Uniform Location Model: Equivariant Estimation based onProgressively Censored Samples

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ABSTRACT

In this paper, we consider uniform model and obtain minimum risk Equivariant estimators of the parameters based on type-II progressively censored samples under Standardized quadratic loss function, Absolute error loss function and Linex loss function. These generalize the corresponding results for Type-II censored samples. Leo Alexander (2000).

Keywords: Equivariant estimation, Location model, Optimal estimation, Progressive censored sampleand Uniform model.

1. INTRODUCTION

In life-testing experiments, the common practice is to terminate the experiment when certain number of items have failed (Type-II censoring) or certain stipulated time has elapsed (Type-I censoring). Type-II progressive censoring involves removing certain fixed number of surviving units at each failure which is an extended version of Type-II censoring scheme. As pointed out by Balakrishanand Sandhu, the scheme of progressive censoring is an attractive feature as it saves both cost and time for the experimenter. Progressively censored samples have previously been considered by Herd, Roberts, Cohen and others. This developments have been summarized in Cohen, Balakrishnan and Cohen, and Balakrishnan and Aggarwala.

Lehmann and Casella discussed the marginal Equivariant estimation of the parameters of location,scale, and location-scale models. Edwin Prabakaran and Chandrasekar developed simultaneous Equivariant estimation approach and illustrated the method with suitable examples. Leo Alexander and Chandrasekarobtained minimum risk Equivariant estimators for the parameters of Exponential models based on Type-II censored samples. Viveros and Balakrishnan developed exact conditional inference based on progressively Type-II censored samples. Tse and Tso compared the expected times under Type-I censoring, Type-II censoring and complete sampling plans in the exponential case. Also, Aggarwala and Balakrishnan established some properties of progressively Type-II censored order statistics from arbitrary continuous distributions. In his development and discussion on generalized order statistics, Kamps has proved some general properties of progressively Type-II censored order statistics as a special case of generalized order statistics. Further, the maximum likelihood estimation of the parameters of an exponential distribution based on progressively Type-II censored samples has been discussed recently by Cramer and Kamps and Balakrishnan et.al. All these developments on progressively Type-II censored order statistics have been synthesized in a recent book by Balakrishnan and Aggarawala.

In this paper, assuming that the sample is obtained through Type-II progressive censoring scheme, we obtain minimum risk Equivariant estimator(s) for the parameter(s) of the uniform model. The paper is organized as follows: Section 3 deals with the problem of Equivariant estimation for the uniform location model.

2. PRELIMINARIES

Let N denote the total number randomly selected items put to test simultaneously and n designate the number of samples specimens which fail. Thus the number of completely determined life spans is n. At the time of the i-th failure, r_i surviving units are randomly withdrawn from the test, i=1,2,…,n. Clearly, $r_n = N - n - \sum_{i=1}^{n-1} r_i$. Let $X_{i:N}, i = 1,2,\dots, n$, denote the failure times of the completely observed times. Then, the joint probability density function (pdf) of $(X_{1:N}, X_{2:N},\dots, X_{n:N})$ is

$$g_\theta(x_1, x_2,\dots,x_n) = \prod_{i=1}^{n}(N - \sum_{j=1}^{i-1} r_j - i + 1)\times f_\theta(x_i)\{1 - F_\theta(x_i)\}^{r_i} \quad \dots(2.1)$$

Here, f_θ *and* F_θ denote the common pdf and distribution function of the N items under life-test. Further, $r_1, r_2,\dots, r_n$ are assumed to be pre-fixed by the experimenter.

3. LOCATION MODEL

In this case the pdf is taken to be

$$f_{\xi}(x)=\begin{cases}1\ , & \xi \le x \le \xi+1\ ;\ \xi \in R\\ 0\ , & otherwise\end{cases} .$$

Thus (2.1) reduces to

$$g_{\xi}(x_1,\ldots,x_n)=\{\prod_{i=1}^{n}(N-\sum_{j=1}^{i-1}r_j-i+1)\}\prod_{i=1}^{n}\{1-x_{i:N}+\xi\}^{r_i},$$
$$\xi < x_{1:N} < \ldots < x_{n:N} < \xi+1;\xi \in R.$$

Thus the joint distribution of $\mathbf{X}=(X_{1:N},\ldots,X_{n:N})$ belongs to a location family with the location parameter ξ. We are interested in deriving MRE estimator of ξ by considering three loss functions. Following Lehmann and Casella (1998), the MRE estimator of ξ is given by

$$\delta^{*}(\mathbf{X})=\delta_0(\mathbf{X})-v^{*}(\mathbf{Y}) \quad ,$$

where δ_0 is a location equivariant estimator, $v(\mathbf{y})=v^{*}(\mathbf{y})$ minimizes

$$E_0[\rho\{\delta_0(\mathbf{X})-v(\mathbf{y})\}\,|\,\mathbf{y}]$$

and E_0 denotes E_{ξ} when $\xi=0$.

Here

$$Y_i = X_{i:N}-X_{1:N}\ ,\ i=2,3,\ldots,n\ ,$$
$$Y=(Y_2,Y_3,\ldots,Y_n)$$

and ρ is an invariant loss function.

Case (i): If the loss is squared error , then

$$v^{*}=E_0(\delta_0\,|\,\mathbf{y}).$$

Take $\delta_0(\mathbf{X})=(X_{1:N}+X_{n:N})/2$. Clearly δ_0 is an equivariant estimator but not complete sufficient. In order to find v^{*}, assume that $\xi=0$ and consider the transformation

$$Y_1=(X_{1:N}+X_{n:N})/2 \quad \text{and}$$
$$Y_i=(X_{i:N}-X_{1:N})\ ,\ i=2,3,\ldots,n.$$

Then $X_{1:N}=Y_1-Y_n/2$ and

$$X_{i:N}=Y_i+Y_1-Y_n/2\ ,\ i=2,3,\ldots,n$$

and the Jacobian of the transformation is given by $J=1$. Thus the joint pdf of $(Y_1,Y_2,\ldots,Y_n)$ is given by

$$h(y_1,\ldots,y_n)$$
$$=\{\prod_{i=1}^{n}(N-\sum_{j=1}^{i-1}r_j-i+1)\}\{1-(y_1-y_n/2)\}^{r_1}\times$$
$$\prod_{i=2}^{n}\{[1-(y_i+y_1-y_n/2)]^{r_i}\},$$
$$0<y_2<y_3<\ldots<y_n,$$
$$y_n/2<y_1<1-(y_n/2),\ 0<y_n<1.$$

Also, the joint pdf of $(Y_2,\ldots,Y_n)$ is given by

$$h_1(y_2,\ldots,y_n)=\{\prod_{i=1}^{n}(N-\sum_{j=1}^{i-1}r_j-i+1)\}\times$$
$$\int_{y_n/2}^{1-y_n/2}\{1-(y_1-y_n/2)\}^{r_1}\times$$
$$\prod_{i=2}^{n}\{[1-(y_i+y_1-y_n/2)]^{r_i}\}\,dy_1\ ,\ldots(3.1)$$
$$0<y_2<y_3<\ldots<y_n<1.$$

Thus the conditional pdf of $\delta_0=Y_1$ given $(Y_2,\ldots,Y_n)$ is given by

$$h_2(y_1 \mid y_2,\dots,y_n)$$
$$=[\{1-(y_1-y_n/2)\}^{r_1}\prod_{i=2}^{n}\{[1-(y_i+y_1-y_n/2)]^{r_i}\}]/$$
$$[\int_{y_n/2}^{1-y_n/2}\{1-(y_1-y_n/2)\}^{r_1}\prod_{i=2}^{n}\{[1-(y_i+y_1-y_n/2)]^{r_i}\}\,dy_1]$$
$$,y_n/2<y_1<1-(y_n/2). \qquad \dots(3.2)$$

Now

$$v^*=E_0(\delta_0 \mid \mathbf{y})$$
$$=\frac{\int_{y_n/2}^{1-y_n/2} y_1\{1-(y_1-y_n/2)\}^{r_1}\prod_{i=2}^{n}\{[1-(y_i+y_1-y_n/2)]^{r_i}\}dy_1}{\int_{y_n/2}^{1-y_n/2}\{1-(y_1-y_n/2)\}^{r_1}\prod_{i=2}^{n}\{[1-(y_i+y_1-y_n/2)]^{r_i}\}dy_1}$$

in view of (3.2).

$$v^*=\frac{\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2} y_1\prod_{i=1}^{n}\{1-x_{i:N}+(x_{1:N}+x_{n:N})/2-y_1\}^{r_i}dy_1}{\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2}\prod_{i=1}^{n}\{1-x_{i:N}+(x_{1:N}+x_{n:N})/2-y_1\}^{r_i}dy_1} \dots(3.3)$$

Therefore the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X}) = \delta_0(\mathbf{X})-E_0(\delta \mid \mathbf{y})$$
$$= (X_{1:N}+X_{n:N})/2-v^* \quad , \dots(3.4)$$

where v^* is as given in(3.3).

If $N=3, n=2, r_1=1, r_2=0$, then from (3.3) we obtain

$$v^*=\frac{\int_{(x_{2:3}-x_{1:3})/2}^{1-(x_{2:3}-x_{1:3})/2} y_1\{1-(x_{1:3}-x_{2:3})/2-y_1\}dy_1}{\int_{(x_{2:3}-x_{1:3})/2}^{1-(x_{2:3}-x_{1:3})/2}\{1-(x_{1:3}-x_{2:3})/2-y_1\}dy_1}$$
$$=\frac{[\{1-(x_{1:3}-x_{2:3})/2\}\times\{\int_{(x_{2:3}-x_{1:3})/2}^{1-(x_{2:3}-x_{1:3})/2} y_1\,dy_1-\int_{(x_{2:3}-x_{1:3})/2}^{1-(x_{2:3}-x_{1:3})/2} y_1^2\,dy_1\}]}{[\{1-(x_{1:3}-x_{2:3})/2\}\times\{\int_{(x_{2:3}-x_{1:3})/2}^{1-(x_{2:3}-x_{1:3})/2} dy_1-\int_{(x_{2:3}-x_{1:3})/2}^{1-(x_{2:3}-x_{1:3})/2} y_1\,dy_1\}]}$$
$$=\frac{[3\{1-(x_{1:3}-x_{2:3})/2\}(1-x_{1:3}+x_{2:3})-2\{[1-(x_{2:3}-x_{1:3})/2]^3-[(x_{2:3}-x_{1:3})/2]^3\}]}{3\{1-(x_{1:3}-x_{2:3})^2\}}$$

Therefore the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X})=(X_{1:3}+X_{2:3})/2-\frac{[3\{1-(x_{1:3}-x_{2:3})/2\}(1-x_{1:3}+x_{2:3})-2\{[1-(x_{2:3}-x_{1:3})/2]^3-[(x_{2:3}-x_{1:3})/2]^3\}]}{3\{1-(x_{1:3}-x_{2:3})^2\}}$$

Moreover, when the loss is squared error, the MRE estimator $\delta^*(\mathbf{X})$ can be evaluated more explicitly by the Pitman form (Lehmann 1983 p.160).

Therefore the Pitman estimation of ξ is given by

$$\delta^*(\mathbf{X}) = \frac{\int_{x_{n:N}-1}^{x_{1:N}} u \prod_{i=1}^{n} \{1 - x_{i:N} + u\}^{r_i} du}{\int_{x_{n:N}-1}^{x_{1:N}} \prod_{i=1}^{n} \{1 - x_{i:N} + u\}^{r_i} du} \quad \ldots(3.5)$$

Taking $u = \{(x_{1:N} + x_{n:N})/2 - y_1\}$ in (3.5), $\delta^*(\mathbf{X})$ reduces to the following

$$\delta^*(\mathbf{X}) = \frac{[\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2} \{(x_{1:N} + x_{n:N})/2 - y_1\} \times \prod_{i=1}^{n} \{1 - x_{i:N} + (x_{1:N} + x_{n:N})/2 - y_1\}^{r_i} dy_1]}{\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2} \prod_{i=1}^{n} \{1 - x_{i:N} + (x_{1:N} + x_{n:N})/2 - y_1\}^{r_i} dy_1}$$

$$= \frac{(X_{1:N} + X_{n:N})}{2} - [\frac{\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2} y_1 \prod_{i=1}^{n} \{1 - x_{i:N} + (x_{1:N} + x_{n:N})/2 - y_1\}^{r_i} dy_1}{\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2} \prod_{i=1}^{n} \{1 - x_{i:N} + (x_{1:N} + x_{n:N})/2 - y_1\}^{r_i} dy_1}] \quad \ldots(3.6)$$

which coincides with the one given in (3.4).

Remark 3.1:

If $r_k = 0$, $k = 1,2,\ldots, n-1 \, and \; r_n = N - n$, then the above estimator in (3.6) reduces to

$$\delta^*(\mathbf{X}) = \frac{(N-n+1)X_{1:N} + X_{n:N} - 1}{(N-n+2)},$$

which is same as the one obtain for type II right censored case given in (Leo Alexander, 2000).

Case (ii) : If the loss is absolute error, then the MRE estimator of ξ is

$$\delta^*(\mathbf{X}) = (X_{1:N} + X_{n:N})/2 - v_0 \ ,$$

where v_0 is the median of the distribution with pdf given in (3.2).

Case (iii): Consider the location invariant Linex loss function (Varian, 1975)

$$L(\xi;\delta) = e^{a(\delta-\xi)} - a(\delta-\xi) - 1 \ , \quad a \in R - \{0\}.$$

In order to find v^*, take $\delta_0(\mathbf{X}) = (X_{1:N} + X_{n:N})/2$, consider

$$R(\delta \mid \mathbf{y}) = E_0[\{e^{a(\delta_0 - v)} - a(\delta_0 - v) - 1\} \mid \mathbf{y}]$$

$$= e^{-av} E_0(e^{a\delta_0} |\mathbf{y}) - aE_0(\delta_0 |\mathbf{y}) + |av - 1$$

$$= e^{-av} \frac{[\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2} e^{ay_1} \times \{\prod_{i=1}^{n} \{1 - x_{i:N} + (x_{1:N} + x_{n:N})/2 - y_1\}^{r_i} dy_1\}]}{[\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2} \prod_{i=1}^{n} \{1 - x_{i:N} + (x_{1:N} + x_{n:N})/2 - y_1\}^{r_i} dy_1]}$$

$$-a\,\frac{\left[\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2} y_1 \times \left\{\prod_{i=1}^{n}\{1-x_{i:N}+(x_{1:N}+x_{n:N})/2-y_1\}^{r_i}\,dy_1\right\}\right]}{\left[\int_{(x_{n:N}-x_{1:N})/2}^{1-(x_{n:N}-x_{1:N})/2}\prod_{i=1}^{n}\{1-x_{i:N}+(x_{1:N}+x_{n:N})/2-y_1\}^{r_i}\,dy_1\right]} + av - 1 \quad ,$$

in view of (3.3). Thus v^* is to be obtain by minimizing $R(\delta\,|\,\mathbf{y})$. Hence the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X}) = (X_{1:N} + X_{n:N})/2 - v^* .$$

Remark 3.2: If $r_k = 0$, $k = 1,2,..., n-1 \, and \;\; r_n = N - n$, then the above estimator reduces to

$$\delta^*(\mathbf{X}) = (X_{1:N} + X_{n:N})/2 - (1/a)\{cgf\ of\ (\delta_0\,|\,\mathbf{y})\ at\ a\} \quad ,$$

which is the one obtain under Linex loss function based on Type-II censored sample (Leo Alexander, 2000).

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