Sequence Space $\overline{sl(p)}$ Generated by an Infinite Diagonal Matrix

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Abstract:

The sequence space $\overline{l(p)} = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \left| t_k(x) \right|^{p_k} < \infty \right\}$ where $t_k(x) = \sum_{i=1}^{k} x_i$ was introduced and studied by B. Choudhary and S.K. Mishra [3]. In the present paper, we generalize the space $\overline{l(p)}$ by means of an infinite diagonal matrix $A = (a_{nk}) = \begin{cases} 2^{-n} & \text{for } n = k \\ 0 & \text{otherwise} \end{cases}$ and introduce a new sequence space $\overline{sl(p)}$. We shall study some properties of $\overline{sl(p)}$ and find its β- dual. Furthermore we characterize the matrix classes $(\overline{sl(p)}, l_\infty)$ and $(\overline{sl(p)}, c)$.

***Keywords*: Paranormed sequence space, β- dual , matrix transformation**

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I.PRELIMINARIES, BACKGROUND and NOTATIONS

By ω, we denote the space of all complex valued sequences . Any vector subspace of ω is called a sequence space. We write l_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g: X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous i.e. $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha \in \mathbb{R}$ and all $x \in X$, where θ is the zero vector in the linear space X . We shall assume here and after $\{p_k\}$ be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. The linear space $l(p)$ was defined by Maddox as follows:

$$l(p) = \left\{ x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty \right\} \tag{1.1}$$

which is a complete space paranormed by

$$g(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M}$$

For simplicity in notation , the summation without limits is assumed to run from 1 to ∞.

Let X and Y be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk} , where $n, k \in \mathbb{N}$. Then we write $Ax = ((Ax)_n)$, the A- transform of x, if $(Ax)_n = \sum_k a_{nk} x_k$ converges for each $n \in \mathbb{N}$. If $x \in X$ implies that $Ax \in Y$ then we say that A defines matrix transformation from X into Y and denote it by $A: X \to Y$. By (X, Y) we mean the class of all infinite matrices A such that $A: X \to Y$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\} \tag{1.2}$$

which is a sequence space.

In 1993, Choudhary and Mishra [3] have defined and studied the sequence space $\overline{l(p)}$ which consists of all sequences such that S-transforms are in $l(p)$. Here $S = (s_{nk})$ is the matrix given by

$$s_{nk} = \begin{cases} 1, & 0 \le k \le n \\ 0 & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

For $p = \{p_k\}$ a bounded sequence of strictly positive real numbers the sequence space $\overline{l(p)}$ is given by

$$\overline{l(p)} = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |t_k(x)|^{p_k} < \infty \right\}$$

where

$$t_k(x) = \sum_{i=1}^{k} x_i$$

When $p_k = p$ for every k , the sequence space $\overline{l(p)}$ is reduced to the sequence space

$$\overline{l_p} = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |t_k(x)|^{p} < \infty \right\}$$

In 2002, Altay and Basar [2] have studied the space $r^t(p)$ which consists of all sequences whose Riesz transforms are in the space $l(p)$.

In 2004, Malkowsky and Savas [5] have defined and studied the sequence space $Z(u, v; p)$ which consists all sequences such that $G(u, v)$ transforms are in $X \in \{l_\infty, c, c_0, l_p\}$. The matrix $G(u, v) = (g_{nk})$ called generalized weighted mean or factorable matrix is given by

$$g_{nk} = \begin{cases} u_n v_k, & 0 \le k \le n \\ 0 & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$; where u_n depends only on n and v_k depends only on k.

With the notation of (1.2) , the spaces $\overline{l(p)}$, $Z(u, v; p)$ and $r^t(p)$ may be represented as

$Z(u, v; p) = [X]_{G(u,v)}$, $\overline{l(p)} = [l(p)]_S$ and $r^t(p) = [l(p)]_{R^t}$

where the matrix $R^t = (r^t_{nk})$ of the Riesz mean (R, t_n) is given by

$$r^t_{nk} = \begin{cases} t_k / \sum_{k=0}^{n} t_k & 0 \le k \le n \\ 0 & k > n \end{cases}$$

with the sequence of positive real (t_k) .

II. NEW PARANORMED SEQUENCE SPACE $\overline{sl(p)}$

Before defining new sequence space, we take an infinite diagonal matrix $A = (a_{nk})$ given by

$$a_{nk} = \begin{cases} 2^{-n}, & n = k \\ 0 & \text{otherwise} \end{cases}$$

Following Choudhary and Mishra [3], Altay and Basar [2] and Malkowsky and Savas [5]

for $p = \{p_k\}$ a bounded sequence of strictly positive real numbers we define the sequence space $\overline{sl(p)}$ by

$$\overline{sl(p)} = \{x = (x_k) : Ax \in \overline{l(p)}\} \tag{2.1}$$

Thus, $\overline{sl(p)}$ is now the set of all sequences $\{v_k\}$ whose A –transforms are in the sequence space $\overline{l(p)}$. Using the notation as in (1.2) $\overline{sl(p)}$ can be represented as

$$\overline{sl(p)} = [\,\overline{l(p)}\,]_A$$

Here the sequence $\{v_k\}$ is given by

$$\{v_k\} = \sum_{r=1}^{k} \frac{1}{2^r} |t_r(x)|^{p_r}$$

Thus $\overline{sl(p)}$ can be rewritten as

$$\overline{sl(p)} = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \frac{1}{2^k} |t_k(x)|^{p_k} < \infty \right\}$$

Also when $p_k = p$ for every k , the sequence space $\overline{sl(p)}$ is reduced to the sequence space

$$\overline{sl_p} = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \frac{1}{2^k} |t_k(x)|^{p} < \infty \right\}$$

We shall now present some properties of $\overline{sl(p)}$ and $\overline{sl_p}$.

Property 2.1 .
$\overline{sl(p)}$ is linearly isomorphic to $\overline{l(p)}$.
Proof:
For each $x \in \overline{sl(p)}$, we have $Ax \in \overline{l(p)}$ where $A = (a_{nk})$ is given by

$$a_{nk} = \begin{cases} 2^{-n}, & n = k \\ 0, & \text{otherwise} \end{cases}$$

Moreover A is linear and bijective.

Also the matrix $B = (b_{nk})$ defined by
$$B = (b_{nk}) = \begin{cases} 2^{n}, & n = k \\ 0, & \text{otherwise} \end{cases}$$
is inverse of A. Thus $\overline{sl(p)}$ is linearly isomorphic to $\overline{l(p)}$.

Corollary 2.1.

$\overline{sl_p}$ and $\overline{l_p}$ are linearly isomorphic.

Proof:

Using the same arguments as given in property (2.1), it can be shown that $\overline{sl_p}$ and $\overline{l_p}$ are linearly isomorphic.

Property 2.2.

$\overline{sl(p)}$ is complete paranormed space paranormed by

$$g(x)=\left(\sum_{k=1}^{\infty}\frac{1}{2^k}|t_k(x)|^{p_k}\right)^{\frac{1}{M}} \text{ where } M=\max\left(\frac{1}{2},\sup_k \frac{p_k}{2^k}\right)$$

Proof:

Since $\overline{sl(p)}$ and $\overline{l(p)}$ are linearly isomorphic and $\overline{l(p)}$ is a complete paranormed space with paranorm

$g(x)=\left(\sum_{k=1}^{\infty}|t_k(x)|^{p_k}\right)^{\frac{1}{M}}$ where $M=\max\left(1,\sup_k p_k\right)$, then from property (2.1) $\overline{sl(p)}$ is a complete paranormed space with paranorm ,

$g(x)=P(Ax)$; where P is usual paranorm on $\overline{l(p)}$.

Property 2.3.

$\overline{sl_p}$ is a Banach space for $1\leq p<\infty$ and $t_0(x)=0$; normed by

$$\|x\|=\left(\sum_{k=0}^{\infty}\frac{1}{2^k}|t_k(x)|^{p}\right)^{\frac{1}{p}}$$

Proof :

The proof follows immediately by using the fact that $\|x\|=\|Ax\|_p$ where $\|.\|_p$ is the usual norm on $\overline{l_p}$.

Property 2. 4.

$\overline{sl_2}$ is a Hilbert space with inner product $\langle x,y\rangle=\sum_{k=1}^{\infty}2^{2k}t_k(x)\overline{t_k(y)}$, where bar denotes the conjugate.

Proof :

We have $\overline{l_2}$ is a Hilbert space with inner product $\langle x,y\rangle=\sum_{k=1}^{\infty}t_k(x)\overline{t_k(y)}$.

Also from property (2.1) for $x\in\overline{sl_2}$, $Ax\in\overline{l_2}$.

Setting $\langle x,y\rangle=\langle Ax,Ay\rangle$ which is usual inner product in $\overline{l_2}$; we can easily see that $\overline{sl_2}$ is also a Hilbert space.

Property 2.5.

If z be a closed subset of $\overline{l(p)}$, then $\frac{z}{2^k}$ is a closed subset of $\overline{sl(p)}$.

Proof :

Since $z\in\overline{l(p)}$, $\frac{z}{2^k}\in\overline{sl(p)}$. Let x belongs to closure of $\frac{z}{2^k}$. Then there exists a sequence $(x^n)\subset\frac{z}{2^k}$ such that (x^n) converges to x.

This implies that $g\left(x^n - x\right)\to 0$ as $n\to\infty$. Hence by definition, from above,

$t_k^{\,n}\left(x\right) - t_k\left(x\right)\to 0$ *i.e.* $t_k^{\,n}\left(x\right)\to t_k\left(x\right)$. This completes the proof.

Now we state a proposition which gives characterization of compact sets on $\overline{sl(p)}$.

Proposition 2.1:

A set $G\subset\overline{sl(p)}$ is compact if and only if

i) *G* is closed and bounded,

ii) Given $\varepsilon>0$, there exists a positive integer n_0 such that $\sum_{k=n+1}^{\infty}\left|t_k(x)\right|^{p_k}<\varepsilon$ for $x\in G$ whenever $n\ge n_0$,

iii) If $d_k:\overline{sl(p)}\to\mathbb{C}$ is given by $d_k(x)=t_k(x)$ for all $x\in\overline{sl(p)}$, then $d_k(G)$ is compact for all $k\ge 1$.

Proof:

Following the same arguments as in proposition 4.1.7 in [11] , we can easily prove the proposition.

III. Dual

For a sequence space X we define β-dual of X as

$$X^{\beta}=\left\{a=(a_k):\sum_{k=1}^{\infty}a_kx_k \text{ is convergent for each } x\in X\right\}$$

Theorem 3.1

Let $0<p^k\le\frac{1}{2}$ for every $k\in N$. Then $\overline{sl(p)}^{\beta}=\overline{sl_\infty(p)}$ where

$$\overline{sl_\infty(p)}=\left\{a=(a_k):\sum_{k=1}^{\infty}a_k\left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_v}}+\left(\frac{N^{-2}}{2^N}\right)^{\frac{1}{p_k}}\right)\text{ converges and }\sup\left|\frac{\Delta a_k}{2^k}\right|<\infty\right\};$$

$N\ge 1$, $\Delta a_k=a_k-a_{k+1}$ i.e. the β- dual of $\overline{sl\left(p\right)}$ is $\overline{sl_\infty\left(p\right)}$.

Proof :

Necessary Part:

Let $a\in\overline{sl(p)}^{\beta}$. Then the series $\sum_{k=1}^{\infty}a_kx_k$ converges for each $x\in\overline{sl\left(p\right)}$.

Since, $x=\left\{-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{1/p_v}+\left(\frac{N^{-2}}{2^N}\right)^{1/p_k}\right\}\in\overline{sl\left(p\right)}$; it follows that $\sum_{k=1}^{\infty}a_k\left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{1/p_v}+\left(\frac{N^{-2}}{2^N}\right)^{1/p_k}\right)$ converges.

Next we need to show that $\sup_k\left(\frac{\Delta a_k}{2^k}\right)^{p_k}<\infty$. On the contrary we assume that $\sup_k\left(\frac{\Delta a_k}{2^k}\right)^{p_k}=\infty$. Then

$\Delta a = (\Delta a_k) \notin \overline{l_\infty(p)}$ *i.e.* $\overline{l(p)}^{\beta}$. Hence there exists a sequence $y = (y_k) \in \overline{l(p)}$ such that $\sum_{k=1}^{\infty} \Delta a_k \, y_k$ does not converge.

Whenever we define the sequence $u = (u_k)$ by

$$u_k = \begin{cases} \dfrac{y_1}{2} \text{ for } k = 1 \\ \\ \dfrac{y_k}{2^k} - \dfrac{y_{k-1}}{2^{k-1}} \text{ for } k > 1 \end{cases}$$

Then $u \in \overline{sl(p)}$ and $\sum_{k=1}^{\infty} a_k \, u_k = \sum_{k=1}^{\infty} \Delta a_k \, \dfrac{y_k}{2^k}$.

So , it follows that the series $\sum_{k=1}^{\infty} a_k \, u_k$ does not converge which is the contradiction to the assumption that $a \in \overline{sl(p)}^{\beta}$. Hence we must have, $\sup_k \left(\dfrac{\Delta a_k}{2^k} \right)^{p_k} < \infty$, thereby showing that β- dual of $\overline{sl(p)}$ exists and is $\overline{sl_\infty(p)}$.

Sufficient part :

Let $a \in \overline{sl_\infty(p)}$ and $x \in \overline{sl(p)}$. We can choose a positive integer N ≥ 1 such that

$$\left(\frac{1}{2^k} \left| t_k(x) \right| \right)^{p_k} \le \frac{1}{2^N N^2} \tag{3.1}$$

We have,

$$\sum_{k=1}^{m} a_k x_k = \sum_{k=1}^{m-1} \Delta a_k \, \frac{1}{2^k} t_k(x) + \Delta a_m \frac{1}{2^m} t_m(x) \, , \; m \in N \tag{3.2}$$

so that

$$\left| \sum_{k=1}^{m} a_k x_k \right| \le \sum_{k=1}^{m-1} \left| \Delta a_k \right| \, \frac{1}{2^k} \left| t_k(x) \right| + \left| a_m \right| \frac{1}{2^m} \left| t_m(x) \right|$$

$$\le \sum_{k=1}^{m-1} \left| \Delta a_k \right| \left(\frac{N^{-2}}{2^N} \right)^{1/p_k} + \left| a_m \right| \left(\frac{N^{-2}}{2^N} \right)^{1/p_m} .$$

Since $\left| \Delta a_k \right|^{p_k}$ is bounded , so that for some M > 0 , $\left| \Delta a_k \right|^{p_k} < M \Rightarrow \left| \Delta a_k \right| \le M^{1/p_k}$.

Hence , $\sum_{k=1}^{\infty} \left| \Delta a_k \right| \left(\dfrac{N^{-2}}{2^N} \right)^{1/p_k} \le \sum_{k=1}^{\infty} M^{1/p_k} \left(\dfrac{N^{-2}}{2^N} \right)^{1/p_k} < \infty$

Now the sequence $\left\{ \left(\dfrac{N^{-2}}{2^N} \right)^{1/p_k} \right\} \in \overline{l(p)}$. Also if $\sum_{k=1}^{\infty} M^{1/p_k} \left(\dfrac{N^{-2}}{2^N} \right)^{1/p_k}$ does not converge, then the sequence $\left\{ M^{1/p_k} \right\} \notin \overline{l(p)}^{\beta}$

.

We know $\overline{l(p)}^{\beta} = \overline{l_\infty(p)} = \left\{ a = (a_k) : \sum_{k=1}^{\infty} a_k \left(-\sum_{k=1}^{\infty} (N^{-2})^{1/p_\nu} + (N^{-2})^{1/p_k} \right) < \infty \text{ and } \sup |a_k|^{p_k} < \infty \right\}$.

This implies that $M > \infty$; which is impossible. Hence right hand side of (3.2) is absolutely convergent.

Moreover, $\sum_{k=1}^{m} a_k \left(-\sum_{\nu=1}^{k-1} \left(\frac{N^{-2}}{2^N} \right)^{1/p_\nu} + \left(\frac{N^{-2}}{2^N} \right)^{1/p_k} \right) = \sum_{k=1}^{m-1} \Delta a_k \left(\frac{N^{-2}}{2^N} \right)^{1/p_k} + a_m \left(\frac{N^{-2}}{2^N} \right)^{1/p_m}$; $m \in \mathbb{N}$,

Hence there exists a sequence $\left\{ a_m \left(\frac{N^{-2}}{2^N} \right)^{1/p_m} \right\}$ having a finite limit and hence the series $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in \overline{sl(p)}$

i.e. $a \in \overline{sl_\infty(p)}$.

IV. MATRIX TRANSFORMATION

Let (X, Y) denote the set of all infinite matrices which transforms X into Y. Now we shall provide characterization for the classes $\left(\overline{sl(p)}, l_\infty\right)$ and $\left(\overline{sl(p)}, c\right)$.

Theorem 4.1. :

Let $0 < p_k \le \frac{1}{2}$ for every $k \in N$. Then $A \in \left(\overline{sl(p)}, l_\infty\right)$ if and only if

i) $\sup_n \left| \sum_{k=1}^{\infty} a_{n,k} \left(-\sum_{\nu=1}^{k-1} \left(\frac{N^{-2}}{2^N} \right)^{1/p_\nu} + \left(\frac{N^{-2}}{2^N} \right)^{1/p_k} \right) \right| < \infty;\ N \ge 1$

ii) $\sup_{n,k} |\Delta a_{n,k}|^{p_k} < \infty$ *where* $\Delta a_{n,k} = a_{n,k} - a_{n,k+1}$.

Proof :

Let the conditions hold. Now

$$\left| \sum_{k=1}^{\infty} a_{n,k} \left(-\sum_{\nu=1}^{k-1} \left(\frac{N^{-2}}{2^N} \right)^{1/p_\nu} + \left(\frac{N^{-2}}{2^N} \right)^{1/p_k} \right) \right|$$

$$\le \sup_n \left| \sum_{k=1}^{\infty} a_{n,k} \left(-\sum_{\nu=1}^{k-1} \left(\frac{N^{-2}}{2^N} \right)^{1/p_\nu} + \left(\frac{N^{-2}}{2^N} \right)^{1/p_k} \right) \right| < \infty;$$

i.e. $\sum_{k=1}^{\infty} a_{n,k} \left(-\sum_{\nu=1}^{k-1} \left(\frac{N^{-2}}{2^N} \right)^{1/p_\nu} + \left(\frac{N^{-2}}{2^N} \right)^{1/p_k} \right)$ converges .

It implies that $A_n \in \overline{sl(p)}^{\beta} = \overline{sl_\infty(p)}$ and hence $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $x \in \overline{sl(p)}$ and for each $n \in \mathbb{N}$.

Convergence of $A_n(x)$ implies that $|A_n(x)| = \left| \sum_{k=1}^{\infty} a_{nk} x_k \right| < \infty \Rightarrow Ax \in l_\infty$.

Conversely let $A\in\left(\overline{sl(p)},l_\infty\right)$. Since $\sigma=\left\{-\sum_{\nu=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{1/p_\nu}+\left(\frac{N^{-2}}{2^N}\right)^{1/p_k}\right\}\in\overline{sl(p)}$, we have $A_n\,\sigma<\infty$ for each $n\in N$ and that $A_n\,\sigma\in l_\infty$. So $\sup_n\left|\sum_{k=1}^{\infty}a_{nk}\,\sigma\right|<\infty$.

We now prove the necessity of (ii).

We assume that the necessity of (ii) is false. Let us define the matrix $B=\left(b_{nk}\right)$ by

$b_{nk}=\Delta\,a_{nk}\ ;\ n,k\in\mathbb{N}$.

Then $B\notin\left(\overline{l(p)},l_\infty\right)$ by the fact that when $0<p_k\le 1$ for every $k\in N$, then $A\in\left(\overline{l(p)},l_\infty\right)$ iff $\sup_{n,k}\left|a_{nk}\right|^{p_k}<\infty$. Hence there is a sequence $y=\left(y_k\right)\in\overline{l\left(p\right)}$ such that

$\sum_{k=1}^{\infty}b_{n,k}\,y_k\neq O(1)$. However, if we define the sequence $u=\left(u_k\right)$ by

$$u_k=\begin{cases}\dfrac{y_1}{2} & for\ k=1\\[2ex] \dfrac{y_k}{2^k}-\dfrac{y_{k-1}}{2^{k-1}} & for\ k>1\end{cases}$$

Then $u\in\overline{sl(p)}$ and $\sum_{k=1}^{\infty}a_{n,k}\,u_k=\sum_{k=1}^{\infty}b_{n,k}\,\frac{y_k}{2^k}\neq O(1)$; which is now a contradiction to the fact that $A\in\left(\overline{sl(p)},l_\infty\right)$. Hence we must have,

$\sup_{n,k}\left|\Delta\,a_{n,k}\right|^{p_k}<\infty$.

Theorem 4.2.

Let $0<p_k\le\frac{1}{2}$ for every $k\in N$. Then $A\in\left(\overline{sl(p)},c\right)$ if and only if

i) $A_n\left(-\sum_{\nu=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{1/p_\nu}+\left(\frac{N^{-2}}{2^N}\right)^{1/p_k}\right)\in c,\,N>1$;

ii) $B\in\left(\overline{l(p)},c\right)$ where $B=(b_{nk})=(\Delta a_{nk})\,;\,n,k\in\mathbb{N}$

iii) $\lim_{n\to\infty}\Delta\,a_{nk}=\Delta\,\alpha_k$ (k is fixed).

Proof:

Let us assume that the above conditions hold. Then for any $x=\left(x_k\right)\in\overline{sl(p)}$, $\sum_{k=1}^{\infty}a_{nk}\,x_k$ is absolutely convergent, and that, $\lim_{n\to\infty}\sum_{k=1}^{\infty}a_{nk}\,x_k=\sum_{k=1}^{\infty}\alpha_k\,x_k$. Hence $A\in\left(\overline{sl(p)},c\right)$.

Conversely, let $A\in\left(\overline{sl(p)},c\right)$. Then $\sum_{k=1}^{\infty}a_{nk}\,x_k$ converges for each $x=(x_k)\in\overline{sl(p)}$ and for $n\in\mathbb{N}$. If we define the sequence $v=(v_k)$ by

$$v_k=\begin{cases}\dfrac{z_1}{2} & \text{for } k=1\\[2ex] \dfrac{z_k}{2^k}-\dfrac{z_{k-1}}{2^{k-1}} & \text{for } k>1\end{cases}$$

Then it can easily be verified that $v\in\overline{sl(p)}$ and $\Delta a_{nk}\to\Delta\alpha_k\ (as\, n\to\infty)$.

Since $x=\left(-\sum_{v=1}^{k-1}\left(\frac{N^{-2}}{2^N}\right)^{1/p_v}+\left(\frac{N^{-2}}{2^N}\right)^{1/p_k}\right)\in\overline{sl(p)}$, then the necessity of (i) follows. We need to show that $B\in\left(\overline{l(p)},c\right)$.

On the contrary we assume that $B\notin\left(\overline{l(p)},c\right)$. Following the same arguments as in Theorem 4. 1 , it can easily be verified that ,

$\left(\sum_{k=1}^{\infty}a_{nk}\,u_k\right)=\left(\sum_{k=1}^{\infty}b_{nk}\,y_k\right)\notin$ c , where $y=(y_k)\in\overline{l(p)}$ and $u=(u_k)\in\overline{sl(p)}$.

This is a contradiction to the fact that $B\in\left(\overline{l(p)},c\right)$. This proves the necessity of (ii).

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