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Riesz Operators and their Applications in the Eigen values of Linear Operators

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Abstract: We have introduced Riesz Operators and its various properties. We have developed Eigen value theorems for absolute operator.

Key words: Operators, Linear Operators, Riesz Operators, Absolutely r-summing Operator, Eigen Value, Riesz Decomposition, Pigeon-hole Principle.

I. INTRODUCTION

Linear Operators: Let D and G be two linear space with the same set of Scalars and let y = T(x) be an operator defined on D with range lying in G,

i.e. $T : D \to G$

The Operator T is called linear if

1. T is additive, i.e., $T(x_1 + x_2) = T(x_1) + T(x_2)$ for every $x_1, x_2 \in D$.
2. T is homogeneous, i.e. for any scalar λ $T(\lambda x) = \lambda T(x)$ for every $x \in D$.

The properties (1) and (2) can be put in a combined from

$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ for every $x_1, x_2 \in D$ and every scalars λ_1 λ_2.

⋯ 1. Linear maps are also called linear transformations, linear operators on homeomorphisms.

2.The linear operations of both the linear spaces E and $\bar{E}$ are denoted by the same symbol.

3. For linear mappings, it is customary to write the value of T at x by T x rather than T(x).

Eigen Values of a Linear Operator: Let X be a linear space over the field K and $T \in L(x)$. A scalar $\lambda \in K$ is said to be an Eigen value of T if there exist a non zero vector $x \in X$ such that

$$T x = \lambda x \text{ or } (T - \lambda I) x = 0 \qquad (1)$$

Clearly, λ is Eigen value of T iff the null space of $(T - \lambda I)$ is non-trivial.

Let x be a linear space over the field K and $T \in L(x)$. A vector $x \in X$ is said to be an eigenvector of T if (i) $x \neq 0$ and (ii) $T x = \lambda x$, for some $\lambda \in K$.

Absolutely r-summing operators: An operator $T \in L(D,G)$ is called absolutely r-summing if there exist a constant $C \geq 0$ such that

$$\left\{ \sum_{i=1}^{n} \left(\| Tx_i \|^r \right)^{1/r} \right\} \leq C \sup \left\{ \sum_{i=1}^{n} \left(| \langle x_i , a \rangle |^r \right)^{1/r} : a \in U^0 \right\}$$

For every finite family of elements $x_1, \ldots\ldots, x_n \in D$.

Riesz Operators: An operator $T \in L(E)$ is said to be Riesz if every $\in > 0$ their exist an exponent n and elements u_1, ……, $u_k \in E$. Such that $T^n(U) \subseteq \bigcup_{h=1}^{k} \{u_h + \varepsilon^n U\}$. An operator $T \in L(E)$ is said to be iteratively compact if for every $\in > 0$ there exists an exponent n and element $u_1, \ldots, u_k \in E$ such that $T^n(U) \subseteq \bigcup_{h=1}^{k} \{u_h + \varepsilon^n U\}$. Where U denotes the closed unit ball of the underlying Banach space E.

Riesz Decomposition: A is an arbitrary linear map acting on a linear space E.

Let $N_k(A) = \{x \in E : A^k x = 0\}$ and $M_h(A) = \{A^h x : x \in E\}$

Obiviously, $\{0\} = N_0(A) \subseteq N_1(A) \subseteq$ …..and $\subseteq$ ……. $M_1(A) \subseteq M_0(A) = E$

Therefore, $N_\alpha(A) = \bigcup_{k=0}^{\alpha} N_k(A)$ and $M_\alpha(A) = \bigcap_{h=0}^{\alpha} M_h(A)$.

Furthermore, $n(A) = \dim [N_\alpha(A)]$ and $m(A) = \text{codim} [M_\alpha(A)]$.

If there exist an integer K such that, $N_k(A) = N_{k+1}(A)$, then A is said to have finite ascent. The smallest such K is denoted by $d_N(A)$. Therefore $N_k(A)$ is constant for $K \geq d_N(A)$.

Pigeonhole Principle: Pigeonhole Principle states that if n items are put into m containers, with n > m, then at least one container must contain more than one item.

The following are alternate formulations of the pigeonhole principle:

If *n* objects are distributed over *m* places, and if *n* > *m*, then some place receives at least two objects.

1. (equivalent formulation of 1) If *n* objects are distributed over *n* places in such a way that no place receives more than one object, then each place receives exactly one object.
2. If *n* objects are distributed over *m* places, and if *n* < *m*, then some place receives no object.
3. (equivalent formulation of 3) If *n* objects are distributed over *n* places in such a way that no place receives no object, then each place receives exactly one object.

Strong Form:

Let $q_1, q_2, ..., q_n$ be positive integers. If $q_1 + q_2 + \ldots + q_n - n + 1$ objects are distributed into *n* boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, ..., or the *n*th box contains at least q_n objects.

The simple form is obtained from this by taking $q_1 = q_2 = ... = q_n = 2$, which gives $n + 1$ objects. Taking $q_1 = q_2 = ... = q_n = r$ gives the more quantified version of the principle, namely:

Let *n* and *r* be positive integers. If $n(r - 1) + 1$ objects are distributed into *n* boxes, then at least one of the boxes contains *r* or more of the objects.

This can also be stated as, if *k* discrete objects are to be allocated to *n* containers, then at least one container must hold at least [K/N] objects, where [x] is the ceiling function, denoting the smallest integer larger than or equal to *x*. Similarly, at least one container must hold no more than [K/N] objects, where [x] is the floor function, denoting the largest integer smaller than or equal to *x*.

II. Lemma and Preposition

Lemma1: Let $T \in L(E)$ is iteratively compact. Let (x_i) be any sequence in U. Then for every $\varepsilon > 0$ there exist an exponent n and an infinite subset 1 of N such that $\|T^n x_i - T^n x_j\| \leq \varepsilon$ for all $i, j \in I$.

Proof: Let n and $u_1, \ldots, u_k \in E$ such that $T^n(U) \subseteq \bigcup_{h=1}^{k} \{u_h + \varepsilon/2\, U\}$.

Setting $I_n = \{ i \in N : T^n x_i \in u_h + \varepsilon / 2U\}$ for $h = 1,\ldots.k$, therefore, $\|T^n x_i - T^n x_j\| \leq \varepsilon$ for all $i, j \in I_h$. Furthermore, it follows from $\bigcup_{h=1}^{k} I_h = N$ that at least one of the sets $I_1, \ldots., I_k$ is infinite.

Preposition1: If $T \in L(E)$ is iteratively compact, then all null spaces $N_k(I - T)$ are finite dimensional.

Proof: Since the smallest k for which $N_k(I - T)$ is infinite dimensional. Therefore, from Reisz lemma with $\varepsilon = 1/3$, there exist an elements $x_i \in N_k(I - T)$ such that $\| x_i \| = 1$ and $\| x_i - x \| \geq 3/4$ for all $X \in \text{span}(x_1, \ldots\ldots, x_{i-1}) + N_{k-1}(I - T)$. It follows from $x - T^n x = (I + T + \ldots.\, T^{n-1})(I - T)x$ that $x - T^n x \in N_k(I - T)$ for all $x \in N_k(I - T)$ and $n = 1,2,\ldots\ldots.$

Hence $T^n x_i - T^n x_j \in x_i - x_j + N_{k-1}(I - T)$ which implies that $\|T^n x_i - T^n x_j\| \geq 3/4$ whenever I > j and n = 1,2,…… Therefore I > j and n = 1,2,…… Therefore, by the Pigeon hole principle, there exist an exponent n and different indices I and j such that $\|T^n x_i - T^n x_j\| \leq 1/2$ which is a contradiction, and it is proved.

Preposition2: If $T \in L(E)$ is iteratively compact, then I – T has finite ascent.

Proof : From Reisz lemma with $\varepsilon = 1/3$,

Let $x_k \in N_k(I - T)$ such that $\| x_k \| = 1$ and $\| x_k - x\| \geq 3/4$ for all $x \in N_k(I - T)$. It follows from, $x = (I + T + \ldots + T^{n-1})(I - T)x$ that $x_k - T^n x_k \in N_{k-1}(I - T)$ for all n = 1,2,……. Hence,

$T^n x_h - T^n x_k \in x_h - x_k + N_{h-1}(I - T) + N_{k-1}(I - T)$. Which implies that, $\| T^n x_h - T^n x_k \| \geq 3/4$ whenever h > k and n = 1,2,…. Therefore, by the Pigeon-hole principle, there exist and exponent n and different indices and k such that , $\| T^n x_h - T^n x_k \| \leq 1/2$. Which is contradiction and the preposition is proved.

Lemma2: Let $T \in L(E)$ is iteratively compact, then every bounded sequence (x_i) for which $((I - T)x_i)$ is convergent has a convergent subsequence.

Proof: Let (x_i) is contained in U. Given $\varepsilon > 0$, By the Pigeon-hole principle, there exist an exponent n and an infinite subset I such that $\|T^n x_i - T^n x_j\| \leq \varepsilon$ for all i , j $\in$ I, it follow from, $x = T^n x + (I + T + \ldots + T^{n-1})(I - T)x$ that, $\|x_i - x_j\| \leq \|T^n x_i - T^n x_j\| + \|(I + T + \ldots + T^{n-1})\|$.

$\|(I - T)x_i - (I - T)x_j\|$ thus, since $((I - T)x_i)$ is an infinite subset I_0 of I such that $\|x_i - x_j\| \leq 2\varepsilon$ for all i ,j $\in$ I_0 . Let $(x_i^0) = (x_i)$ and $\varepsilon_m : 2^{-m-1}$ for m = 1,2, …… Therefore a sequence (x_i^m) each of which is a subsequence of its predecessor (x_i^{m-1}) and such that $\|x_i^m - x_j^m\| \leq 2\,\varepsilon_m$ for all i and j. Then the diagonal (x_i^i) is the desired convergence subsequence, because $\|x_i^i - x_j^j\| \leq 2^{-m}$ whenever i , j $\geq$ m. Hence the lemma is proof.

Lemma3: $N_k(A) = N_{k+1}(A)$ implies $N_{k+1}(A) = N_{k+2}(A)$.

Proof: Let $x \in N\, N_{k+2}$. Then, $Ax \in N_{k+1}$.

Therefore, $Ax \in N_k$. Hence, $x \in N_{k+1}$.

Lemma4: If A has finite ascent then, $M_h(A) \cap N_k(A) = \{0\}$. For k = 0,1,….. and h $\geq d_n(A)$.

Proof:Let $y \in M_h \cap N_k$. Because $Y \in M_h$, there exists $x \in E$ with $y = A^h x$. Now, $A^{h+k}x = A^k y = 0 \Rightarrow x \in N_{h+k}$. hence, $x \in N_h$. Which implies that y = 0.

Lemma5: If a has finite ascent, then $n(A) \geq d_n(A)$.

Proof: Since, $\dim(N_k / N_{k-1}) \geq 1$ for K = 1,2, ……$d_n(A)$. Hence,

$$n(A) = \dim [N_\infty(A)] = \sum_{k=1}^{\infty} \dim(N_k / N_{k-1}) \geq d_n(A).$$

Lemma6: If a has finite descent then, $M_h(A) + N_k(A) = E$ for h = 0,1,… and K $\geq d_m(A)$.

Proof: Let $x \in E$, Because $A^k x \in M_k = M_{h+k}$, there exists $x_0 \in E$ such that $A^k x = A^{h+k}x$.

Set $x_m = A^h x_0$ and $x_n = x - A^h x_0$. The $x_m \in M_h$ and $x_n \in N_k$. Thus $x = x_m + x_n$ is a decomposition.

Lemma7: If A has finite descent then $m(A) \geq dm(A)$.

Proof: We observe that, $\dim(M_{h-1} / M_h) \geq 1$ for h = 1, ……. $d_m(A)$. Hence, $m(A) = \text{codim}[N_\infty(A)] = \sum_{k=1}^{\infty} \dim(N_h / N_{h-1}) \geq d_m(A)$. Hence the proof.

Proposition3: If $T \in L(E)$ is iteratively compact, then all ranges $M_h(I - T)$ are closed.

Proof: Let $y = \lim_i y_i$ where(y_i) is contained in $M_h(I - T)$. Set $\rho_i = \inf \{\|x\| : (I - T)^h x = y_i\}$, and let $x_i \in E$ such that $(I - T)^h x_i = y_i$ and $\|x_i\| \leq 2\,\rho_i$. Let $\rho_i \rightarrow \infty$, then $u_i = \rho_i^{-1} x_i$ and $v_i = \rho_i^{-1} y_i$, then $\|u_i\| \leq 2$ and (v_i) tends to zero. A subsequence of (u_i) which converges to some $u \in E$. Therefore, $(I - T)^h u_i = v_i$ implies $(I - T)^h u = 0$. Hence $(I - T)^h (x_i - \rho_i u) = y_i$. Thus, we have

$\| x_i - \rho_i u\| \geq \rho_i$ or $\|ui - u\| \geq 1$. Which shows that (ρ_i)has a bounded subsequence. Hence proved.

Proposition4: An operator $T \in L(E)$ is Riesz if and only if λT is iteratively compact for all $\lambda \in C$.

Proof: Let T is Riesz. Given $\lambda \in C$ and $\varepsilon > 0$, therefore $\beta = \min(\varepsilon,1) / 1 + |\lambda|$. Then, there exist n and $u_1, \ldots, u_k \in E$ such that $T^n(U) \subseteq \bigcup_{h=1}^{k} \{u_h + \beta^n U\}$. It follows from $|\lambda|\, \beta \leq \varepsilon$.

Hence, $(\lambda T)^n(U) \subseteq \bigcup_{h=1}^{k} \{\lambda^n u_h + \varepsilon U\}$. It follows from $\lambda\,\varepsilon \geq 1$ and $\lambda \geq 1$ that $\lambda^n \varepsilon^{n-1} \geq 1$. Hence

$\lambda^{n-1} \varepsilon \leq \varepsilon^n$, which implies that $T^n(U) \subseteq \bigcup_{h=1}^{k} \{\lambda^{n-1} u_h + \varepsilon U\}$. Thus T is Riesz.

Lemma8: Let $T \in L(E)$ is Riesz operator. If $\lambda_n(T) \neq 0$ then there exists an n-dimensional T-invariant subspace En such that the operator $T_n \in L(E_n)$ induced by T has precisely $\lambda_1(T) \ldots\ldots \lambda_n(T)$ as its eigen values.

Proof: Let $\{\lambda_1, \ldots\ldots \lambda_m\}$ the set of distinct complex numbers appearing in $\{\lambda_1(T) \ldots\ldots \lambda_n(T)\}$. In particular, Let $\lambda_m = \lambda_n(T)$. since $\lambda_m(I - T)$ is nilpotent on $N_\infty(\lambda_m(I - T))$, Let $K = n - \sum_{i=1}^{m-1} n(\lambda_i (I - T))$. Then $1 \le K \le n(\lambda_m (I - T))$, and $E_n = \sum_{i=1}^{m-1} N_\infty(\lambda_i (I - T)) + \text{span}(x_1,\ldots x_k)$ is T-invariant subspace.

Proposition5: An operator $T \in L(E)$ is iteratively compact if and only if T is iteratively compact.

Proof: If T is supposed to be iteratively compact then for every $\varepsilon > 0$ there exist n and $(u_1, \ldots, \ldots u_h) \in E$ such that $T^n(U) \subseteq \bigcup_{i=1}^{h} \{u_i + \varepsilon U\}$. Let $u_i = Tx_i$ with $x_i \in U$. Since, $X \in L(l_1(h), E)$ then

$X = \sum_{i=1}^{h} e_i \otimes x_i$. Since, $X'(T')^n$ has finite rank, Let $a_1,\ldots,a_k \in U^0$ such that,

$X'(T')^n U^0 \subseteq \bigcup_{i=1}^{h} \{ X'(T')^n a_j + \varepsilon U_\infty(h)\}$. Where U^0 and $U_\infty(h)$ is the closed unit balls of E' and $l_\infty(h)$, respectively.

Since $X' a = (\langle x_i, a\rangle)$ for all $a \in E'$. Given $a \in U^0$, there exists a_j with

$$|\langle x_i,(T')^n a - (T')^n a_j\rangle| \le \varepsilon \text{ for } I = 1,\ldots,h. \quad \ldots\ldots\ldots(1)$$

Next, some $x \in U$ such that,

$$\|(T')^n a - (T')^n a_j\| \le 2|\langle x_i,(T')^n a - (T')^n a_j\rangle| \quad \ldots\ldots\ldots\ldots(2)$$

Finally, Let x_i satisfies $\|T^n x - T^n x_i\| \le \varepsilon$. $\quad \ldots\ldots\ldots\ldots..(3)$

Combining (1),(2) and (3) then,

$$\begin{aligned}(T')^n a - (T')^n a_j\| &\le 2|\langle x_i,(T')^n a - (T')^n a_j\rangle| = 2\,|\langle T^n x , a - a_j \rangle| \\ &\le 2\,|\langle T^n x , a - a_j \rangle| + 2\,|\langle T^n x - T^n x_i, a - a_j \rangle| \\ &\le 2\,|\langle x_i,(T')^n a - (T')^n a_j\rangle| + 4\,\| T^n x - T^n x_i\| \le 6\varepsilon.\end{aligned}$$

This proves that

$(T')^n U^0 \subseteq \bigcup_{j=1}^{k} \{(T')^n a_j + 6\varepsilon U^0\}$. Thus T' is iteratively compact.

Conversely, if T' is iteratively compact, then so is T'' and it follows from $K_S T^n = (T'')^n K_E$ that T is iteratively compact.

III. Theorems

Theorem1: Decomposition Theorem- If the linear map A has finite ascent and descent, then the linear space E is the direct sum of the A-invariant linear subsets $S_\infty(A)$ and $T_\infty(A)$, Moreover the following holds:

1. The restriction of A to $S_\infty(A)$ is invertible.
2. The restriction of A to $T_\infty(A)$ is nilpotent of order d(A)

Proof: Let $d = d(A)$ then $S_\infty(A) = S_d$ and $T_\infty(A) = T_d$. Then E is indeed the direct sum of $S_\infty(A)$ and $T_\infty(A)$. Obviously both linear subsets are invariant under A. Since $y \in S_\infty(A)$, there exist $x_0 \in E$, such that $y = A^{d+1}x_0$, Hence $y = Ax$, where $x : A^d x_0 \in S_d$. Which proves that A maps $S_\infty(A)$. furthermore, $S_d \cap T_d = \{0\}$. Thus the restriction of A to $S_\infty(A)$ is one-to-one. Lastly, it shows that $A^d x = 0$ for all $x \in T_\infty(A)$.

Theorem2: Decomposition Theorem- If $T \in L(E)$ is iteratively compact, then Banach Space E is direct sum of the T-invariant subspaces $S_\infty(I – T)$ and $T_\infty(I – T)$, the later being finite dimensional. Moreover, the following holds:

1. The restriction of $I – T$ to $S_\infty(I – T)$ is invertible.
2. The restriction of $I – T$ to $T_\infty(I – T)$ is nilpotent of order $d(I – T)$.

Proof: Let $S_\infty(I – T)$ is continuously invertible. Since $S_\infty(I – T)$ is closed. Then, the normalized element $x_i \in S_\infty(I – T)$ such that $((I – T) x_i)$ tends to zero. Hence, there exists a subsequence converging to an element $x_i \in S_\infty(I – T)$. $\| x_i \| = 1$.

Furthermore, $((I – T) x_i) \to 0$ yields $(I – T)x = 0$.

Hence, $x_i \in S_\infty(I – T) \cap T_\infty(I – T)$. Thus it shows that $x = 0$.

Theorem3: Let $T \in L(E)$ is Riesz operator. Then, for every $\rho > 0$, the set of all Eigen value λ with $|\lambda| \ge \rho$ is finite.

Proof: Let T possesses a sequence of distinct Eigen values $\lambda_1, \lambda_2, \ldots$ .Such that $|\lambda_k| \ge \rho$. Let any sequence of associated Eigen elements $u_1,u_2,\ldots \in E$. It follows that $E_k : \text{span}(u_1,\ldots,u_k)$ is K-dimensional, $E_0 = \{0\}$. By Riesz lemma with $\varepsilon = 1/3$. Let an elements $x_k \in E_k$ such that

$\| x_k \| = 1$ and $\| x_k - x\| \ge 3/4$ for all $x \in E_{k-1}$. There exists co-efficient α_k for which $x_k – \alpha_k u_k \in E_{k-1}$.

Hence, $T^n x_k – \alpha_k \lambda_k^n u_k \in E_{k-1}$ and $\lambda_k^n x_k - \alpha_k \lambda_k^n u_k \in E_{k-1}$.

This implies that $T^n x_k - \lambda_k^n x_k \in E_{k-1}$ for n = 1,2,….. consequently, $T^n x_h - T^n x_k \in \lambda_h^n x_k - \lambda_k^n x_k + E_{h-1} + E_{k-1}$ and $\| T^n x_h - T^n x_k \| \geq 3/4\ \rho^n$ whenever h > k and n = 1,2,……. On the other hand, by the Pigeon-hole principle to the operator $\rho^{-1}T$. Let an exponent n and different indices h and k such that $\| T^n x_h - T^n x_k \| \leq 1/2\ \rho^n$. This contradiction completes the proof.

IV. Concluding Remarks

An operator $T \in L(E)$ is said to be Riesz if every $\in > 0$ their exist an exponent n and elements $u_1, \ldots\ldots, u_k \in E$. If $T \in L(E)$ is iteratively compact, then all null spaces $N_k(I - T)$ are finite dimensional and also I –T has finite ascent and many Lemmas and Prepositions has been proved. We also developed Riesz decomposition and decomposition theorems. Also proved, Let $T \in L(E)$ is Riesz operator. Then, for every $\rho > 0$, the set of all Eigen value λ with $|\lambda| \geq \rho$ is finite. Thus we have introduced Riesz operators and its various properties in the Eigen values of linear operators.

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