FIXED POINT THEOREM FOR WEAKLY INWARD NONSELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract

In this paper, we established some weak and strong convergence theorems for common fixed points of three nonself asymptotically Banach spaces. Our results extended and improve the result announ- ed by Wang[6] [Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, J. Math. Anal. Appl., 323(2006)550-557.] and Wei-QiDeng, Lin Wang and Yi-Juan Chen[13] [Strong and Weak Convergence Theorems for common fixed points of two asymptotically nonexpansive mappings in Banach spaces, International Mathematical Forum, Vol. 7, 2012, no. 9, 407 – 417.]

For a smooth banach space E, let us assume that K is a nonempty closed convex subset of with P as a sunny nonexpansive retraction. Let, T_1, T_2, T_3 : K → E be three weakly inward nonself asymptotically nonexpansive mappings with respect to P with three sequences

$\{k_n{}^{(i)}\}$ ⊂ [1,∞) satisfying $\sum_{n=1}^{\infty}(k_n{}^{(i)} - 1) < \infty$,(i=1,2,3) and $F(T_1) \cap F(T_2) \cap F(T_3) = \{x \epsilon k, T_1x = T_2x = T_3x = x\}$ respectively .

For any given $x_1 \in k$, suppose that $\{x_n\}$ is sequence generated iteratively by

$x_{n+1}=a_{n1}x_n+b_{n1}(PT_1)^n y_n+c_{n1}(PT_2)^n y_n+d_{n1}(PT_3)^n y_n$

$y_n=a_{n2}x_n+b_{n2}(PT_1)^n y_n+c_{n2}(PT_2)^n y_n+d_{n2}(PT_3)^n y_n$

$z_n=a_{n3}x_n+b_{n3}(PT_1)^n y_n+c_{n3}(PT_2)^n y_n+d_{n3}(PT_3)^n y_n$

where, $\{a_{ni}\}, \{b_{ni}\}, \{c_{ni}\}, \{d_{ni}\}$ for i=$(1,2,3)$ are sequences in $[a, 1-a]$ for some a∈ $(0,1)$

satisfying $a_{ni} + b_{ni} + c_{ni} + d_{ni} = 1$ (i=1,2,3). some a ∈ (0,1), Under some suitable conditions, the strong and weak convergence theorems of $\{x_n\}$ to a common fixed point of T_1, T_2 and T_3 are obtained.

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Nonself asymptotically nonexpansive mapping, Strong and weak convergence, Common fixed point.

1 INTRODUCTION

For a self-mapping T : K → K, nonexpansive mapping is defined as || T x – Ty || ≤ || x – y|| for all x, y ∈ K and asymptotically nonexpansive if there exists a sequence $\{k_n\}$ ⊂ [1, ∞) with $k_n \to 1$ as n → ∞ such that for all n ∈N ,where N stands for set of natural number,

$||T^n x - T^n y|| \leq k_n || x - y ||$ (1.1)
for all x, y ∈ K. T is called uniformly L-Lipschitzian if there exists a real number L>0suchthat $||T^n x - T^n y|| \leq L|| x - y ||$ for all x, y ∈ K. and integers n≥ 1. (1.2)

As a generalization of the class of nonexpansive maps, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] in 1972, who proved that if K is a nonempty bounded closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point. Recently, Chidume et al.[1] further generalized the class of asymptotically nonexpansive mappings introduced by Goebel and Kirk [4], and proposed the concept of nonself asymptotically nonexpansive mapping defined as follows:

Definition1.1.[2] Let K be a nonempty subset of real normed linear space E. Let P : E → K be the nonexpansive retraction of E onto K. (1) A nonself mapping T : K → E is called asymptotically nonexpansive if there exist sequences $\{k_n\} \in [1,\infty)$ with $k_n \to 1$ as n →∞such that

$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq k_n || x - y ||$ for all x, y ∈ K.

(2)A nonself mapping T : K → E is said to be uniformly L-Lipschitzian if there exists a constant L ≥ 0 such that

$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq L || x - y ||$ for all x, y ∈ K. (1.4)

By using the following iterative algorithm:

$x_1 \in k, x_{n+1} = P\left((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\right),$

∀ n≥1 (1.5)

Some authors [2,6 ,7 ,11] have studied the strong and weak convergence theorem for such mappings.

As a matter of fact, if T is a self-mapping, then P is a identity mapping. Thus (1.3) and (1.4) reduce to (1.1) and (1.2) as T is a self-mapping, respectively. In addition, if T : K → E is asymptotically nonexpansive in light of (1.3) and P : E → K is a nonexpansive retraction, then PT : K → K is asymptotically nonexpansive in light of (1.1). Indeed, for all x,y ∈ K and n ≥ 1, by (1.3), it follows that

$||(PT)^n x - (PT)^n y|| = ||PT(PT)^{n-1}xPT(PT)^{n-1}y||$
$\leq || PT(PT)^{n-1}x - PT(PT)^{n-1}y||$

$\leq k_n || x - y ||$ (1.6)

Conversely, it may not be true. Therefore, Zhou et al.[13] introduced the following generalized definition recently.

Definition 1.2.[9] Let K be a nonempty subset of real normed linear space E. Let P : E → K be a nonexpansive retraction of E onto K.

(1) A nonself mapping T : K → E is called asymptotically nonexpansive with respect to P if there exist sequences$\{k_n\} \in [1,\infty)$ with $k_n \to 1$ asn →∞ such that ,$|| (PT)^n x - (PT)^n y|| \leq k_n || x - y ||$ $\forall x$, y ∈ K , n≥1(1.7)

(2) A nonself mapping T : K → E is said to be uniformly L-Lipschitzian with respect to P if there exists a constant L ≥ 0 such that

$|| (PT)^n x - (PT)^n y|| \leq L || x - y ||$ (1.3)

∀ x, y ∈ K , n≥1 (1.8)

Furthermore, by studying the following iterative process:

$x_{n+1} = \alpha_n \beta_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n$ $\forall x_1 \in k$, n≥1 (1.9)

where $\{\alpha_n\}$,$\{\beta_n\}$,and $\{\gamma_n\}$ are three sequences in [a,1 − a] for some a ∈ (0,1), satisfying $\alpha_n + \beta_n + \gamma_n = 1$, Zhou et al.[3] obtained some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to P in uniformly convex Banach spaces. As a consequence, the main results of Chidume et al. [1] are deduced.

Inspired and motivated by those work mentioned above and three step iteration method proposed by Noor[8], in this paper, we construct a three step iteration scheme for approximating common fixed points of three nonself asymptotically nonexpansive mappings with respect to P and to prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

2. PRELIMINARIES

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E with retraction P. Let $T_1, T_{2,}T_3$: K → E be three nonself asymptotically nonexpansive mappings with respect to P. For approximating common fixed points of such mappings, we further generalize the iteration scheme(1.9) as follows:

$$x_1 \in k$$

$$x_{n+1}=a_{n1}x_n+b_{n1}(PT_1)^n\ y_n+c_{n1}(PT_2)^n\ y_n+d_{n1}(PT_3)^n\ y_n$$

$$y_n=a_{n2}x_n+b_{n2}(PT_1)^n\ y_n+c_{n2}(PT_2)^n\ y_n+d_{n2}(PT_3)^n\ y_n \quad (2.1)$$

$$z_n=a_{n3}x_n+b_{n3}(PT_1)^n\ y_n+c_{n3}(PT_2)^n\ y_n+d_{n3}(PT_3)^n\ y_n$$

Where,$\{a_{ni}\}, \{b_{ni}\}, \{c_{ni}\}, \{d_{ni}\}$,i={1,2,3}are sequences in [0,1] satisfying

$a_{ni} + b_{ni} + c_{ni} + d_{ni}$ for {1,2,3}

Let E be a Banach space with dimension E ≥ 2. The modulus of E is the function $\delta_E(\varepsilon)$: (0 ,2] → [0,1] defined by

$$\delta_E(\varepsilon)=\inf\left\{\ 1-||\tfrac{1}{2}(x+y)||;||x||1,||y||=1,\varepsilon=||x-y||\ \right\}$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon)\ > 0$ for all $\varepsilon \in (0,2]$. Let E be a Banach space and S(E)={x ∈ E : x =1 }. The space E is said to be smooth if

$$\lim_{t\to 0}\frac{||x+ty||-||x||}{t}$$

exists for all x, y ∈ S(E).

A subset K of E is said to be retract if there exists continuous mapping P : E → K such that Px = x for all x ∈ K. A mapping P : E → E is said to be a retraction if P^2= P. Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny if P(Px+ t(x−Px)) = Px for x ∈ C with Px+ t(x−Px) ∈ C and t ≥ 0. Note that, if mapping P is a retraction, then Pz= z for every z ∈ R(P) (the range of P). It is well-known that every closed convex subset of a uniformly convex Banach space is a retract. For any x ∈ K, the inward set $I_k(x)$ is defined as follows: $I_k(x)$ ={y ∈ E : y = x + λ(z − x),z∈ K,λ≥ 0}. A mapping T : K → E is said to satisfy the inward condition if $T_x \in I_k(x)$ for all x ∈ K. T is said to satisfy the weakly inward condition if, for each x ∈ K, T_x ∈ cl $I_k(x)$((cl $I_k(x)$ is the closure of : $I_k(x)$).

A Banach space E is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in E, x_n →x implies that

$$\lim_{n\to\infty} sup||x_n-x|| < \lim_{n\to\infty} sup||x-y||$$

for all y ∈ E with $y \neq x$, where , x_n → x denotes that $\{x_n\}$ converges weakly to x .

Let K be a nonempty closed subset of a real Banach space E. T : K → E is said to be demicompact if, for any sequence

$\{x_n\} \subset k$ with $||x_n - Tx_n||$ →0 $(n\to\infty)$ their exists subsequence $\{x_{nj}\}$ of $\{x_n\}$ such that $\{x_n\}$ converges strongly to $x^* \in k$

A mapping T with domain D(T) and range R(T) inE is said to be demi- closed at p if whenever $\{x_n\}$ is a sequence in D(T) such that $\{x_n\}$ converges weakly to x^* ∈ D(T) and $\{Tx_n\}$ converges strongly to p, then Tx^*= p

Lemma 2.1. [12] Let $\{a_n\}$,$\{\delta_n\}$,and$\{b_n\}$be sequences of nonnegative real numbers satisfying

$a_n + 1 \leq (1+\delta_n)a_n + b_n$, ∀n ≥ 1, if

$\sum_{n=1}^{\infty}\delta_n < \infty\ and \sum_{n=1}^{\infty} b_n < \infty$ then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2. [6] Let E be a real uniformly convex Banach space and let B_r (0) be the closed ball of E with center at the origin and radius r ≥ 0. Then, there exists a continuous strictly increasing convex function g : [0 ,∞) → [0,∞) with g(0) = 0 such that

$$||\lambda x+\mu y+\gamma z||^2 \leq \lambda||x||^2+\mu||y||^2+\gamma||z||^2-\lambda\mu g(||x-y||)$$

For all x,y,zϵB_r (0) and $\lambda, \mu, \gamma \epsilon [0,1]$ with $\lambda+\ \mu+\gamma$=1

Lemma 2.3. [7] Let E be a real smooth Banach space, let K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let T : K → E be a mapping satisfying weakly inward condition. Then F(PT)= F(T).

Lemma 2.4. [3] Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let T : K → K be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$

⊂[1,∞) such that $\{k_n\}\to 1$ as n →∞. Then I−T is demiclosed at zero, that is, for each sequence $\{x_n\}$ }in K, if the sequence $\{x_n\}$ converges weakly to q ∈ K and {(I−T) x_n converges strongly to 0, then (I −T)q =0.

3 MAIN RESULTS

Lemma 3.1. Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let, T_1, T_2, T_3 : K → E be three nonself asymptotically non expansive mappings with respect to P with three sequences $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ (i=1,2, 3), respectively. Suppose that $\{x_n\}$ is defined by (2.1), where $\{a_{ni}\}$,$\{b_{ni}\}$ $\{c_{ni}\}$ and $\{d_{ni}\}$, (i =1 ,2,3) are sequences in [m,1−m] for some m∈ (0,1). If F = F(T_1)∩F(T_2))∩F(T_3) = ∅, then

(1) $\lim_{n\to\infty}||x_n-q||$ exists, ∀q∈F;
(2) $\lim_{n\to\infty} d(x_n,F)$ exists ,where $d(x_n,F)=inf_{q\in F}||x_n-q||$;
(3) $\lim_{n\to\infty}||x_n-(PT_i)x_n||=0$ (i=1,2,3)

Proof: Setting $k_n = \max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}\}$ since, $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ $(i=1,2,3)$

So, $\sum_{n=1}^{\infty}(k_n-1)<\infty$.

For any q∈F, by (2.1) we have

$$||z_n-q||=||a_{n3}(x_n-q)+b_{n3}((PT_1^{\,n})x_n-q)+c_{n3}((PT_2^{\,n})x_n-q)+d_{n3}((PT_3^{\,n})x_n-q)||$$

$$\le a_{n3}||x_n-q||+b_{n3}k_n||x_nq||+c_{n3}k_n||x_n-q||+d_{n3}k_n||x_n-q||$$

$$\le k_n||x_n-q||. \qquad (3.1)$$

By (2.1) and (3.1) we have

$$||y_n-q||=||a_{n2}(x_n-q)+b_{n2}((PT_1^{\,n})z_n-q)+c_{n2}((PT_2^{\,n})z_n-q)+d_{n2}((PT_3^{\,n})z_n-q)||$$

$$\le k_n^{\,2}||x_n-q||. \qquad (3.2)$$

And hence, it follows from (2.1) and (3.2)

$$||x_{n+1}-q||=||a_{n1}(x_n-q)+b_{n1}((PT_1^{\,n})y_n-q)+c_{n1}((PT_2^{\,n})y_n-q)+d_{n1}((PT_3^{\,n})y_n-q)||$$

$$\le a_{n1}||x_n-q||+b_{n1}k_n^{\,3}||x_nq||+c_{n1}k_n^{\,3}||x_n-q||+d_{n1}k_n^{\,3}||x_n-q||$$

$$\le k_n^{\,3}||x_n-q|| \qquad (3.3)$$

Where, $\delta_n = k_n^{\,3}-1$ satisfying $\sum_{n=1}^{\infty}\delta_n<\infty$, since $\sum_{n=1}^{\infty}(k_n-1)<\infty$ is equivalent to

$\sum_{n=1}^{\infty}(k_n^{\,3}-1)<\infty$ .Thus by (3.3) and lemma (2.1) , we obtain that

$$\lim_{n\to\infty}||x_n-q|| \text{ exists, } \forall q\in F$$

(2)This conclusion can be easily shown by taking infimum in (3.3) for all q∈F.

(3) Assume, by conclusion of (1), $\lim_{n\to\infty}||x_n-q||$=d and from lemma (2.2) ,we have,

$$||x_n-q||^2=$$

$$||a_{n1}(x_n\text{-}q)+b_{n1}((PT_1^{\,n})y_n\text{-}q)+c_{n1}((PT_2^{\,n})y_n\text{-}q)^2+d_{n1}((PT_3^{\,n})y_n\text{-}q||$$

$$\le a_{n1}||x_n-q||^2+b_{n1}||((PT_1^{\,n})y_n-q)||^2+c_{n1}||(PT_2^{\,n})y_n-q||^2+d_{n1}||(PT_3^{\,n})y_n-q||^2-a_{n1}b_{n1}c_{n1}g_1||x_n-(PT_1^{\,n})y_n||$$

$$\le a_{n1}||x_n-q||^2+(b_{n1}+c_{n1}+d_{n1})\,k_n^{\,2}||y_n\text{-}q||^2\text{-}m^3g_1||x_n\text{-}(PT_1^{\,n})y_n||$$

$$\le (a_{n1}+(b_{n1}+c_{n1}+d_{n1})\,k_n^{\,4})\,||x_n-q||^2-m^3g_1||x_n-(PT_1^{\,n})y_n||$$

$$\le k_n^{\,4}||x_n-q||^2-m^3g_1||x_n-(PT_1^{\,n})y_n||$$

which implies that $g_1||x_n-(PT_1^{\,n})y_n||\to 0$ asn →∞. Sinceg_1 : [0 ,∞) →[0 ,∞) with g_1 (0)=0 is a continuous strictly increasing convex function, it follows that

$$\lim_{n\to\infty}||x_n-(PT_1^{\,n})y_n||=0 \qquad (3.4)$$

Similarly we have ,

$$\lim_{n\to\infty}||x_n-(PT_2^{\,n})y_n||= \qquad (3.5)$$

And

$\lim_{n\to\infty}||x_n - (\mathrm{PT_3}^{\mathrm{n}})y_n||$=0 (3.6)

Noting that,

$$||x_n - q||=||x_n - (\mathrm{PT_1}^{\mathrm{n}})y_n||+||(\mathrm{PT_1}^{\mathrm{n}})y_n - \mathrm{q}||$$
$$\le ||x_n - (\mathrm{PT_1}^{\mathrm{n}})y_n||+k_n||y_n - q||$$

we obtain from (3.4) that, by taking liminf on both sides in the inequality above,

$$\mathrm{d}= \liminf_{n\to\infty}||x_n - q|| \le \liminf_{n\to\infty} k_n||y_n - q|| = \liminf_{n\to\infty}||y_n - q||$$

In addition, it follows from (3.2) that $\limsup_{n\to\infty}||y_n - q|| \le$d, thus

$\lim_{n\to\infty}||y_n - q|| = d$ (3.7)

Hence, by (2.1), (3.1), (3.6) and Lemma 2.2, we have

$$||y_n - q||^2=||\mathrm{a_{n2}}(x_n - q) + \mathrm{b_{n2}}((\mathrm{PT_1}^{\mathrm{n}})z_n - q) + \mathrm{c_{n2}}((\mathrm{PT_2}^{\mathrm{n}})z_n - q) + \mathrm{d_{n2}}((\mathrm{PT_3}^{\mathrm{n}})z_n - \mathrm{q}||^2$$

$$\le \mathrm{a_{n2}}||x_n - q||^2 + \mathrm{b_{n2}}||((\mathrm{PT_1}^{\mathrm{n}})z_n - q)||^2 + \mathrm{c_{n2}}||(\mathrm{PT_2}^{\mathrm{n}})z_n - q||^2 + \mathrm{d_{n2}}||(\mathrm{PT_3}^{\mathrm{n}})z_n - \mathrm{q}||^2 - \mathrm{a_{n2}b_{n2}c_{n2}}g_2||x_n - (\mathrm{PT_1}^{\mathrm{n}})z_n||$$

$$\le \mathrm{a_{n2}}||x_n - q||^2 + (\mathrm{b_{n2}} + \mathrm{c_{n2}} + \mathrm{d_{n2}})\, k_n{}^2||z_n - q||^2 - m^3 g_2||x_n - (\mathrm{PT_1}^{\mathrm{n}})z_n||$$

$$\le (\mathrm{a_{n2}} + (\mathrm{b_{n2}} + \mathrm{c_{n2}} + \mathrm{d_{n2}})\, k_n{}^3)\, ||x_n - q||^2 - m^3 g_2||x_n - (\mathrm{PT_1}^{\mathrm{n}})z_n||$$

$$\le k_n{}^3||x_n - q||^2 - m^3 g_1||x_n - (\mathrm{PT_1}^{\mathrm{n}})z_n||$$

which implies that $g_2||x_n - (\mathrm{PT_1}^{\mathrm{n}})y_n|| \to 0$ as$n \to\infty$. Since$g_2 : [0,\infty) \to [0,\infty)$ with $g_1(0) = 0$ is a continuous strictly increasing convex function, it follows that

$\lim_{n\to\infty}||x_n - (\mathrm{PT_1}^{\mathrm{n}})z_n||$=0 (3.8)

Similarly we have ,

$\lim_{n\to\infty}||x_n - (\mathrm{PT_2}^{\mathrm{n}})z_n||$=0 (3.9)

And

$\lim_{n\to\infty}||x_n - (\mathrm{PT_3}^{\mathrm{n}})z_n||$=0 (3.10)

Noting that,

$$||x_n - q||=||x_n - (\mathrm{PT_1}^{\mathrm{n}})z_n||+||(\mathrm{PT_1}^{\mathrm{n}})z_n - \mathrm{q}||$$
$$\le ||x_n - (\mathrm{PT_1}^{\mathrm{n}})y_n||+k_n||z_n - q||$$

we obtain from (3.4) that, by taking liminf on both sides in the inequality above,

$$\mathrm{d}=\liminf_{n\to\infty}||x_n - q|| \le \liminf_{n\to\infty} k_n||z_n - q||$$
$$= \liminf_{n\to\infty}||z_n - q||$$

In addition, it follows from (3.2) that $\limsup_{n\to\infty}||z_n - q|| \le$d, thus

$\lim_{n\to\infty}||z_n - q|| = d$ (3.11)

Next, it follows from (2.1), (3.11) and Lemma 2.2 that

$$||z_n - q||^2=||\mathrm{a_{n3}}(x_n - q) + \mathrm{b_{n3}}((\mathrm{PT_1}^{\mathrm{n}})x_n - q) + \mathrm{c_{n3}}((\mathrm{PT_2}^{\mathrm{n}})x_n - q) + \mathrm{d_{n3}}((\mathrm{PT_3}^{\mathrm{n}})x_n - \mathrm{q}||^2$$

$$\le \mathrm{a_{n3}}||x_n - q||^2 + \mathrm{b_{n3}}||((\mathrm{PT_1}^{\mathrm{n}})x_n - q)||^2 + \mathrm{c_{n3}}||(\mathrm{PT_2}^{\mathrm{n}})x - q||^2 + \mathrm{d_{n3}}||(\mathrm{PT_3}^{\mathrm{n}})x_n - \mathrm{q}||^2 - \mathrm{a_{n3}b_{n3}c_{n3}}g_3||x_n - (\mathrm{PT_1}^{\mathrm{n}})x_n||$$

$$\le \mathrm{a_{n3}}||x_n - q||^2 + (\mathrm{b_{n3}} + \mathrm{c_{n3}} + \mathrm{d_{n3}})\, k_n||x_n - q||^2 - m^3 g_2||x_n - (\mathrm{PT_1}^{\mathrm{n}})x_n||$$

$$\le (\mathrm{a_{n3}} + (\mathrm{b_{n3}} + \mathrm{c_{n3}} + \mathrm{d_{n3}})\, k_n{}^2)\, ||x_n - q||^2 - m^3 g_2||x_n - (\mathrm{PT_1}^{\mathrm{n}})x_n||$$

$$\le k_n{}^2||x_n - q||^2 - m^3 g_3||x_n - (\mathrm{PT_1}^{\mathrm{n}})x_n||$$

which implies that $g_2||x_n - (\mathrm{PT_1}^{\mathrm{n}})x_n|| \to 0$ as$n \to\infty$. Since$g_3 : [0,\infty) \to [0,\infty)$ with $g_3(0) = 0$ is a continuous strictly increasing convex function, it follows that

$\lim_{n\to\infty}||x_n - (\mathrm{PT_1}^{\mathrm{n}})x_n||$=0 (3.12)

Similarly we have ,

$\lim_{n\to\infty}||x_n - (\mathrm{PT_2}^{\mathrm{n}})x_n||$=0 (3.13)

And

$\lim_{n\to\infty}||x_n - (\mathrm{PT_3}^{\mathrm{n}})z_n||$=0 (3.14)

Furthermore, we claim that $||x_{n+1} - x_n|| \to 0$, as n →∞. In fact, by (2.1), we have

$||x_{n+1} - x_n||$=$||\mathrm{b_{n1}}((\mathrm{PT_1}^{\mathrm{n}})y_n - x_n) + \mathrm{c_{n1}}((\mathrm{PT_2}^{\mathrm{n}})y_n - x_n) + \mathrm{d_{n1}}((\mathrm{PT_3}^{\mathrm{n}})y_n - x_n)||$

≤ $\mathrm{b_{n1}}||(\mathrm{PT_1}^{\mathrm{n}})y_n - x_n||$+$\mathrm{c_{n1}}||(\mathrm{PT_2}^{\mathrm{n}})y_n - x_n||$+$\mathrm{d_{n1}}||(\mathrm{PT_3}^{\mathrm{n}})y_n - x_n||$

Hence, it follows from (3.4) , (3.5) and(3.6)

$\lim_{n\to\infty}||x_{n+1} - x_n||$=0 (3.15)

Since any asymptotically nonexpansive mapping with respect to P must be uniformly L-Lipschitzian with respect to P, where L = $sup_{n\geq1}\{k_n\} \geq 1$ we have,

$$\left||x_{n+1} - (\mathrm{PT}_i)x_{n+1}\right||$$

$$\leq \left||x_{n+1} - (\mathrm{PT}_i)^{\mathrm{n+1}}x_{n+1}\right|| + ||(\mathrm{PT}_i)x_{n+1} - (\mathrm{PT}_i)^{\mathrm{n+1}}x_{n+1}||$$

$$\leq \left||x_{n+1} - (\mathrm{PT}_i)^{\mathrm{n+1}}x_{n+1}\right||\text{+L}\left||x_{n+1} - (\mathrm{PT}_i)^{\mathrm{n}}x_{n+1}\right||$$

$\leq \left||x_{n+1} - (\mathrm{PT}_i)^{\mathrm{n+1}}x_{n+1}\right||$+L$\left||x_n - (\mathrm{PT}_i)^{\mathrm{n}}x_{n+1}\right||$+L$||x_{n+1} - x_n||$

$\leq \left||x_{n+1} - (\mathrm{PT}_i)^{\mathrm{n+1}}x_{n+1}\right||$+L$\left||x_n - (\mathrm{PT}_i)^{\mathrm{n}}x_{n+1}\right||$+L$(L + 1)||x_{n+1} - x_n||$

Consequently, by (3.13), (3.14), and (3.15), it can be obtained that,

$\lim_{n\to\infty}\left||x_{n+1} - (\mathrm{PT}_i)x_{n+1}\right||$=0

(i=1,2,3) (3.17)

This completes the proof.

Theorem 3.2. Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. LetT_1, T_2, T_3 : K → E be three weakly inward nonself asymptotically nonexpansive mappings with respect to P with two sequences $\{k_n{}^{(1)}\}$,$\{k_n{}^{(2)}\}$,$\{k_n{}^{(3)}\}$,⊂[1,∞) satisfying satisfying $\sum_{n=1}^{\infty}(k_n{}^{(i)} - 1) < \infty$, (i=1,2,3) respectively.suppose that sequence $\{x_n\}$ defined by (2.1)

where $\{a_{ni}\}$,$\{b_{ni}\}$ $\{c_{ni}\}$ and$\{d_{ni}\}$, (i =1 ,2,3) are sequences in [m,1−m] for some m∈ (0,1).

If $PT_1$and PT_2 and PT_3 satisfy Condition (B) with respect to the sequence $\{x_n\}$, i.e., there exists a nondecreasing function f : [0 ,∞) → [0,∞) with f(0) = 0 and f(r) > 0 for all r ∈ (0,∞) such that f(d(x_n,F_1)) $\leq max_{1\leq i\leq 3}\left||x_n - (\mathrm{PT}_i)x_n\right||andF_1$ =F(PT_1)∩F(PT_2) ∩ F(PT_3) ={x ∈ K : PT_1x =$PT_2x = PT_3x$ = x} = ∅, then $\{x_n\}$ converges strongly to a common fixed point of T_1 , T_2 and T_3.

Proof. It follows from Lemma 2.3 that F_1 = F, where F is the common fixed point set of T_1, $T_2$and T_3. Since PT_1 , PT_2 $and_1 PT_3$ satisfy Condition (B) with respect to the sequence $\{x_n\}$, that is to say

f(d(x_n,F)) $\leq max_{1\leq i\leq 3}\left||x_n - (\mathrm{PT}_i)x_n\right||$

Taking limsup as n→∞on both sides in the inequality above, we get

$\lim_{n\to\infty}\mathrm{f}(\mathrm{d}(x_n, F))$ =0

which implies $\lim_{n\to\infty}\mathrm{f}(\mathrm{d}(x_n, F))$ =0 , by the definition of the function f.

Now we show that $\{x_n\}$ is a Cauchy sequence. By (3.3), we may assume that $\sum_{n=0}^{\infty}\delta_n$ =M≥ 0Since $\lim_{n\to\infty}$ (d(x_n,F) = 0, then for any ε>0, there exists a positive integer N such that d(x_n,F) <$\frac{\varepsilon}{2e^M}$ for all n ≥ N. On the other hand, there exists a p ∈ F such that $\left||x_N - \mathrm{P}\right||$ = d(x_N,F) <$\frac{\varepsilon}{2e^M}$ because d(x_N,F) =$inf_{q\in F}\left||x_N - \mathrm{q}\right||$ and F is closed. Thus, for any n>N, it follows from (3.3) that $\left||x_n - \mathrm{p}\right||$=(1+$\delta_n$) $\left||x_n - \mathrm{p}\right|| \leq \prod_{i=1}^{n}(1+\delta_i)\left||x_N - \mathrm{p}\right||$

$\leq e^{\sum_{i=1}^{n}(1+\delta_i)}\left||x_N - \mathrm{p}\right||$

$\leq e^{M}\left|\left|x_N - \mathrm{p}\right|\right|$

Hence, for any n,m>N

$\left|\left|x_n - x_m\right|\right| \leq \left|\left|x_n - \mathrm{p}\right|\right| + \left|\left|x_m - \mathrm{p}\right|\right|$

$\leq 2e^{M}\left|\left|x_N - \mathrm{p}\right|\right| < \varepsilon$

This implies that $\{x_n\}$ is a Cauchy sequence. Thus, there exists a x ∈ K such that $x_n \to$ x as n →∞, since E is complete. Then, $\lim_{n\to\infty} \mathrm{d}(x_n, F)$ = 0 yields that d(x,F) = 0. Further, it follows from the closedness of F that x ∈ F. This completes the proof.

Theorem 3.3. Let K be a nonempty closed convex subset of a uniformly convex and smooth Banach space E satisfying Opial's condition with P as a sunny nonexpansive retraction. LetT_1, T_2, T_3 : K → E be two weakly inward nonself asymptotically nonexpansive mappings with respect to P with two sequences $\{k_n^{(1)}\}$,$\{k_n^{(2)}\}$,$\{k_n^{(3)}\}$,⊂[1,∞) satisfying $\sum_{n=1}^{\infty}\left(k_n^{(i)} - 1\right) < \infty$, (i=1,2,3) respectively.suppose that sequence $\{x_n\}$ defined by (2.1)where $\{a_{ni}\}$,$\{b_{ni}\}$ $\{c_{ni}\}$ and$\{d_{ni}\}$, (i =1 ,2,3) are sequences in [m,1−m] for some m∈ (0,1).

If F := F(T_1)∩F(T_2) ∩ F(T_3) = ∅, then $\{x_n\}$ converges weakly to some common fixed point of T_1 , T_2 and T_3.

Proof. For any q ∈ F, by Lemma 3.1, we know that $\lim_{n\to\infty}\left|\left|x_n - \mathrm{q}\right|\right|$ exists. We now prove that $\{x_n\}$ has a unique weakly subsequential limit in F. First of all, since PT_1, PT_2 and PT_3 are self-mappings from K into itself, therefore, Lemmas 2.3, 2.4, and 3.1 guarantee that each weakly subsequential limit of $\{x_n\}$ is a common fixed point of T_1 , T_2 and T_3. Secondly, Opial's condition guarantees that the weakly subsequential limit of$\{x_n\}$ is unique. Consequently, $\{x_n\}$ converges weakly to a common fixed point ofT_1, T_2 and T_3 . This completes the proof.

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