Some Inequality on Some Subsequence of the Sequence of Prime Numbers

Arto Adili[#1], Artur Baxhaku[*2]

[#]Department of Mathematics, "Fan S. Noli" University, Shëtitorja "Rilindasit", Korçë, Albania

[*]Department of Mathematics, University of Tirana, Bulevardi "Zogu i Parë", Tiranë, Albania

***Abstract*— In this paper we will present some inequality on some subsequence of the sequence of prime numbers such that $\equiv 1(\bmod 2^{k+1})$, where k is positive integer. We prove that for every positive integer k there exists a positive constant real number κ_k, such that for every real number $\lambda > \kappa_k$, there exists a natural number n_λ, such that for every natural number $n > n_\lambda$, it is true the inequality $\prod_{i=1}^{n} \frac{p_{ki}}{p_{ki}-2^k} < \frac{\lambda p_{kn}}{2^k \cdot n}$, where p_{kn} is the n-th prime number $\equiv 1(\bmod 2^{k+1})$. The constant κ_k is equal with $e^{2^k \cdot M(2^{k+1},1)+2^{2k}\cdot\sigma_k}$, where $M(2^{k+1},1)$ is the Merten's constant and σ_k is the sum of the convergent series $\sum_{p\equiv 1(\bmod 2^{k+1})} \frac{1}{p(p-2^k)}$. We find approximate values of constant κ_1 and κ_2.**

***Keywords*— prime number, inequality, series of prime number, prime number theorem.**

I. Introduction

Let's note $p_{k1}, p_{k2}, p_{k3}, \ldots$ the subsequence of the sequence of prime numbers, such that $\equiv 1(\bmod 2^{k+1})$. In this way for $k=1$ we have the subsequence of the sequence of prime numbers $p_{11}, p_{12}, p_{13}, \ldots$, such that $\equiv 1(\bmod 4)$, namely $5, 13, 17, 29, 37, \ldots$. Also, for $k=2$ we have the subsequence of the sequence of prime numbers $p_{21}, p_{22}, p_{23}, \ldots$, such that $\equiv 1(\bmod 8)$, namely $17, 41, 73, 89, 97, \ldots$.

We will show that for every positive integer k there exists a positive constant real number κ_k, such that for every real number $\lambda > \kappa_k$, there exists a natural number n_λ, such that for every natural number $n > n_\lambda$, it is true the inequality

$$\frac{p_{k1}}{p_{k1}-2^k} \cdot \ldots \cdot \frac{p_{kn}}{p_{kn}-2^k} < \frac{\lambda p_{kn}}{2^k \cdot n}.$$

We will show that $\sum_{p\equiv 1(\bmod 2^{k+1})} \frac{1}{p(p-2^k)}$ is convergent and if we note σ_k its sum, then

$$\kappa_k = e^{2^k \cdot M(2^{k+1},1)+2^{2k}\cdot\sigma_k},$$

where $M(2^{k+1},1)$ is Merten's constant.

We find that $\sigma_1 \approx 0.0822$, $\sigma_2 \approx 0.0059$ and since $M(4,1) \approx -0.2867$, $M(8,1) \approx -0.2864$, then the approximate values of the constant real numbers κ_1 and κ_2 are:

$$\kappa_1 = e^{2M(4,1)+4\sigma_1} \approx 0.783$$

and

$$\kappa_2 = e^{4M(8,1)+16\sigma_2} \approx 0.3495.$$

II. Prove the Inequality

We note that in general, the method which we use to prove the inequality is the same method we used in prove the inequality with prime numbers in [1].

Let's k positive integer. For every positive real number x we have known inequality $1+x<e^x$. Since for every prime number p_{ki} we have

$$\frac{p_{ki}}{p_{ki}-2^k}=1+\frac{2^k}{p_{ki}-2^k},$$

and $\frac{2^k}{p_{ki}-2^k}>0$, then for every index n it is true the inequality

$$\prod_{i=1}^{n}\frac{p_{ki}}{p_{ki}-2^k}=\prod_{i=1}^{n}\left(1+\frac{2^k}{p_{ki}-2^k}\right)<e^{2^k\sum_{i=1}^{n}\frac{1}{p_{ki}-2^k}}. \tag{1}$$

From the identity

$$\frac{1}{p_{ki}-2^k}=\frac{1}{p_{ki}}+\frac{2^k}{p_{ki}(p_{ki}-2^k)},$$

we have

$$\sum_{i=1}^{n}\frac{1}{p_{ki}-2^k}=\sum_{i=1}^{n}\frac{1}{p_{ki}}+2^k\sum_{i=1}^{n}\frac{1}{p_{ki}(p_{ki}-2^k)},$$

consequently, based on (1) for every natural number n it is true the inequality:

$$\prod_{i=1}^{n}\frac{p_{ki}}{p_{ki}-2^k}<e^{2^k\sum_{i=1}^{n}\frac{1}{p_{ki}}}\cdot e^{2^{2k}\sum_{i=1}^{n}\frac{1}{p_{ki}(p_{ki}-2^k)}}. \tag{2}$$

Let's prove now that the series $\sum\frac{1}{p_{ki}(p_{ki}-2^k)}$ is convergent. Indeed, since the terms of the series are positive and since for every natural number n we have:

$$\sum_{i=1}^{n}\frac{1}{p_{ki}(p_{ki}-2^k)}=\frac{1}{p_{k1}(p_{k1}-2^k)}+\frac{1}{p_{k2}(p_{k2}-2^k)}+\dots+\frac{1}{p_{kn}(p_{kn}-2^k)}=$$

$$=\frac{1}{2^k}\left(\frac{1}{p_{k1}-2^k}-\frac{1}{p_{k1}}\right)+\frac{1}{2^k}\left(\frac{1}{p_{k2}-2^k}-\frac{1}{p_{k2}}\right)+\dots+\frac{1}{2^k}\left(\frac{1}{p_{kn}-2^k}-\frac{1}{p_{kn}}\right)=$$

$$=\frac{1}{2^k(p_{k1}-2^k)}-\frac{1}{2^k}\left(\frac{1}{p_{k1}}-\frac{1}{p_{k2}-2^k}\right)-\dots-\frac{1}{2^k}\left(\frac{1}{p_{k(n-1)}}-\frac{1}{p_{kn}-2^k}\right)-\frac{1}{2^k\cdot p_{kn}}<\frac{1}{2^k(p_{k1}-2^k)},$$

then its partial sums are upper bounded, consequently this series is convergent. Let's note

$$\sum_{p\equiv1(\mathrm{mod}\,2^{k+1})}\frac{1}{p(p-2^k)}=\sigma_k.$$

From Dirichlet theorem [2] we have

$$\sum_{i=1}^{n}\frac{1}{p_{ki}}=\frac{\ln(\ln p_{kn})}{\varphi(2^{k+1})}+M(2^{k+1},1)+O\left(\frac{1}{\ln p_{kn}}\right),$$

where $M(2^{k+1},1)$ is Merten's constant [3]. Since $\varphi(2^{k+1})=2^k$, then based on (2) for every natural number n it is true the inequality

$$\prod_{i=1}^{n}\frac{p_{ki}}{p_{ki}-2^k}<e^{\ln(\ln p_{kn})+2^k\cdot M(2^{k+1})+2^k\cdot O\left(\frac{1}{\ln p_{kn}}\right)}\cdot e^{2^{2k}\cdot\sigma_k}=e^{2^k\cdot M(2^{k+1},1)+2^{2k}\cdot\sigma_k}\cdot e^{2^k\cdot O\left(\frac{1}{\ln p_{kn}}\right)}\cdot\ln p_{kn},$$

otherwise, for every natural number n it is true the inequality

$$\prod_{i=1}^{n}\frac{p_{ki}}{p_{ki}-2^k}<\kappa_k\cdot e^{2^k\cdot O\left(\frac{1}{\ln p_{kn}}\right)}\cdot\ln p_{kn}, \tag{3}$$

where $\kappa_k=e^{2^k\cdot M(2^{k+1},1)+2^{2k}\cdot\sigma_k}$.

Let's now consider a real number λ, such that $\lambda > \kappa_k$.

Since $\lim\limits_{n\to\infty} 2^k O\left(\frac{1}{\ln p_{kn}}\right) = 0$, then

$$\lim_{n\to\infty} e^{2^k O\left(\frac{1}{\ln p_{kn}}\right)} = 1. \tag{4}$$

From theorem of distribution of prime number in arithmetic progressions [4] we have

$$\lim_{n\to\infty} \frac{2^k \cdot n \cdot \ln p_{kn}}{p_{kn}} = 1. \tag{5}$$

Consequently, from (4) and (5) we have

$$\lim_{n\to\infty} \frac{e^{2^k \cdot O\left(\frac{1}{\ln p_{kn}}\right)} \cdot 2^k \cdot n \cdot \ln p_{kn}}{p_{kn}} = 1.$$

Moreover, since $\kappa_k < \lambda$ we can write

$$\lim_{n\to\infty} \frac{\kappa_k \cdot e^{2^k \cdot O\left(\frac{1}{\ln p_{kn}}\right)} \cdot 2^k \cdot n \cdot \ln p_{kn}}{\lambda \cdot p_{kn}} = \frac{\kappa_k}{\lambda} = \theta < 1.$$

Then, for $\varepsilon = 1-\theta$ there exists a natural number n_λ, such that for every natural number $n > n_\lambda$, it is true the inequality

$$\frac{\kappa_k \cdot e^{2^k \cdot O\left(\frac{1}{\ln p_{kn}}\right)} \cdot 2^k \cdot n \cdot \ln p_{kn}}{\lambda \cdot p_{kn}} - \theta < \varepsilon.$$

Thus, for every natural number $n > n_\lambda$ it is true the inequality

$$\frac{\kappa_k \cdot e^{2^k \cdot O\left(\frac{1}{\ln p_{kn}}\right)} \cdot 2^k \cdot n \cdot \ln p_{kn}}{\lambda \cdot p_{kn}} < \theta + \varepsilon = 1,$$

or otherwise

$$\kappa_k \cdot e^{2^k \cdot O\left(\frac{1}{\ln p_{kn}}\right)} \cdot \ln p_{kn} < \lambda \frac{p_{kn}}{2^k \cdot n}. \tag{6}$$

Finally, by (3) and (6) we can say that: there exists a natural number n_λ, such that for every natural number $n > n_\lambda$, is true the inequality

$$\prod_{i=1}^{n} \frac{p_{ki}}{p_{ki} - 2^k} < \kappa_k \cdot e^{2^k \cdot O\left(\frac{1}{\ln p_{kn}}\right)} \cdot \ln p_n < \lambda \frac{p_{kn}}{2^k \cdot n},$$

namely

$$\frac{p_{k1}}{p_{k1} - 2^k} \cdot \frac{p_{k2}}{p_{k2} - 2^k} \cdot \ldots \cdot \frac{p_{kn}}{p_{kn} - 2^k} < \frac{\lambda p_{kn}}{2^k \cdot n}. \tag{7}$$

Let's find now the approximated values of constants κ_k for $k=1$ and $k=2$.

For $k=1$ we have:

$$\sigma_1 = \sum_{p\equiv 1 (\mathrm{mod}\, 4)} \frac{1}{p(p-2)} \approx 0.0822,$$

and from [5]

$$M(4,1) \approx -0.2867,$$

consequently we have

$$\kappa_1 = e^{2M(4,1)+4\sigma_1} = e^{-0.2446} \approx 0.783.$$

For $k = 2$ we have:

$$\sigma_2 = \sum_{p \equiv 1 (\mathrm{mod}\, 8)} \frac{1}{p(p-4)} \approx 0.0059,$$

and from [5]

$$M(8,1) \approx -0.2864,$$

consequently we have

$$\kappa_2 = e^{4M(8,1)+16\sigma_2} = e^{-1.0512} \approx 0.3495.$$

Note.

For $k = 0$ we get the constant $\kappa = \kappa_0 = 2.812$ for which have spoken in [1].

REFERENCES

[1] Adili A, Milo E, "An inequality with the sequence of prime numbers", *International Journal of Mathematics Trends and Technology*, Volume 15 Number 1, November 2014, pp 45-48.

[2] Tom M. Apostol, "An asymptotic formula for the partial sums $\Sigma_{p\leq x}(1/p)$", *Introduction to Analytic Number Theory*, pp 156, Springer-Verlag, New York Heidelberg Berlin 1976.

[3] A. Languasco, A. Zaccagnini, A note of Merten's formula for arithmetic progressions, *Jurnal of Number Theory*, Volume 127, Issue 1, November 2007, pp 37-46.

[4] Tom M. Apostol, "Distribution of prime numbers in arithmetic progressions", *Introduction to Analytic Number Theory*, pp 154, Springer-Verlag, New York Heidelberg Berlin 1976.

[5] A. Languasco, A. Zaccagnini, Computation of the Mertens constant for the sum, March 2009, pp 3, 7.