Characteristics of P-Semi pseudo Symmetric Ideals in Ternary Semiring

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Abstract: In this paper we introduce and study about P-semipseudo symmetric ideals in ternary semi rings and characterized p-semipseudo symmetric ideals in ternary semirings.

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1.INTRODUCTION:

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and the like. The theory of ternary algebraic systems was introduced by D. H. Lehmer [5]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. After that W. G. Lister[6] studied about ternary semirings. About T. K. Dutta and S. Kar [1, 3] introduced and studied some properties of ternary semirings which is a generalization of ternary rings. Dheena. P, Manvisan. S [2] made a study on P-prime and small P-prime ideals in semirings. S. Kar [4] investigated on quasi ideals and bi-ideals in ternary semirings.

D. Madhusudhana Rao, A. Anjaneyulu and A. Gangadhara Rao [7] in 2011 introduced the notion of pseudo symmetric ideals in Γ-Semigroups and they [8] are also introduced the notion of semipseudo symmetric Γ-ideals in Γ-semigroups. In 2012 Y. Sarala, A. Anjaneyulu and D. Madhusudhana Rao [15] introduced the same concept to the ternary semigroups. In 2014 D. MadhusudhanaRao and G. Srinivasa Rao [9, 10] investigated and studied about classification of ternary semirings and some special elements in a ternary semirings. D. Madhsusudhana Rao and G. Srinivasa Rao [11] introduced and investigated structure of certain ideals in ternary semiring. D. Madhusudhana Rao and G. Srinivasa Rao[12] also introduced the structure of completely P-prime, P-prime, Completely P-Semiprime and P-semiprime ideals in Ternary semiring. After that they [13] made a study and investigated prime radicals in ternary semiring. They also introduced [14] the notion of P-Pseudo Symmetric Ideals in Ternary Semiring. Our main purpose in this paper is to introduce the Structure of P-semipseudo symmetric Ideals in ternary Semiring.

2.PRELIMINARIES :

Definition 2.1[6] **:** A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by [ ] is said to be a ***ternary semiring*** if T is an additive commutative semigroup satisfying the following conditions :

i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,
ii) $[(a + b)cd] = [acd] + [bcd]$,
iii) $[a(b + c)d] = [abd] + [acd]$,
iv) $[ab(c + d)] = [abc] + [abd]$ for all $a; b; c; d; e \in$ T.

Throughout Twill denote a ternary semiring unless otherwise stated.

Note 2.2 : For the convenience we write $x_1x_2x_3$ instead of $\left[x_1x_2x_3\right]$

Note 2.3 : Let T be a ternary semiring. If A,B and C are three subsets of T , we shall denote the set ABC = $\left\{\Sigma abc : a \in A, b \in B, c \in C\right\}$.

Note 2.4 : Let T be a ternary semiring. If A,B are two subsets of T , we shall denote the set A + B = $\left\{a+b : a \in A, b \in B\right\}$.

Note2.5 : Any semiring can be reduced to a ternary semiring.

Example 2.6 [6] : Let T be an semigroup of all $m \times n$ matrices over the set of all non negative rational numbers. Then T is a ternary semiring with matrix multiplication as the ternary operation.

Example 2.7 [6]: Let $S = \{\ldots, -2i, -i, 0, i, 2i, \ldots\}$ be a ternary semiring withrespect to addition and complex triple multiplication.

Definition 2.8 [6]: A ternary semiring T is said to be ***commutative ternary semiring*** provided$abc = bca = cab = bac = cba = acb$for all $a,b,c \in$ T.

Definition 2.9 [8]: A nonempty subset A of a ternary semiring T is said to be ***ternary ideal*** or simply an ***ideal*** of T if

(1) $a, b \in$ A implies $a + b \in$ A

(2) $b, c \in$ T , $a \in$ A implies $bca \in$ A, $bac \in$ A, $abc \in$ A.

Definition 2.10 [9]:An ideal A of a ternary semiring T is said to be a ***completely prime ideal*** of T provided $x, y, z \in$ T and $xyz \in$ A implies either $x \in$ A or $y \in$ A or $z \in$ A.

Definition 2.11 [11]:An ideal A of a ternary semiring T is said to be a ***completely P-prime ideal*** of T provided $x, y, z \in$ T and xyz+ P$\subseteq$A implies either $x \in$ A or $y \in$ A or $z \in$ A for any ideal P.

Definition 2.12 [11]: An ideal A of a ternary semiring T is said to be a P-***prime ideal*** of T provided X,Y,Z are ideals of T and XYZ + P$\subseteq$ A $\Rightarrow$ X $\subseteq$A or Y $\subseteq$A or Z $\subseteq$A for any ideal P.

Theorem 2.13[11]: Every completely P-prime ideal of a ternary semiring T is a P-prime ideal of T.

Theorem 2.14[11] : Every completely P-semiprime ideal of a ternary semiring T is a P-semiprime ideal of T.

Theorem 2.15[11] : If T is a globally idempotent ternary semiring then every maximal ideal of T is a P-prime ideal of T.

Definition 2.16 [11]: An ideal A of a ternary semiring T is said to be a ***completely P-semiprime ideal*** provided $x \in$T, $x^n + p \in$A for some odd natural number n>1 and $p \in$ P implies $x \in$A.

Definition 2.17 [11]: An ideal A of a ternary semiring T is said to be ***semiprime ideal*** provided X is an ideal of T and $X^n \subseteq$ A for some odd natural number n implies X ⊆ A.

Definition 2.18 [11]: An ideal A of a ternary semiring T is said to be P-***semiprime ideal*** provided X is an ideal of T and X^n+ P $\subseteq$ A for some odd natural number n implies X ⊆ A.

Theorem 2.19[11] **: Let A be a P-prime ideal of a ternary semiring T. If A is completely P-Semiprime ideal of T then A is completely P-prime.**

Theorem 2.20 [11] **: Every completely P-Semiprime ideal of a ternary semiring T is a P-Semiprime ideal of T.**

Theorem 2.21 [11] **: Every P-prime ideal of a ternary semiring T is P-Semiprime.**

Corollary 2.22[11] : If an ideal A of a ternary semiring T is completely P-semiprime then x, y, $z \in$ T,$p \in$ P andxyz+ $p \in$ A $\Rightarrow$< x > < y > < z > + P ⊆ A.

Theorem 2.23[11] : Every completely P-prime ideal of a ternary semiring T is a completely P-Semiprime ideal of T.

Notation 2.24 [11]: If A is an ideal of a ternary semiring T, then we associate the following four types of sets.

A_1 = The intersection of all completely prime ideals of T containing A.

A_2 = $\{x \in T: x^n \in A$ for some odd natural numbers $n\}$

A_3 = The intersection of all prime ideals of T containing A.

A_4 = $\{x \in T: < x >^n \subseteq$ A for some odd natural number $n\}$

Theorem 2.25 [11]**: If A is an ideal of a ternary semiring T, then** A$\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$**.**

Corollary 2.26: If $a \in \sqrt{A}$, then there exist a positive integer n such that $a^n \in$A for some odd natural number $n \in$ N.

Theorem 2.27 : An ideal Q of ternary semiring T is a semiprime ideal of T if and only if $\sqrt{Q}$ =Q.

Definition 2.28 : An ideal A of a ternary semiring T is said to be ***pseudo symmetric*** provided $x, y, z \in$T, $xyz \in$A implies $xsytz \in$A for all $s, t \in$T.

Definition 2.29 : A pseudo symmetric ideal A of a ternary semiring T is said to be P-***pseudo symmetric ideal*** provided $x, y, z \in$T and P is an ideal of T, $xyz + p \in$A implies $xsytz + p \in$A for all $s, t \in$T and $p \in$ P.

Note 2.30 : A pseudo symmetric ideal A of a ternary semiring T is said to be P-***pseudo symmetric ideal*** provided $x, y, z \in$T and P is an ideal of T, $xyz + P \subseteq$ A implies $xsytz + P \subseteq$ A for all $s, t \in$T.

Corollary 2.31 : Let A be a P-pseudo symmetric ideal in a ternary semiring T, then for any odd natural number n, $a^n + p \in$A implies $< a >^n$ + P$\subseteq$A.

Theorem 2.32 : Let A be an ideal of a ternary semiring T. Then A is completely P-prime iff A is P-prime and P-pseudo symmetric.

Corollary 2.33 : Let A be an ideal of a ternary semiring T. Then A is completely P-semiprime iff A is P-semiprime and P-pseudo symmetric.

3. P-SEMIPSEUDO SYMMETRIC TERNARY IDEALS

We now introduce the notion of P-semipseudo symmetric ideal of a ternary semiring

Definition 3.1 : An ideal A in a ternary semiring T is said to be P-***semipseudo symmetric*** provided for any odd natural number n, $x \in$ T and P is an ideal of T, x^n + P $\subseteq$ A $\Rightarrow < x >^n$ + P $\subseteq$ A.

Theorem 3.2 : Every P-pseudo symmetric ideal of a ternary semiring is a P-semipseudo symmetric ideal.

Proof **:** Let A be a P-pseudo symmetric ideal of a ternary semiring T.
Let $x \in$ T and x^n + P $\subseteq$ A for some odd natural number n. Since A is P-pseudo symmetric, by corollary 2.31, x^n + P $\subseteq$ A$\Rightarrow < x >^n$ + P $\subseteq$ A. Therefore A is a P-semipseudo symmetric ideal.

Note 3.3 : The converse of theorem 3.2, is not true. i.e. a P-semipseudo symmetric ideal of a ternary semiring need not be a P-pseudo symmetric ideal.

Example 3.4 : Let T be a free ternary semiring over the alphabet $\{a, b, c, d, e\}$ and P is any ideals of T. Let A = $<abc>+<bca>+<cab>$ + P. Since abc + P $\subseteq$ A and $adbec$ + P $\not\subseteq$ A, A is not P-pseudo symmetric. Suppose $x^n + P \in$ A for some odd natural number n. Now the word x contains abc or bca or cab and hence $< x >^n$ + P $\subseteq$ A. Therefore x^n + P $\subseteq$ A for some odd natural number $n \Rightarrow < x >^n$ + P $\subseteq$ A. Therefore A is a P-semipseudo symmetric ideal.

Theorem 3.5 : Every P-semiprime ideal Q minimal relative to containing a P-semipseudo symmetric ideal A in a ternary semiring T is completely P-semiprime.

Proof **:** Write S={x^n : $x \in$ T\Q for any odd natural number n}. First we show that $A \cap S = \emptyset$.

If $A \cap S \neq \emptyset$, then there exists an element $x \in T \setminus Q$ such that x^n + P $\subseteq A$ where n is odd natural number. Since A is a P-semipseudo symmetric ideal, $< x >^n$ + P $\subseteq A \subseteq Q$ $\Rightarrow$ $< x >^n$ + P $\subseteq Q \Rightarrow$ $x \in$ Q. It is a contradiction. Thus $A \cap S = \emptyset$. Consider the set $\Sigma = \{B : B$ is an ideal in T containing A such that $B \cap S = \emptyset\}$. Since $A \in \Sigma$, Σ is nonempty. Now Σ is a poset under set inclusion and satisfies the hypothesis of Zorn's lemma. Thus by Zorn's lemma, Σ contains a maximal element, say M. Suppose $< a >^3$ + P $\subseteq$ M and $a \notin$ M. Then M $\cup < a >$ is an ideal containing A. Since M is maximal in Σ, we have $(M \cup < a >) \cap S \neq \emptyset$.
Then there exists $x \in$ T\Q such that $x^n \in < a > \cap S$ for some odd natural number n.
Therefore $x^{3n} \in$ < a $>^3 \cap S \subseteq M \cap S \Rightarrow x^{3n} \in M \cap S$. It is a contradiction.
Therefore M is a P-semiprime ideal containing A.
Now, $A \subseteq M \subseteq T \setminus S \subseteq Q$. Since Q is a minimal P-semiprime ideal relative to containing A,
we have $M = T \setminus S = Q$. Let $x \in$ S, x^m + P $\subseteq$ Q. Suppose if possible $x \notin$ Q.
Now $x \notin$ Q $\Rightarrow x \in$ S$\Rightarrow x^m \in$ S. It is a contradiction. Therefore $x \in$ Q.
Hence Q is a completely P-semiprime ideal.

Corollary 3.6 : Every P-prime ideal Q in a ternary semiring T minimal relative to containing a P-semipseudo symmetric ideal A is completely P-prime.

Proof **:** Since every P-prime ideal is a P-semiprime ideal, by theorem 3.5, we have Q is a completely P-semiprime ideal and by theorem 2.19, P is a completely P-prime ideal.

Corollary 3.7 : Every P-prime ideal minimal relative to containing a P-pseudo symmetric ideal A in a ternary semiring T is completely P-prime.

Proof **:** Let P be a P-prime ideal containing a P-pseudo symmetric ideal A of a ternary semiring T. By theorem 3.2, every P-pseudo symmetric ideal is a P-semipseudo symmetric ideal, by corollary 3.6, P is a completely P-prime ideal of T.

Theorem 3.8 : If A is an ideal in a ternary semiring T, then the following are equivalent.
1) A is completely P-semiprime.
2) A is P-semiprime and P-pseudo symmetric.
3) A is P-semiprime and P-semipseudo symmetric.

Proof **:** (1) $\Rightarrow$(2) : Suppose A is a completely P-semiprime ideal of T. By theorem 2.20, A is a P-semiprime ideal of T and by theorem 2.33, A is a P-pseudo symmetric ideal of T.
(2) $\Rightarrow$(3) : Suppose A is P-semiprime and P-pseudo symmetric. By theorem 3.2, A is a P-semipseudo symmetric ideal. Hence A is P-semiprime and P-semipseudo symmetric.
(3) $\Rightarrow$ (1) : Suppose A is P-semiprime and P-semipseudo symmetric.
Let $x \in$ T, x^3 + P $\subseteq$ A. Since A is P-semipseudo symmetric, $x \in$ T, x^3 + P $\subseteq$ A
$\Rightarrow < x >^3$ + P $\subseteq$ A. Since A is P-semiprime, by definition 2.17, $< x >^3$ + P $\subseteq$ A $\Rightarrow < x > \subseteq$ A.
$\therefore$ A is completely P-semiprime.

Corollary 3.9: If A is an ideal in a ternary semiring T, then the following are equivalent.
1) A is completely semiprime.
2) A is semiprime and pseudo symmetric.
3) A is semiprime and semipseudo symmetric.

Theorem 3.10 : If A is an ideal of a semi simple ternary semiring T, then the following are equivalent.
1) A is completely P-semiprime.
2) A is P-pseudo symmetric.
3) A is P-semipseudo symmetric.
Proof **:** (1) $\Rightarrow$(2) : Suppose that A is completely P-semi prime. By corollary 2.20, A is P-pseudo symmetric.

(2) $\Rightarrow$ (3) : Suppose that A is P-pseudo symmetric. By theorem 3.2, A is P-semi pseudo symmetric.
(3) $\Rightarrow$ (1) : Suppose that A is P-semipseudo symmetric. Let $x \in$ T, $x^3 \in$ A. Since A is P-semipseudo symmetric, x^3 + P $\subseteq$ A $\Rightarrow < x >^3$ + P $\subseteq$ A. Since T is semi simple, x is a semi simple element. Therefore $x \in < x >^3 \subseteq$ A. Thus A is completely P-semiprime.

Theorem 3.11 : If A is an ideal of a ternary semiring T, then the following are equivalent.
1) **A is completely P-prime.**
2) **A is P-prime and P-pseudo symmetric.**
3) **A is P-prime and P-semipseudo symmetric.**

***Proof* : (1) $\Rightarrow$ (2) :** Suppose that A is completely P-prime. By theorem 2.32, A is P-prime and P-pseudo symmetric.
(2) $\Rightarrow$ (3) : Suppose A is prime and pseudo symmetric. Since A is P-pseudo symmetric by theorem 3.2, A is P-semipseudo symmetric.
(3) $\Rightarrow$ (1) : Suppose A is P-prime and P-semipseudo symmetric. Since A is P-prime by theorem 2.21, A is P-semiprime. Since A is P-semiprime and P-semipseudo symmetric, by theorem 3.8, A is completely P-semiprime. Since A is P-prime and completely P-semiprime by theorem 2.19, A is completely P-prime.

The following theorem is an analogue of KRULL's Theorem.

THEOREM 3.12 : Let A be a P-semipseudo symmetric ideal of a ternary semiring T. Then the following are equivalent.
1) **A_1=The intersection of all completely prime ideals of T containing A.**
2) **A_1^1 = The intersection of all minimal completely prime ideals of T containing A.**
3) **A_1^{11} = The minimal completely semi prime ideal of T relative to containing A.**
4) **A_2= {$x \in$ T : $x^n \in$A for some odd natural number n}**
5) **A_3= The intersection of all prime ideals of T containing A.**
6) **A_3^1 = The intersection of all minimal prime ideals of T containing A.**
7) **A_3^{11} = The minimal semiprime ideal of T relative to containing A.**
8) **A_4 = {$x \in$ T : $< x >^n \subseteq$A for some odd natural number n}**

Proof: Since completely P-prime ideals containing A and minimal completely P-prime ideals containing A and hence completely prime ideals containing A and minimal completely prime ideals containing A and minimal completely semiprime ideal relative to containing A are coincide, it follows that $A_1 = A_1^1 = A_1^{11}$. Since P-prime ideals containing A and minimal P-prime ideals containing A and hence prime ideals containing A and minimal prime ideals containing A and the minimal semiprime ideal relative to containing A are coincide, it follows that $A_3 = A_3^1 = A_3^{11}$. Since A is P-semipseudo symmetric ideal. Therefore A is semipseudo symmetric ideal, we have $A_2 = A_4$. Now by theorem 3.15, we have $A_1^{11} = A_3^{11}$. Therefore $A_1 = A_1^1 = A_1^{11} = A_3 = A_3^1 = A_3^{11}$ and $A_2 = A_4$. Hence the given conditions are equivalent.

Corollary 3.13 : Let A be a P-pseudo symmetric ideal of a ternary semiring T. Then the following are equivalent.
1) **A_1=The intersection of all completely prime ideals of T containing A.**
2) **A_1^1 = The intersection of all minimal completely prime ideals of T containing A.**
3) **A_1^{11} = The minimal completely semiprime ideal of T relative to containing A.**
4) **A_2= {$x \in$ T : $x^n \in$A for some odd natural number n}**

5) A_3= The intersection of all prime ideals of T containing A.
6) A_3^1= The intersection of all minimal prime ideals of T containing A.
7) A_3^{11}= The minimal semiprime ideal of T relative to containing A.
8) A_4 = {$x \in$ T : $<x>^n \subseteq$A for some odd natural number n}

***Proof* :** By theorem 3.2, every P-pseudo symmetric ideal is a P-semi pseudo symmetric ideal of T. Hence the proof follows from theorem 3.12.

Theorem 3.14 : If M is a maximal ideal of a ternary semiring T with $M_4 \neq$T, then the following are equivalent.
1) M is completely P-prime.
2) M is completely P-Semiprime.
3) M is P-pseudo symmetric.
4) M is P-semipseudo symmetric.

***Proof* :** (1) ⇒ (2) : Suppose that M is completely P-prime. By theorem 2.22, M is completely semiprime.
(2) ⇒ (3) : Suppose that M is completely P-Semiprime. By theorem 2.33, M is P-pseudo symmetric.
(3) ⇒ (4) : Suppose that M is P-pseudo symmetric. By theorem 3.2, M is P-semi pseudo symmetric.
(4) ⇒ (1) : Suppose M is P-semi pseudo symmetric. By the theorem 3.12, M $\subseteq M_4 \subseteq$ T. Since M is maximal ideal and $M_4 \neq$ T, it implies that M = M_4. Let $x \in$ T, $x^3 \in$ M. Since M is P-semi pseudo symmetric, $< x >^3 \subseteq$ M. Then $x \in M_4$ = M. ∴ M is completely P-semiprime.
Let $x, y, z \in$ T, xyz + P ⊆ M. Since M is completely P-semiprime, by corollary 2.21, xyz + P ⊆ M ⇒ $< x > < y > < z >$ + P ⊆ M ⇒ $< x > < y > < z > \subseteq$ M. Suppose if possible $x \notin$ M, $y \notin$ M, $z \notin$ M. Then M ∪$< x >$, M ∪$< y >$, M ∪$< z >$ are ideals of T and M ∪$< x >$ = M ∪$< y>$ = M ∪$< z >$ = T, Since M is maximal, $y, z \in$ M ∪$< x >$, $x, z \in$ M ∪$< y >$ and $x, y \in$ M ∪$< z >$ ⇒ $y, z \in< x >$, $x, z \in< y >$, $x, y\in< z >$ ⇒ $< x > = < y > = < z >$. Now $< x > < y > < z > \subseteq$ M ⇒ $< x > < y > < z > = < x >^3 \subseteq$ M ⇒$x^3 \in$ M ⇒$x\in$ M. It is a contradiction. ∴ either $x\in$ M or $y\in$ M or $z\in$ M. ∴ M is completely P-prime.

We now introduce the notion of a P-semi pseudo symmetric ternary semiring.

Definition 3.15 : A ternary semiring T is said to be a P-***semi pseudo symmetric ternary semiring*** provided every ideal of T is P-semi pseudo symmetric.

Theorem 3.16 : A ternary semiring T is P-semi pseudo symmetric iff every principal ideal is P-semi pseudo symmetric.
***Proof* :** Suppose a ternary semiring T is P-semi pseudo symmetric. Then every ideal of T is P-semi pseudo symmetric. Hence every principal ideal of T is P-semi pseudo symmetric.

Conversely suppose that every principal ideal of T is P-semi pseudo symmetric. Let A be any ideal of T. For $x \in$ T, x^n + P ⊆ A for an odd natural number n. Since $< x^n >$ is a P-semi pseudo symmetric ideal, $< x >^n$ + P $\subseteq<x^n>$. Now $<x >^n$ + P $\subseteq<x^n>\subseteq$ A for an odd natural number n. ∴$<x>^n$ + P ⊆ A for an odd natural number n. ∴ A is a P-semi pseudo symmetric ideal. Hence T is a P-semi pseudo symmetric semiring.

Theorem 3.17 : In a P-semipseudo symmetric ternary semiring T, an element *a* is semi simple iff *a* is lateral regular.

***Proof* :** Let T be a P-semipseudo symmetric ternary semiring. Suppose an element $a \in$ T is semi simple. Then $a \in< a >^3$. Since T is P-semipseudo symmetric, $< a^3 >$ is a P-semipseudo symmetric ideal. Thus a^3 + P $\subseteq <a^3>$⇒ $a^3 \in< a^3 >$ and P $\subseteq < a^3 >$ ⇒ $< a >^3 \subseteq< a^3 >$ ⇒$a \in<a >^3 \subseteq<a^3>$. Therefore $a = sa^3t$ for some $s, t \in$ T and hence a is lateral regular.

Conversely suppose that $a \in$ T is lateral regular. Then $a = xa^3y$ for some $x, y \in$ T and hence $a \in < a^3 >$. Therefore a is semi simple.

Definition 3.18: A ternary semiring T is said to be an ***Archimedean ternary semiring*** provided for any $a, b \in$ T there exists an odd natural number n such that $a^n \in$ TbT.

Definition 3.19: A ternary semiring T is said to be a ***strongly Archimedean ternary semiring*** provided for any $a, b \in$ T there exists an odd natural number n such that $< a >^n \subseteq < b >$.

Theorem 3.20: Every strongly Archimedean ternary semiring is an Archimedean ternary semiring.

***Proof*:** Suppose that T is strongly Archimedean ternary semiring. Let $a, b \in$ T. Since T is Strongly Archimedean ternary semiring, there is an odd natural number n such that $< a >^n \subseteq < b >$. Now $a^n \in < a >^n \subseteq < b > \Rightarrow a^{n+2} \in$ T$< b >$T $\subseteq$ TbT. Therefore T is an Archimedean ternary semiring.

Theorem 3.21 : If T is a P-semipseudo symmetric ternary semiring, then the following are true.

1) **S = {$a \in$ T : $\sqrt{< a >} \neq$ T} is either empty or a completely P-prime ideal.**
2) **T\S is either empty or an Archimedean subsemiring of T.**

***Proof* :** (1) If S is an empty set, then there is nothing to prove. If S is nonempty, then clearly S is an ideal of T. Let $a, b, c \in$ T, P is any ideals of T and abc + P $\subseteq$ S.
Suppose if possible $a \notin$ S, $b \notin$ S, $c \notin$ S. Then $\sqrt{< a >}$ = T, $\sqrt{< b >}$ = T and $\sqrt{< c >}$ = T.
Since abc + P $\subseteq$ S and hence $abc \in$ S, P $\subseteq$ S. Since $abc \in$ S, then $\sqrt{< abc >} \neq$ T.
Now T = $\sqrt{<a>} \cap \sqrt{<b>} \cap \sqrt{<c>} = \sqrt{< abc >} \neq$ T. It is a contradiction.
Thus $a \in$ S or $b \in$ S or $c \in$ S. $\therefore$ S is a completely P-prime ideal.
(2) Since S is a completely P-prime ideal, T\S is either empty or a ternary subsemigroup of T. Let $a, b, c \in$ T\S. Then $\sqrt{< a >} = \sqrt{< b >} = \sqrt{< c >}$= T. Now $b, c \in \sqrt{<a>}$, $c, a \in \sqrt{<b>}$, $c, a \in \sqrt{<c>}$ by the corollary 2.26. $b^n \in < a >$ for some odd natural number n. So $b^{n+2} \in$ TaT $\Rightarrow b^{n+2} = sat$ for some $s, t \in$ T.
If either s or $t \in$S, then $b^{n+2} \in$S and hence $b \in$ S. It is a contradiction. Hence $s, t \in$ T\S.
Now $b^{n+2} = sat \in$ (T\S) a (T\S). Hence T\S is an archimedean ternary subsemiring of T.

Theorem 3.22 : If T is a P-semipseudo symmetric ternary semiring, then the following are equivalent.

1) **T is a strongly Archimedean semiring.**
2) **T is an Archimedean semiring.**
3) **T has no proper completely P-prime ideals.**
4) **T has no proper completely P-semiprime ideals.**
5) **T has no proper P-prime ideals.**
6) **T has no proper P-semiprime ideals.**

***Proof* :** (1) $\Rightarrow$ (2) : Suppose that T is a strongly Archimedean ternary semiring. By theorem 3.20, T is an Archimedean ternary semiring.
(2) $\Rightarrow$(3) : Suppose that T is an Archimedean ternary semiring. Let Q be any completely P-prime ideal of T. Let $a \in$T, $b \in$Q. Since T is an Archimedean ternary semiring, there exists a odd natural number n such that $a^n \in$ TbT $\subseteq$ Q $\Rightarrow a^n \in$ Q $\Rightarrow a \in$ Q. $\therefore$ T $\subseteq$ Q. Clearly Q $\subseteq$ T. Thus Q = T. $\therefore$ T has no proper completely P-prime ideals.
By theorem 2.24, corollary 2.25, and theorem 2.27; (3), (4), (5) and (6) are equivalent.
(5) $\Rightarrow$(1) : T has no proper P-prime ideals. Let $a, b \in$ T. Since T has no proper P-prime ideals, $\sqrt{<b>}$ = T. Now $a \in$ T = $\sqrt{< b >} \Rightarrow a^n \in < b >$ for some odd natural number n. Since T is a P-semipseudo symmetric semiring, $< b >$ is a P-semipseudo symmetric ideal and hence
$a^n \in < b > \Rightarrow < a >^n \subseteq < b >$. Thus T is a strongly Archimedean ternary semiring. Hence the given conditions are equivalent.

Corollary 3.23: If T is a P-pseudo symmetric ternary semiring, then the following are equivalent.

1. **T is strongly Archimedean ternary semiring**
2. **T is an Archimedean ternary semiring**
3. **T has no proper completely P-prime ideals.**
4. **T has no proper completely P-semiprime ideals.**
5. **T has no proper P-prime ideals.**
6. **T has no proper P-semiprime ideals.**

Proof: Every P-pseudo symmetric ternary semiring is a P-semipseudo symmetric ternary semiring. Therefore by theorem 3.22, (1) to (6) are equivalent.

Corollary 3.24 : A commutative ternary semiring T is Archimedean iff T has no proper P-prime ideals.

Proof **:** Since T is a commutative ternary semiring, T is a P-semipseudo symmetric semiring. By theorem 3.22, T is Archimedean iff T has no proper P-prime ideals.

Theorem 3.25 : If M is a nontrivial maximal ideal of a P-semipseudo symmetric ternary semiring T then M is P-prime ideal of T .

Proof **:** Suppose if possible M is not P-prime. Then there exists $a, b, c \in$ T\M such that $<a> <b> <c> + P \subseteq M$ where P is any ideal of T. Then $<a> <b> <c> \subseteq M$ and $P \subseteq M$. Now $<a> <b> <c> \subseteq M$, then for any $x \in$ T\M, we have T = M +$<b>$ = M +$<c>$ = M +$<x>$. Since $b, c, x \in$ T\M, we have $b, c \in <x>$ and $x \in <b>$, $x \in <c>$. So $<b> = <c> = <x>$.
Therefore $<b>^3 \subseteq M$, $<c>^3 \subseteq M$.

If $a \neq b$. Then $a = \sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} a r_j s_j + \sum_{k=1}^{n} t_k a u_k + \sum_{l=1}^{n} v_l w_l a x_l y_l + na$

for some $p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l, y_l, n \in T^e$.

So $a \in <s><b><t>$. If either $s \in M$ or $t \in M$ then $a \in M$. It is a contradiction.
If $s \notin M$ and $t \notin M$, then $<s><b><t> \subseteq <b>^3 \subseteq M$. $\therefore a \in <s><b><t> \subseteq M$.
$\therefore a \in M$. It is a contradiction. Thus $a = b$ and hence M is trivial, which is not true.
So M is P-prime.

Theorem 3.26 : If T is a P-semipseudo symmetric ternary semiring and contains a nontrivial maximal ideal then T contains semisimple elements.

Proof **:** Let M be a nontrivial maximal ideal of T. By theorem 3.25, M is P-prime.
Let $a \in$ T\M. Then $<a> \nsubseteq M$. Since M is maximal, $M \cup <a> = T$.
If $<a>^3 \subseteq M$ then $<a> \subseteq M$ which is not true. So $<a>^3 \nsubseteq M$.
Since M is maximal, $M \cup <a>^3 = T$. Now $M \cup <a> = M \cup <a>^3$.
Therefore $a \in <a>^3$ and hence a is semisimple.

Theorem 3.27 : Let T be a semipseudo symmetric archimedean ternary semiring. Then an ideal M is maximal iff it is trivial, and T has no maximal ideals if T = T^3.

Proof **:** If M is trivial, then clearly M is maximal ideal. Conversely suppose that M is maximal. Suppose if possible M is nontrivial. By theorem 3.25, M is P-prime. Since T is an Archimedean semiring, by theorem 3.22, S has no P-prime ideals. It is a contradiction. So M is trivial. If T=T^3, then by theorem 2.15, every maximal ideal is P-prime and hence T has no maximal ideals.

Theorem 3.28 : Let T be a P-semipseudo symmetric ternary semiring containing maximal ideals. If either T has no semisimple elements or T is an Archimedean ternary semiring, then $T \neq T^3$ and $T^3 = M^*$ where M^* denotes the intersection of all maximal ideals.

***Proof* :** Suppose that T has no semisimple elements. Then by theorem 3.25, every maximal ideal is trivial. So if M is maximal, then T = M + $\{a\}$, $a \notin$ M.
Suppose $a \in T^3$. Then $a \in T^3 \Rightarrow a = \sum_{finite} bcd$ for some $b, c, d \in$ T.
If $b \neq a$ then $b \in$ Mand hence $\sum_{finite} bcd \in$ M (Since M is Maximal) $\Rightarrow a \in$ M. It is a contradiction. $\therefore b = a$. Similarly we can prove $c = a$ and $d = a$. $\therefore a = \sum_{finite} bcd = a^3$. $\therefore a$ is semisimple. It is a contradiction. $\therefore a \notin T^3$. $\therefore T \neq T^3$ and $T^3 \subseteq$ M. Let $t \in M^*$ and $t \notin T^3$.
Let $a \in T \backslash \{t\} \Rightarrow \sum_{finite} ars \neq t, \sum_{finite} ras \neq t, \sum_{finite} rsa \neq t$ for all $r, s \in$ T
$\Rightarrow \sum_{finite} ars, \sum_{finite} ras, \sum_{finite} rsa \in T \backslash \{t\} \Rightarrow T \backslash \{t\}$ is an ideal. Then $T \backslash \{t\}$ is a maximal ideal. Hence $t \in T \backslash \{t\}$, it is a contradiction. $\therefore M^* \subseteq T^3$. $\therefore T^3 = M^*$. Now suppose that T is an archimedean ternary semiring. Since T has maximal ideals, by theorem 3.27, $T \neq T^3$. Suppose if possible $x \in T^3 \backslash M^*$. Then there exists a maximal ideal M, such that $x \notin$ M. So by theorem 3.27, M=T$\backslash\{x\}$. Since $x \in T^3$, $x = \sum_{finite} rst$ for some$r, s, t \in$ T.
If either r or s or $t \in$M, then $x \in$ M. It is a contradiction. Therefore $r = s = t = x$ and hence $x = x^3$. Let $a, b, c \in$ T, $abc \in$ M. Suppose if possible $a \notin$ M, $b \notin$ M, $c \notin$ M. Then $a = x$, $b = x$, $c = x$. Therefore $abc = xxx = x \notin$ M.It is a contradiction. Thus M is P-prime.
By theorem 3.22, S has no proper P-prime ideals. It is a contradiction. Thus $T^3 \subseteq M^*$.
As above, we can show that $M^* \subseteq T^3$. Therefore $T^3 = M^*$.

Corollary 3.29 : Let T be a commutative ternary semiring containing maximal ideals. If either T has no idempotent or T is an Archimedean ternary semiring, then $T \neq T^3$ and $T^3 = M^*$ where M^* denotes the intersection of all maximal ideals of T.

***Proof* :** Suppose that T has no idempotent. If T contains a semi simple element a then a is regular. Hence there exists $x, y \in$ T such that $axaya = a$. Now a is an idempotent in T. It is a contradiction. So S has no semi simple elements. Then by theorem 3.28, we have $T \neq T^3$ and $T^3 = M^*$.

CONCLUSION :

In this paper mainly we studied about the P-semipseudo symmetric ideals in ternary semirings.

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