Farey Matrix

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Abstract— **We define two new sequences, namely Farey N-Subsequence and Modified Farey N-Subsequence. Also, we introduce the concept of Farey graph and Farey matrix from Farey sequence and present a few observations.**

Keywords— **Farey Sequence, Farey subsequence, Farey fractions, Farey graph, Farey matrix.**

I. INTRODUCTION

Farey fractions are often better suited than regular continued fractions in all kind of approximation problems. They are a useful tool in a variety of domains, especially in the circle method (started by Hardy and Littlewood in the early 1920's and significantly enhanced over the years, and in the rational approximation to irrationals.

The Farey Sequence in mathematics is a sequence in which you are able to find all fractions that are larger than 0 but less than 1. The Farey$N$-subsequence is obtained from Farey sequence in [0, 1] by a sequence of deletions of fractions except the fractions whose denominatorsare equal to the order of their size. The cardinality of Farey sequenceis$|F_N| = 1 + \sum_{j=1}^{N} \varphi(j)$, the cardinality of Farey N-subsequence is $|F'_N| = \varphi(N)$ and the cardinality of modified reduced Farey sequence is $|F_N^M| = \frac{\varphi(N)}{2}$.

The Farey graph is formed from Farey sequence and the Farey matrix from Farey graphs..

Farey sequence

The sequence of all reduced fractions with denominators not exceeding N listed in order of their size is called the Farey sequence of order N.

For example:- The Farey Sequence of order 5 is

$$F_5 = \left[\frac{0}{1} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{1}{1}\right]$$

Farey$N$-Subsequence$\langle F'_N \rangle$

The subsequence of a Farey sequence of order N whose denominators is equal to N is named as Farey N-subsequence$\langle F'_N \rangle$.

$$\langle F'_N \rangle = \left\{\frac{u_i}{N} / 0 \le u_i \le N, 0 \le i \le N\right\}$$

For example:

$$\langle F'_5 \rangle = \left[\frac{0}{1} < \frac{1}{5} < \frac{2}{5} < \frac{3}{5} < \frac{4}{5} < \frac{1}{1}\right] \text{and}$$

$$\langle F'_6 \rangle = \left[\frac{0}{1} < \frac{1}{6} < \frac{5}{6} < \frac{1}{1}\right]$$

ModifiedFarey$N$-Subsequence$\langle F_N^M \rangle$

The ratio of terms in Farey$N$-Subsequence equidistant from the ends is called modified Farey fractions.In symbols,

$$\langle F_N^M \rangle = \left\{ \frac{u_i}{u_{N-i}} / \ i = 1,2, \dots, \left[\frac{N-1}{2}\right] \right\}$$

Clearly $u_{N-i} + u_i = N$ and $(u_{N-i}, u_i) = 1$. It is called Modified Farey$N$-Subsequence.

Theorem:

For all primes$p \geq 3$, the sum of reciprocal of fractions in modified Farey sequence is

$$\delta \cong p\left[\ln\left(\frac{p-1}{2}\right) + \gamma + \frac{1}{p-1}\right] - \frac{\varphi(p)}{2}$$

Where $\gamma \approx 0.5772156649$ is the Euler Mascheroni constant.

Proof:

Consider the reciprocal of the set of all fractions in modified Farey sequence of order p and all the fractions are in the form $\frac{u_j}{u_i}$; $u_i \in \left[0, \frac{1}{2}\right]$, $u_j \in \left[\frac{1}{2}, 1\right]$

$$\delta = \sum_{i=1}^{\frac{p-1}{2}} \left[\frac{u_j}{u_i}\right]; \qquad j = p - i;$$

$$= \sum_{i=1}^{\frac{p-1}{2}} \left(\frac{p-i}{i}\right)$$

$$= p\sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i} - \frac{\varphi(p)}{2}$$

$$\cong p\left[\ln\left(\frac{p-1}{2}\right) + \gamma + \mathcal{E}_{\left(\frac{p-1}{2}\right)}\right] - \frac{\varphi(p)}{2}$$

$$\delta \cong p\left[\ln\left(\frac{p-1}{2}\right) + \gamma + \frac{1}{p-1}\right] - \frac{\varphi(p)}{2}$$

Example:

The Farey 11-Subsequence is

$$\langle F'_{11} \rangle = \left[\frac{0}{1} < \frac{1}{11} < \frac{2}{11} < \frac{3}{11} < \frac{4}{11} < \frac{5}{11} < \frac{6}{11} < \frac{7}{11} < \frac{8}{11} < \frac{9}{11} < \frac{10}{11} < \frac{1}{1}\right]$$

The corresponding modifiedFarey subsequence is derived from $\langle F'_{11} \rangle$.

$$\langle F_{11}^M \rangle = \left[\frac{1}{10} < \frac{2}{9} < \frac{3}{8} < \frac{4}{7} < \frac{5}{6}\right]$$

Then the reciprocal sum of modified Farey subsequence is$\left(\frac{10}{1} + \frac{9}{2} + \frac{8}{3} + \frac{7}{4} + \frac{6}{5}\right) = 20.12$.

By using the equation

$$\delta \cong p\left[\ln\left(\frac{p-1}{2}\right) + \gamma + \frac{1}{p-1}\right] - \frac{\varphi(p)}{2}$$

the approximate value of the reciprocal sum of modified Farey subsequence $\delta = 20.15$.

Proposition:

In modified Farey subsequence of order N, where N is a prime the fraction is such that $u_{N-i} = -u_i(modN)$and the sum of the fraction is equal to $-\left(\frac{\emptyset(p)}{2}\right)$.

Proof:

$$\psi = \sum_{I=1}^{\frac{P-1}{2}} \frac{u_i}{u_{N-i}}$$
$$= \sum_{I=1}^{\frac{P-1}{2}} (-1)$$
$$= -\left(\frac{P-1}{2}\right)$$
$$\psi = -\left(\frac{\emptyset(p)}{2}\right)$$

Theorem:

The sum of Farey fractions in Farey Sequence on [0, 1] satisfy the recurrence relation

$$S_N = S_{N-1} + \frac{\emptyset(N)}{2}; N \geq 2;$$

Proof:

Let S_N denotes the sum of the Farey fractions of order N. Proof by induction.

Trivially, $S_1 = 1$

$$S_2 = S_1 + \frac{\emptyset(2)}{2}$$
$$S_3 = S_1 + \frac{\emptyset(2)}{2} + \frac{\emptyset(3)}{2}$$
$$S_3 = S_2 + \frac{\emptyset(3)}{2}$$

Assume the result for $N = k$

$$S_k = S_{k-1} + \frac{\emptyset(k)}{2}$$
$$S_{k+1} = \eta_1 + \eta_2 + \cdots + \eta_k + \eta_{k+1}$$

η_k −Sum of the fractions in Farey k-subsequence.

$$S_{k+1} = S_1 + \eta_2 + \cdots + \eta_k + \eta_{k+1}$$
$$= S_1 + \sum_{i=2}^{k+1} \eta_i$$
$$= S_1 + \sum_{i=2}^{k+1} \frac{\phi(i)}{2}$$
$$= S_1 + \frac{\emptyset(2)}{2} + \frac{\emptyset(3)}{2} + \cdots \frac{\emptyset(k)}{2} + \frac{\emptyset(k+1)}{2}$$
$$= S_2 + \frac{\emptyset(3)}{2} + \cdots \frac{\emptyset(k)}{2} + \frac{\emptyset(k+1)}{2}$$
$$= S_3 + \frac{\emptyset(4)}{2} + \cdots \frac{\emptyset(k)}{2} + \frac{\emptyset(k+1)}{2}$$

Continuing this process, we get

$$S_{k+1} = S_k + \frac{\emptyset(k+1)}{2}$$

Hence it is true for N

$$\therefore S_N = S_{N-1} + \frac{\phi(N)}{2}$$

Results:

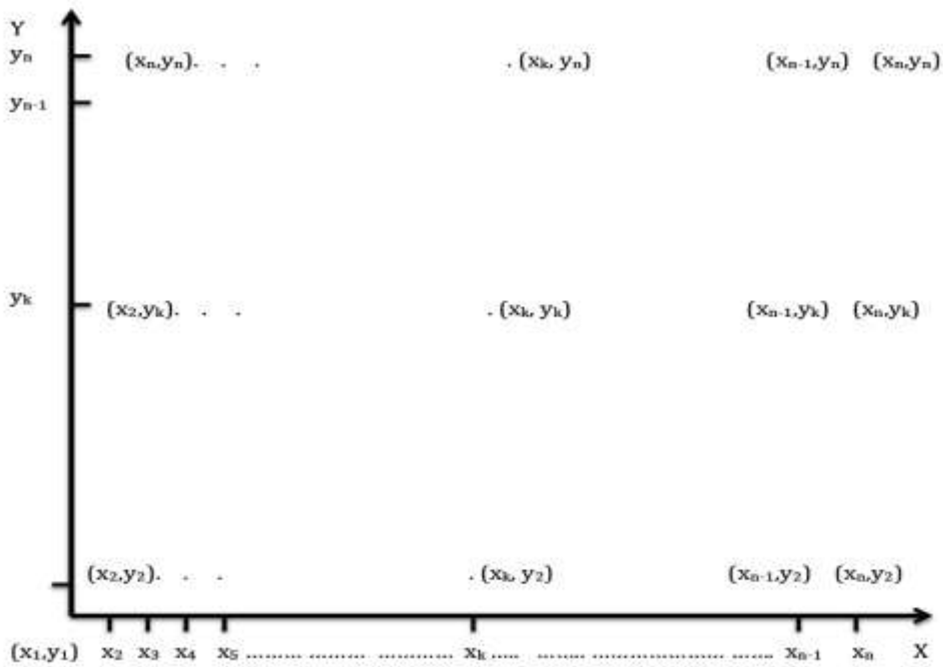
1. In modifiedFarey subsequence of order N, the fraction $\frac{u_i}{u_{N-i}}$ with $u_{N-i} = -u_i(modN)$ then the product $\prod_{i=1}^{\left[\frac{N-1}{2}\right]} \frac{u_i}{u_{N-i}} =$ $1;\ N-12 is\ even-1; N-12 is\ odd$

2. In a modified Farey subsequence of prime order N,the product of the reciprocal fraction$\prod_{i=1}^{n} \frac{u_{N-i}}{u_i} = (n-1)_{c_{\left(\frac{n-1}{2}\right)}}$
3. If N is prime, the product of fractions in Farey N subsequence is equal to $\frac{(N-1)!}{N^{N-1}}$.
4. The sum of Farey fractions in Farey Sequence on [0, 1] is

$$S_N = \frac{|F_N|}{2}$$

Farey Graph:

$X = (x_1, x_2, x_3, \dots, x_n)$,$Y = (y_1, y_2, y_3, \dots, y_n)$where$x_i$,$y_j \in F_N[0,1]$. The Farey graph is a graph of vertices (x_i, y_j) and it forms a grid whose graphical representation is as given below.



FAREY GRAPH

Farey Matrix:

A Farey matrix denoted by FM is defined as the square matrix of order n, whose elements are the sum of Farey fractions in the Farey graph. In other words,

$$FM = \begin{bmatrix} x_1 + y_1 & x_1 + y_2 & \cdots & x_1 + y_n \\ x_2 + y_1 & x_2 + y_2 & \cdots & x_2 + y_n \\ \vdots & \vdots & \cdots & \vdots \\ x_n + y_1 & x_n + y_2 & \cdots & x_n + y_n \end{bmatrix} = [x_i + y_j]_{n\times n}$$

Observations:

1. Farey matrix is always symmetric.
2. The order of the matrix is equal to the cardinality of Farey sequence.

$$n = |F_N| = 1 + \sum_{j=1}^{N} \varphi(j)$$

3. In Fareygraph, the fractions$x_i = y_j$ whenever $i = j$.

Theorem:

The sum of the fractions (row sum or column sum) in Farey matrix is equal to the square of the dimension of the matrix.

Proof:

The Farey matrix,

$$FM = \begin{bmatrix} x_1 + y_1 & x_1 + y_2 & \cdots & x_1 + y_n \\ x_2 + y_1 & x_2 + y_2 & \cdots & x_2 + y_n \\ \vdots & \vdots & \cdots & \vdots \\ x_n + y_1 & x_n + y_2 & \cdots & x_n + y_n \end{bmatrix}$$

Where $x_i \in F_N$, $y_j \in F_N$ and $i = 1,2, \dots, n, j = 1,2, \dots, n$.

SinceA_F,is symmetric the sum of row fractions and column fractions are equal.

Sum of all Farey fractions in this matrix is

$$\eta = \sum_{i=1}^{n}\sum_{j=1}^{n}[x_i + y_j]$$

$$= \sum_{i=1}^{n}[x_i + y_1] + \sum_{i=1}^{n}[x_i + y_2] + \cdots + \sum_{i=1}^{n}[x_i + y_n]$$

$$= n[x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_n] = n[2S_N]$$

$$\eta = n^2$$

Numerical Illustration:

The sum of the fractions in Farey matrix of order 11 is equal to $11^2 = 121$.

Solution:

Consder the Farey matrix of order 11

$$FM = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{9}{20} & \frac{8}{15} & \frac{3}{5} & \frac{7}{10} & \frac{4}{5} & \frac{13}{15} & \frac{19}{20} & 1 & \frac{6}{5} \\ \frac{1}{4} & \frac{9}{20} & \frac{1}{2} & \frac{7}{12} & \frac{13}{20} & \frac{3}{4} & \frac{17}{20} & \frac{11}{12} & 1 & \frac{21}{20} & \frac{5}{4} \\ \frac{1}{3} & \frac{8}{15} & \frac{7}{12} & \frac{2}{3} & \frac{11}{15} & \frac{5}{6} & \frac{14}{15} & 1 & \frac{13}{12} & \frac{17}{15} & \frac{4}{3} \\ \frac{2}{5} & \frac{3}{5} & \frac{13}{20} & \frac{11}{15} & \frac{4}{5} & \frac{9}{10} & 1 & \frac{16}{15} & \frac{23}{20} & \frac{6}{5} & \frac{7}{5} \\ \frac{1}{2} & \frac{7}{10} & \frac{3}{4} & \frac{5}{6} & \frac{9}{10} & 1 & \frac{11}{10} & \frac{7}{6} & \frac{5}{4} & \frac{13}{10} & \frac{3}{2} \\ \frac{3}{5} & \frac{4}{5} & \frac{17}{20} & \frac{14}{15} & 1 & \frac{11}{10} & \frac{6}{5} & \frac{19}{15} & \frac{27}{20} & \frac{7}{5} & \frac{8}{5} \\ \frac{2}{3} & \frac{13}{15} & \frac{11}{12} & 1 & \frac{16}{15} & \frac{7}{6} & \frac{19}{15} & \frac{4}{3} & \frac{17}{12} & \frac{22}{15} & \frac{5}{3} \\ \frac{3}{4} & \frac{19}{20} & 1 & \frac{13}{12} & \frac{23}{20} & \frac{5}{4} & \frac{27}{20} & \frac{17}{12} & \frac{3}{2} & \frac{31}{20} & \frac{7}{4} \\ \frac{4}{5} & 1 & \frac{21}{20} & \frac{17}{15} & \frac{6}{5} & \frac{13}{10} & \frac{7}{5} & \frac{22}{15} & \frac{31}{20} & \frac{8}{5} & \frac{9}{5} \\ 1 & \frac{6}{5} & \frac{5}{4} & \frac{4}{3} & \frac{7}{5} & \frac{3}{2} & \frac{8}{5} & \frac{5}{3} & \frac{7}{4} & \frac{9}{5} & 2 \end{bmatrix}$$

In this matrix,

The sum of the fraction inFM $= \frac{11}{2} + \frac{77}{10} + \frac{33}{4} + \frac{55}{6} + \frac{99}{10} + 11 + \frac{121}{10} + \frac{77}{6} + \frac{55}{4} + \frac{143}{10} + \frac{33}{2} = 11^2 = 121.$

From the above illustration, it is noted thatin aFarey matrix,

1. The sum of the Fractions in middle row (column) is equal to the order of the matrix.
2. The sum of fractions with equidistant from middle to the left and right rows (columns) is equal to twice the order of the matrix.

Conclusion:

In this paper we have introduce two new sequences, from the concept of two new sequences, Also, Farey graph and Farey matrix are introduced. To conclude, one may search for the other properties of Farey matrices.

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