D(x, z, z)] = 0$ (6)

Since char R $\neq$ 3

Replacing z by –z In (6) , comparing with (6), and Since char R $\neq$ 2we get

$[a, x]D(x, x, z) + x [a, D(x, x, z)] + [a, z]D(x, z, z) + z [a, D(x, z, z)] = 0$ (7)

Replacing z by x In (7) and Since char R $\neq$ 2 , we get

$[a, x]D(x, x, x) + x [a, D(x, x, x)] = 0$

$[a, x]d(x) + x [a, d(x)] = 0$

$axd(x) = xd(x)a$ for all $x \in R$ (8)

Replacing xd(x) by xrd(x) for all $r \in R$ in (8) and using (8), we get

$axrd(x) = xrd(x)a$

$axrd(x) = xard(x)$

$[a, x] r d(x) = 0$

Since R is a prime ring and d $\neq$ 0, we have $a \in Z(R)$

Lemma 3.9: Let R be a prime ring with char R $\neq$ 2,3 , D(. , . , .) be permuting tri-derivation of R , 0 $\neq$ d be the trace of D(. , . , .). (f– d $)_r$ be right almost Jordan generalized derivations of R. If $[f(x), r] = 0$ for all x , r $\in R$, then R is commutative ring.

Proof : If $[f(x), r] = 0$ for all x , r $\in R$

Replacing x by x^2 , x $\in R$ and using hypothesis, we get

$f(x) [x, r] + [x, r] d(x) + x [d(x), r] = 0$ (9)

Replacing x by x + z in (9) and using (9), we have

$f(x+z) [x+z, r] + [x+z, r] d(x+z) + (x+z) [d(x+z), r] = 0$

By expanding the above equation, we have

$f(x)xr + f(x)zr - f(x)rx - f(x)rz + f(z)xr + f(z)zr - f(z)rx - f(z)rz + xrd(x) + xrd(z) + 3\,xrD(x,x,z) + 3xrD(x,z,z) + zrd(x) + zrd(z) + 3\,zrD(x,x,z) + 3\,zrD(x,z,z) - rxd(x) - rxd(z) - 3\,rxD(x,x,z) - 3rxD(x,z,z) - rzd(x) - rzd(z) - 3rzD(x,x,z) - 3rzD(x,z,z) + xd(x)r + xd(z)r + 3xD(x,x,z)r + 3xD(x,z,z)r - xrd(x) - xrd(z) - 3\,xrD(x,x,z) - 3\,xrD(x,z,z) + zd(x)r + zd(z)r + 3zD(x,x,z)r + 3zD(x,z,z)r - zrd(x) - zrd(z) - 3zrD(x,x,z) - 3zrD(x,z,z) = 0$

Since char R $\neq$ 3, we get

$[x, r]D(x, x, z) + [x, r]D(x, z, z) + x [D(x, x, z), r] + x [D(x, z, z), r] + [z, r]D(x, x, z) + [z, r]D(x, z, z) + z [D(x, x, z), r] + z [D(x, z, z), r] = 0$ (10)

Replacing z by –z in (10) , comparing with (10) and Since char R $\neq$ 2, we get

$[x, r]D(x, x, z) + x [D(x, x, z), r] + [z, r]D(x, z, z) + z [D(x, z, z), r] = 0$ (11)

Replacing z by x in (11) and Since char R $\neq$ 2 , we get

$[x, r]D(x, x, x) + x [D(x, x, x), r] = 0$

$[x, r]d(x) + x [d(x), r] = 0$

$xd(x)r = rxd(x)$ for all $x, r \in R$ (12)

Replacing xd(x) by xzd(x) for all $z \in R$ in (12) and using (12), we get

$xzd(x)r = rxzd(x)$

xrzd(x) = rxzd(x)

[x , r] z d(x) = 0

Since R is a prime ring and $d \neq 0$, R is a commutative ring.

Theorem 3.10: Let R be a prime ring with char $R \neq 2,3$, $D_1(.,.,.)$ and $D_2(.,.,.)$ be permuting tri-derivations of R, $0 \neq d_1$ and $0 \neq d_2$ be the traces of $D_1(.,.,.)$ and $D_2(.,.,.)$ respectively ,$(f_1 - d_1)_r$ and $(f_2 - d_2)_r$ be right almost Jordan generalized derivations of R.
If $af_1(x) = f_2(x)a$, $\forall x \in R$, then $a \in Z(R)$

Proof : $a f_1(x) = f_2(x) a$, $\forall x \in R$ (13)

Replace x by x^2 in (1) , we get

$a f_1(x^2) = f_2(x^2) a$

$a (f_1(x)x + xd_1(x)) = (f_2(x)x + xd_2(x))a$

$af_1(x)x+axd_1(x)=f_2(x)xa+xd_2(x)a$ (14)

Replacing x by x + z in (14) and using (14), we have

$a f_1(x+z)(x+z) +a(x+z)d_1(x+z) = f_2(x+z) (x+z) a + (x+z)d_2(x+z) a$

By expanding the above equation, we have

$a f_1(x)x + a f_1(x)z + a f_1(z)x a f_1(z)z + a xd_1(x) + a xd_1(z) + 3axD_1(x,x,z)+3axD_1(x,z,z)+azd_1(x)+azd_1(z) + 3 azD_1(x,x,z) + 3 azD_1(x,z,z) = f_2(x)xa + f_2(x)za + f_2(z)xa + f_2(z)za + xd_2(x)a + xd_2(z)a + 3 xD_2(x,x,z)a + 3 xD_2(x,z,z)a+ zd_2(x)a + zd_2(z)a + 3 zD_2(x,x,z)a + 3 zD_2(x,z,z)a$

Since char $R \neq 3$, we get

$axD(x,x,z)+axD(x,z,z)+azD(x,x,z)+azD(x,z,z) = xD(x,x,z)a + xD(x,z,z)a+ zD(x,x,z)a+zD(x,z,z)a$ (15)

Replacing z by -z in (15) ,comparing with (15)and since char $R \neq 2$, we get

$axD_1(x,x,z)+azD_1(x,z,z)=xD_2(x,x,z)a+zD_2(x,z,z)a$ (16)

Replace z by x in (16), we have

$axD_1(x,x,x)+ axD_1(x,x,x) = xD_2(x,x,x)a + xD_2(x,x,x)a$

$axD_1(x,x,x) = xD_2(x,x,x)a$

$axd_1(x)=xd_2(x)a \quad \forall x \in R$ (17)

Replacing $xd_1(x)$ by $xrd_1(x)$ and $xd_2(x)$ by $xrd_2(x)$ in (17) and using (17), we get

$a xrd_1(x) = xrd_2(x)a \quad \forall x, r \in R$

$a xrd_1(x) = xard_1(x) \quad \forall x, r \in R$

$[a , x] rd_1(x) = 0 \quad \forall x, r \in R$

Since $d_1 \neq 0$ and R is a prime ring, we get $[a , x] = 0$ for all $x \in R$

i.e., $a \in Z(R)$

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